The asymptotic distribution of elements in automatic sequences

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Abstract

In an automatic sequence an element need not have an asymptotic density. In this paper a necessary and sufficient criterion is proved for the existence of the asymptotic density of a given element. If it does not exist the asymptotic distribution of the element can be described in terms of a function $H$ whose graph is self-similar. An algorithm is given to decide whether $H$ is piecewise continuously differentiable, and in this case it can be computed effectively. Finally, it is shown that the $H^\infty$-density of an element in an automatic sequence always exists and equals its logarithmic density.

1. Introduction

Automatic sequences are among the simplest non-trivial sequences that can be generated by a computing device. Yet they can exhibit an amazingly complex behaviour and are connected to many different parts of mathematics. For an overview, see [1,2,6,16]. One of the most striking results is the following: a sequence $(a_n)_{n \geq 0}$ in the finite field $\mathbb{F}_q$ with $q$ elements is $q$-automatic iff the formal power series $\sum_{n \geq 0} a_n X^n$ is algebraic over the field $\mathbb{F}_q(X)$ of rational functions with coefficients in $\mathbb{F}_q$ (see [4]).

In his excellent paper [5], Cobham proved many distributional properties of automatic sequences. Among others, he showed that (1) every element in an automatic sequence has a logarithmic density,
(2) if an element in an automatic sequence has an asymptotic density then this density is rational,
(3) if an element in an automatic sequence has asymptotic density 0 then there is precise information about the asymptotic distribution of this element.

In the present paper, this investigation is carried on. We prove a simple necessary and sufficient criterion for the existence of an asymptotic density for a given element (Theorem 1.2). If there is no asymptotic density the asymptotic distribution of the element can be described in terms of a Lipschitz continuous 1-periodic function $H$ (Theorem 1.1). Further properties of $H$ are given in Theorem 1.4. There is an algorithm to decide effectively whether $H$ is piecewise continuously differentiable, and in this case $H$ can be computed effectively (Theorem 5.8 and the following remark). The graph of $H$ is always self-similar in the sense that at every point in the unit interval there is a “microscope” focusing at this point such that at every magnification step the graph of $H$ looks the same (Theorem 1.3). Finally, it is shown that the $H^{\infty}$-density of an element in an automatic sequence always exists and equals the logarithmic density (Theorem 1.5). This is a generalization of “Benford’s law” in the form proved by Flehinger [10].

In Section 7 four examples are given which illustrate the different types of behaviour of $H$.

Cobham [5], Theorem 3, proved that a sequence is automatic iff it is a uniform tag sequence. The latter characterization will be used here. Let $m \in \mathbb{N}$, $m \geq 2$, $B$ a finite internal alphabet, $A$ a finite external alphabet, $b_1 \in B$ an initial symbol, $h : B \rightarrow A$ an output map and $w : B \rightarrow B^m$ a production map such that $w(b_1)$ begins with $b_1$. Then $\mathcal{T} := (\mathcal{B}, b_1, w, h, A)$ is called an $m$-tag system. For a finite or infinite word or sequence $x$ and $i \in \mathbb{N}_0$, let $x_i$ be the $i$th letter in $x$. For a finite or infinite word or sequence $y$ over $A$, define $w(y) := w(y_0)w(y_1)\ldots$. Since $b_1$ begins $w(b_1)$ the word $w^k(b_1)$ begins $w^{k+1}(b_1)$ for all $k \in \mathbb{N}_0$. Thus the sequence $y := \lim_{k \rightarrow \infty} w^k(b_1)$ is well defined. It is called the internal sequence intseq($\mathcal{T}$) of $\mathcal{T}$. It is the uniquely determined sequence $y$ over $B$ such that each $w^k(b_1)$, $k \in \mathbb{N}_0$, begins $y$. Alternatively, it is the uniquely determined sequence over $B$ which begins with $b_1$ and has the property $w(y) = y$. The sequence $x = \text{seq}(\mathcal{T})$ over $A$ defined by $x_i := h(y_i)$, $i \in \mathbb{N}_0$, is called the external sequence of $\mathcal{T}$. A sequence $x$ over some finite alphabet $A$ is called a uniform $m$-tag sequence if there is an $m$-tag system $\mathcal{T}$ with output alphabet $A$ such that $x = \text{seq}(\mathcal{T})$.

In this paper we are interested in the uniform tag sequences themselves but not in the tag systems which generate them. Thus we may apply transformations to $\mathcal{T}$ which do not change seq($\mathcal{T}$). The most important one is powering with some $l \in \mathbb{N}$: Let $\mathcal{T}^l := (\mathcal{B}, b_1, w^l, h, A)$. Then $\mathcal{T}^l$ is an $m^l$-tag system with intseq($\mathcal{T}^l$) = intseq($\mathcal{T}$) and seq($\mathcal{T}^l$) = seq($\mathcal{T}$).

The fundamental idea in Cobham’s paper is to associate a stochastic matrix $\mathcal{M}(\mathcal{T})$ to $\mathcal{T}$ in the following way: Let $\mathcal{B} = \{b_1, \ldots, b_t\}$. For $1 \leq i, j \leq t$, define

$$m_{ij} := \frac{1}{m} \# \{0 \leq v < m \mid w(b_1)_v = b_j\}.$$ 

Then $\mathcal{M}(\mathcal{T}) := (m_{ij})_{1 \leq i, j \leq t} \in \mathbb{R}^{t \times t}$ is a row-stochastic matrix. An easy computation shows that $\mathcal{M}(\mathcal{T}^l) = \mathcal{M}(\mathcal{T})^l$, $l \in \mathbb{N}$.
The Perron–Frobenius theory of non-negative matrices (see, e.g., [11,15]) shows that every stochastic matrix $M$ has the eigenvalue 1, every eigenvalue $\lambda$ of $M$ with $|\lambda|=1$ is a root of unity and its geometric and algebraic multiplicities coincide, and all other eigenvalues $\lambda$ have $|\lambda|<1$. Thus there is some $l\in\mathbb{N}$ such that all eigenvalues $\lambda \neq 1$ of $M'$ have $|\lambda|<1$. Such an $l$ can be effectively computed from $M$. In fact, if $M \in \mathbb{R}^{t \times t}$ is stochastic then $l(M):=\text{lcm}\{1 \leq h \leq t \mid X^h-1 \mid p_M\}$ is the smallest $l$ with this property. Here $p_M$ is the characteristic polynomial of $M$.

An integer $m \geq 2$ is called admissible for a uniform tag sequence $x$ if there is an $m$-tag system $\mathcal{T}$ with $\text{seq}(\mathcal{T})=x$ such that every eigenvalue $\lambda \neq 1$ of $\mathcal{M}(\mathcal{T})$ has $|\lambda|<1$. Then the system $\mathcal{T}$ is also called admissible (for $x$). It follows that for every $m$-tag sequence $x$ there is some $l \in \mathbb{N}$ such that $m^l$ is admissible for $x$.

If $\mathcal{T}$ is admissible for $x$ then $\mathcal{M}(\mathcal{T})^\infty := \lim_{k \to \infty} \mathcal{M}(\mathcal{T})^k$ exists. It is an astonishing fact that many asymptotic properties of $x$ can be expressed in terms of $\mathcal{M}(\mathcal{T})^\infty$. This matrix can be computed effectively from $\mathcal{M}(\mathcal{T})$, see [11, Chap. XIII, Section 7].

For a sequence $x$ and a symbol $a$, define

$$\pi(a,x;z):=\#\{0 \leq i \leq z \mid x_i = a\}, \quad z \geq 1.$$ 

In the following theorems always assume that $\mathcal{T}=(\mathcal{B},b_1,w,h,\mathcal{A})$ is an admissible $m$-tag system with $t:=|\mathcal{B}|$ and $y:=\text{intseq}(\mathcal{T})$. If $\mathcal{M}(\mathcal{T})$ has eigenvalues $\in \{0,1\}$, define

$$r(\mathcal{T}):=\max\{|\lambda| \mid \lambda \text{ eigenvalue of } \mathcal{M}(\mathcal{T}), \lambda \neq 1\}$$

and $\delta(\mathcal{T}):=\log(1/r(\mathcal{T}))/\log(m/r(\mathcal{T}))$. Then $0 < r(\mathcal{T}), \delta(\mathcal{T}) < 1$. Otherwise define $r(\mathcal{T}):=0, \delta(\mathcal{T}):=1$.

**Theorem 1.1.** For $b \in \mathcal{B}$, there is a Lipschitz continuous 1-periodic function $H(\cdot;b)$ with

$$\pi(b,y;z) = H(\log_m z;b)z + O((\log z)^{1-\delta(\mathcal{T})}), \quad z \geq m.$$ 

Furthermore, either $H(\cdot;b) \equiv 0$ or $H(\cdot;b) > 0$. The Fourier series of $H(\cdot;b)$ converges absolutely and uniformly to $H(\cdot;b)$.

A related result was proved by Dumont and Thomas [7]. They consider (not necessarily uniform) tag sequences whose matrix $\mathcal{M}(\mathcal{T})$ has the simple eigenvalue 1 and for which there is a second eigenvalue $\theta_2 > 1/m$ such that $|\lambda| < \theta_2$ for all eigenvalues $\lambda \neq 1, \theta_2$. In this case the function $H$ is constant and a second oscillating main term appears after a reduction of the error term.

The next theorem gives an easy-to-check criterion for the existence of an asymptotic density.

**Theorem 1.2.** Assume that every $b \in \mathcal{B}$ occurs in $\text{intseq}(\mathcal{T})$. Let $a \in \mathcal{A}$ and $d \in \mathbb{R}$. Then $a$ has the asymptotic density $d$ in $\text{seq}(\mathcal{T})$ iff for all $1 \leq i \leq t$, we have $\sum_{b \in h^{-1}(a)} \mathcal{M}(\mathcal{T})_{ib}^\infty = d$. 

The self-similarity of the graph of \( H(\cdot; b) \) is contained in

**Theorem 1.3.** For every \( b \in B \) and \( \Lambda^{(\infty)}(0,1) \) there is a sequence of open intervals \((I_n)_{n \geq 1}\) with the properties:

1. \((1, m) \supseteq I_1 \supseteq I_2 \supseteq \cdots \), \( \text{length}(I_n) \to 0 \) as \( n \to \infty \), \( m^{\delta} \in \bigcap_{n \geq 1} I_n \).
2. The graphs of \( xH(\log_m x; b) \upharpoonright I_n, n \geq 1 \), can be transformed into one another by affine linear maps.

The functions \( H(\cdot; b) \) is very simple on intervals where it is continuously differentiable.

**Theorem 1.4.** Let \( b \in B \) and assume that \( H(\cdot; b) \) is continuously differentiable on \((\alpha, \beta) \subset (0,1)\). Then there are constants \( c, c' \in \mathbb{R} \) with \( H(y; b) = c + c'm^{-\Delta} \), \( \Delta \in (\alpha, \beta) \).

Define the \( k \)th Hölder mean of the characteristic function of \( b \) in \( y \) recursively as follows:

\[
M_1(z; b) := \frac{1}{z} \sum_{1 \leq j \leq z; y_j = b} 1,
\]

\[
M_{k+1}(z; b) := \frac{1}{z} \sum_{1 \leq j \leq z} M_k(j; b), \quad k \in \mathbb{N}, \ z \geq 1.
\]

**Theorem 1.5.** For \( b \in B, k \geq 3, z \geq 2 \), we have

\[
M_k(z; b) = \int_0^1 H(\Delta; b) d\Delta + O_{\delta} \left( \left( \frac{\log m}{\log^2 m + 4\pi^2} \right)^k + (\log z)^{z^{-\delta}c_0^k} \right),
\]

where \( 0 < \delta < 1 \) is chosen such that \( \delta \leq \delta(\mathcal{F}) \) and \( c_0 = c_0(\delta) := 7m + m(m^{1-\delta} - 1)^{-1} \). In particular,

\[
\lim_{k \to \infty} \limsup_{z \to \infty} \left| M_k(z; b) - \int_0^1 H(\Delta; b) d\Delta \right| = 0.
\]

The last equation means that the \( H^\infty \)-density of \( b \) in \( y \) exists and equals \( \int_0^1 H(\Delta; b) d\Delta \). For the special case that \( y_n \) is the first digit in the decimal expansion of \( n \) this was proved by Flehinger [10]; see also [12,13] for generalizations. Applying partial summation to Theorem 1.1 shows that the logarithmic density

\[
\lim_{z \to \infty} \frac{1}{\log z} \sum_{1 \leq j \leq z; y_j = b} \frac{1}{j}
\]
exists and equals $\int_0^1 H(A; b) \, dA$, too. The existence of the logarithmic density was first proved by Cobham [5]. For an interesting connection with Dirichlet series, see [3]. They give a formula for the logarithmic density in terms of an infinite series. Except in particular cases, it seems not to be known how to sum this series in closed form.

A result related to Theorem 1.5 was proved in [8]. Under the assumption that 1 is a simple eigenvalue of $\mathcal{M}(T)$ and $1/m$ is another eigenvalue such that $|\lambda| < 1/m$ for all eigenvalues $\lambda \neq 1/m$, a refinement of Proposition 6.1 for $k=2$ is given. In this special case, the function $H_2$ is constant and there are additional, oscillating main terms and a sharper error term.

Software related to the topics of this paper can be downloaded from http://web.mathematik.uni-freiburg.de/mi/zahlen/home/peter.

2. An asymptotic formula

In the following, let $\mathcal{T} = (\mathcal{B}, b_1, w, h, \mathcal{A})$ be an admissible $m$-tag system and $y := \text{intseq}(\mathcal{T})$. Primarily we are interested in its internal sequence which is generated by successive applications of $w$ to $b_1$. But in order to prove the theorems on self-similarity and effective computability we must consider also the sequences of words $w^k(b')$, $k \geq 0$, were $b' \in \mathcal{B}$ is not necessarily $b_1$. In general, however, they cannot be found as the words beginning some fixed infinite sequence, since $w^k(b')$ does not necessarily begin $w^{k+1}(b')$.

Let $b, b' \in \mathcal{B}$, $v \in \mathbb{N}$, $u \in \mathbb{N}_0$, $m^u \leq v \leq m^{u+1}$. Define

$$G(v, u; b, b') := \frac{1}{v} \sum_{0 \leq j < v; b'' := w^{u+1}(b')} \mathcal{M}(\mathcal{T})_{b''}^\infty$$

(2.1)

and

$$G(v; b) := G(v, [\log_m v]; b, b_1) = \frac{1}{v} \sum_{0 \leq j < v} \mathcal{M}(\mathcal{T})_{y}^\infty.$$

Lemma 2.1. Let $b, b' \in \mathcal{B}$. Uniformly in $v \in \mathbb{N}$, $u \in \mathbb{N}_0$, $m^u \leq v \leq m^{u+1}$, $k \in \mathbb{N}$, we have

$$\frac{1}{vm^k} \# \{0 \leq i < vm^k | w^{u+k+1}(b') = b\} = G(v, u; b, b') + \begin{cases} O(k^t r(\mathcal{T})^t), & r(\mathcal{T}) > 0, \\ 0, & r(\mathcal{T}) = 0, \; k \geq t + 1. \end{cases}$$
In particular,
\[
\frac{1}{vm^k} \# \{0 \leq i < vm^k \mid y_j = b \} = G(v; b) + \begin{cases} \mathrm{O}(k' r(\mathcal{F})^k), & r(\mathcal{F}) > 0, \\ 0, & r(\mathcal{F}) = 0, \ k \geq t + 1. \end{cases}
\]

**Proof.** Let \( k \in \mathbb{N} \). For \( 0 \leq i < vm^k \), write \( i = jm^k + i', \ 0 \leq j < v, \ 0 \leq i' < m^k \). Then
\[
w_{u+k+1}(b') = w^k(w_{u+1}(b'))_i = w^k(w_{u+1}(b'))_{i'}.
\]
Thus
\[
\# \{0 \leq i < vm^k \mid w^{k+u+1}(b')_i = b\} = \sum_{0 \leq j < v} \# \{0 \leq i' < m^k \mid w^k(w_{u+1}(b'))_{i'} = b\}.
\]
The \( j \)th summand equals \( m^k \mathcal{M}(\mathcal{F})_{y_{j}b} \) where \( b' := w_{u+1}^{u+1}(b') \). Let \( \| \cdot \| \) be the maximum-norm on the space of complex \( t \times t \)-matrices. It follows that
\[
\left| \frac{1}{vm^k} \# \{0 \leq i < vm^k \mid w^{k+u+1}(b')_i = b\} - G(v,u;b,b') \right| \leq \| \mathcal{M}(\mathcal{F})^k - \mathcal{M}(\mathcal{F})^\infty \|.
\]
The eigenvalue 1 of \( \mathcal{M}(\mathcal{F}) \) has the same algebraic and geometric multiplicities. All eigenvalues \( \lambda \neq 1 \) have \( |\lambda| < 1 \). Transforming \( \mathcal{M}(\mathcal{F}) \) to Jordan normal form thus gives
\[
\| \mathcal{M}(\mathcal{F})^k - \mathcal{M}(\mathcal{F})^\infty \| \leq k'r(\mathcal{F})^k
\]
if \( \mathcal{M}(\mathcal{F}) \) has eigenvalues \( \neq 0, 1 \) and
\[
\mathcal{M}(\mathcal{F})^k = \mathcal{M}(\mathcal{F})^\infty \quad \text{for} \ k \geq t + 1
\]
otherwise. From this the first part follows.

If \( b' = b_1 \) the sequence \( y \) begins with \( w^{k+u+1}(b_1) \). Thus the second part follows. \( \square \)

The next lemma interpolates the previous one.

**Lemma 2.2.** Let \( b, b' \in \mathcal{B} \). There is a function \( H(\cdot; b,b') \) on \([0, 1]\) such that uniformly for \( A \in [0, 1], \ k \in \mathbb{N} \), we have
\[
\frac{1}{m^{d+k}} \# \{0 \leq i < m^{d+k} \mid w^{k+1}(b')_i = b\} = H(A; b,b') + \mathrm{O}(k'm^{-d(\mathcal{F})^k}).
\]
In particular, for \( v \in \mathbb{N}, \ u \in \mathbb{N}_0, \ m^u \leq v \leq m^{u+1}, \) we have
\[
H \left( \log_m \left( \frac{v}{m^u} \right); b,b' \right) = G(v,u;b,b').
\]
Proof. Let \( A \in [0,1] \), \( u \in \mathbb{N}_0 \) and \( k \geq (u+t+1)/\delta(\mathcal{T}) \). Then \( u \in [0,k-t-1] \). Choose \( v \in \mathbb{N} \) such that
\[
1 \leq vm^{-u} \leq m^A \leq (v+1)m^{-u} \leq m. \tag{2.2}
\]

Then
\[
N(k,A) := \# \{ 0 \leq i < m^{A+k} | w^{A+1}(b'), i = b \} \leq \# \{ 0 \leq i \leq (v+1)m^{k-u} | w^{(k-u)+u+1}(b'), i = b \}.
\]

Lemma 2.1 shows that the right-hand side equals
\[
(v + 1)m^{k-u}(G(v + 1, u; b, b') + O((k - u)r(\mathcal{T})^{k-u}))
\]
if \( r(\mathcal{T}) > 0 \); if \( r(\mathcal{T}) = 0 \), the error term is 0. Thus in any case,
\[
m^{-A-k}N(k,A) \leq (v + 1)m^{-A-u}G(v + 1, u; b, b') + O(k'm^{-\delta(\mathcal{T})k}). \tag{2.3}
\]

The same reasoning gives
\[
m^{-A-k}N(k,A) \geq vm^{-A-u}G(v, u; b, b') + O(k'm^{-\delta(\mathcal{T})k}). \tag{2.4}
\]

Now we apply (2.3) and (2.4) in two steps. First, fix \( u \in \mathbb{N}_0 \), \( A \in [0,1] \) and \( v \in \mathbb{N} \) with (2.2). Then (2.3) and (2.4) show that
\[
vm^{-A-u}G(v, u; b, b') \leq H^- := \liminf_{k \to \infty} m^{-A-k}N(k,A)
\]
\[
\leq H^+ := \limsup_{k \to \infty} m^{-A-k}N(k,A)
\]
\[
\leq (v + 1)m^{-A-u}G(v + 1, u; b, b').
\]

From (2.1) it follows that \( 0 \leq H^+ - H^- \leq m^{-A-u} \). This holds for arbitrary \( u \in \mathbb{N}_0 \). Thus \( H^+ = H^- \), meaning that the limit
\[
H(\Delta; b, b') := \lim_{k \to \infty} m^{-\Delta-k}N(k,A)
\]
exists for all \( \Delta \in [0,1] \). Furthermore, for \( \Delta \in [0,1] \), \( u \in \mathbb{N}_0 \), and \( v \in \mathbb{N} \) with (2.2), we have
\[
H(\Delta; b, b') = vG(v, u; b, b')m^{-A-u} + O(m^{-A-u}). \tag{2.5}
\]

In the second step, let \( A \in [0,1] \), \( k \geq (t+1)/\delta(\mathcal{T}) \) and \( u := [\delta(\mathcal{T})k] - t - 1 \). Choose \( v \in \mathbb{N} \) with (2.2). Then (2.3), (2.4) and (2.5) give
\[
m^{-A-k}N(k,A) = H(\Delta; b, b') + O(m^{-A-u} + k'm^{-\delta(\mathcal{T})k})
\]
\[
= H(\Delta; b, b') + O(k'm^{-\delta(\mathcal{T})k}),
\]
which proves the first part of the lemma. A comparison with Lemma 2.1 gives the second part. \( \square \)
Next we collect some properties of \( H \).

**Lemma 2.3.** Let \( b, b' \in \mathcal{B} \).

1. The function \( H(\cdot; b, b') \) is Lipschitz continuous on \([0, 1]\).
2. The function \( H(\cdot; b) := H(\cdot; b, b_1) \) can be continued to a 1-periodic function such that for all \( \Delta \in \mathbb{R} \),
   \[
   \lim_{k \to \infty} \frac{1}{m^{k+\Delta}} \# \{0 \leq i < m^{k+\Delta} \mid y_i = b\} = H(\Delta; b).
   \]
   It is the absolute and uniform limit of its Fourier series. If it has a zero then it vanishes everywhere.

**Proof.** (1) For all \( 0 \leq \Delta_1 \leq \Delta_2 \leq 1 \),
   \[
   0 \leq m^{\Delta_2} H(\Delta_2; b, b') - m^{\Delta_1} H(\Delta_1; b, b')
   = \lim_{k \to \infty} \frac{1}{m^k} \# \{ m^{\Delta_1+k} \leq i < m^{\Delta_2+k} \mid w^{k+1}(b')_i = b \}
   \leq m^{\Delta_2} - m^{\Delta_1} \leq \Delta_2 - \Delta_1.
   \]
   Consequently, the function \( \Delta \mapsto m^\Delta H(\Delta; b, b') \) is Lipschitz continuous on \([0, 1]\). The same holds for \( \Delta \mapsto m^{-\Delta} \) and for the product of these two functions.

(2) Let \( \Delta \in \mathbb{R} \). Then Lemma 2.2 with \( b' := b_1 \) gives
   \[
   \lim_{k \to \infty} \frac{1}{m^{k+\Delta}} \# \{0 \leq i < m^{k+\Delta} \mid y_i = b\}
   = \lim_{k \to \infty} \frac{1}{m^{k+\{\Delta\}}} \# \{0 \leq i < m^{k+\{\Delta\}} \mid y_i = b\}
   = H(\{\Delta\}; b).
   \]
   In particular, we have \( H(1; b) = H(0; b) \). Since \( H(\cdot; b) \) is 1-periodic and Lipschitz continuous, the statement about the Fourier series follows from classical theorems (see, e.g., [9, Remark 2 to Section 10.1.4 and the Remarks to Section 10.6.2]. If \( \Delta_0 \in \mathbb{R} \) with \( H(\Delta_0; b) = 0 \) then
   \[
   \liminf_{k \to \infty} \frac{1}{n} \# \{0 \leq i < n \mid y_i = b\}
   \leq \lim_{k \to \infty} \frac{1}{m^{k+\Delta_0}} \# \{0 \leq i < m^{k+\Delta_0} \mid y_i = b\}
   = H(\Delta_0; b) = 0,
   \]
   i.e. the lower density of \( b \) in \( y \) is 0. From [5], Theorem 11, it follows that the density of \( b \) in \( y \) is 0, i.e. for all \( \Delta \in \mathbb{R} \), we have \( H(\Delta; b) = 0 \). \( \square \)
Corollary 2.4. For $b \in \mathcal{B}$ and $z \geq m$, we have
\[
\pi(b, y; z) = H(\log_m z; b)z + O((\log z)^{1-\delta(\mathcal{F}))}.
\]

Proof. Define $A := \{\log_m n\}$, $k := [\log_m z]$, $b' := b_1$. Then Lemma 2.2 and the 1-periodicity of $H(\cdot; b)$ give the result. □

Lemma 2.3 and Corollary 2.4 together give Theorem 1.1.

Next we prove Theorem 1.2. Let $x := \text{seq}(\mathcal{F})$. From Corollary 2.4 it follows that
\[
\pi(a, x; z) = \sum_{b \in h^{-1}(a)} \pi(b, y; z) = \left( \sum_{b \in h^{-1}(a)} H(\log_m z; b) \right) z + O((\log z)^{1-\delta(\mathcal{F}))}, \quad z \geq m.
\]
Consequently, the symbol $a$ has the asymptotic density $d$ in $x$ iff
\[
d = \lim_{z \to \infty} \frac{1}{z} \pi(a, x; z) = \lim_{z \to \infty} \left( \sum_{b \in h^{-1}(a)} H(\log_m z; b) \right).
\]
Since the functions $H(\cdot; b)$ are 1-periodic this is equivalent to
\[
\sum_{b \in h^{-1}(a)} H(\Delta; b) = d \quad \text{for all } \Delta \in [0, 1].
\]
Since the functions $H(\cdot; b)$ are continuous and the set $\{\log_m (vm^{-u}) \mid v \in \mathbb{N}, \ u \in \mathbb{N}_0, \ m^u \leq v \leq m^{u+1}\}$ is dense in $[0, 1]$, the symbol $a$ has the asymptotic density $d$ in $x$ iff
\[
d = \sum_{b \in h^{-1}(a)} H\left(\log_m \left(\frac{v}{m^u}\right) ; b\right) = \sum_{b \in h^{-1}(a)} G(v; b)
\]
for $v \in \mathbb{N}$, $u \in \mathbb{N}_0$, $m^u \leq v \leq m^{u+1}$. This is equivalent to
\[
dv = \sum_{0 \leq j < v} \sum_{b \in h^{-1}(a)} \mathcal{M}(\mathcal{F})_{y, b} \text{ for all } v \in \mathbb{N}.
\]
Since the inner sum only depends on $j$ but not on $v$, this is equivalent to
\[
d = \sum_{b \in h^{-1}(a)} \mathcal{M}(\mathcal{F})_{y, b} \text{ for all } j \in \mathbb{N}_0.
\]
Since by assumption every $b' \in \mathcal{B}$ occurs in $y$, the symbol $a$ has the asymptotic density $d$ in $x$ iff
\[
d = \sum_{b \in h^{-1}(a)} \mathcal{M}(\mathcal{F})_{y, b} \text{ for all } b' \in \mathcal{B},
\]
which proves Theorem 1.2.
3. Further properties of \(H\)

In this section those properties of \(H(\cdot;b,b')\) are investigated which come from self-reference. The key tool for this is the following lemma. For \(b,b' \in B\), define

\[
K(\Delta;b,b') := m^\Delta H(\Delta;b,b'), \quad \Delta \in [0,1].
\]

Furthermore, define the map \(T:B \to B\) by \(T(b) := w(b)_0\), \(b \in B\).

**Lemma 3.1.** Let \(b,b' \in B\), \(u \in \mathbb{N}_0\), \(v \in \mathbb{N}\), \(m^u \leq v < m^{u+1}\), \(k \in \mathbb{N}\). Then

\[
K(\Delta;b,b') = K\left( \log_m \left( \frac{v}{m^u} \right); b,b' \right) + m^{-u-k} K\left( \log_m \left( \left( m^\Delta - \frac{v}{m^u} \right) m^{k+u} \right); b, T^{k-1}(w^{u+1}(b)_c) \right)
\]

for all \(\Delta \in [\log_m(vm^u + m^{-u-k}), \log_m(vm^u + m^{1-u-k})]\).

**Proof.** For \(\Delta\) in the given interval, Lemma 2.2 gives

\[
K(\Delta;b,b') - K\left( \log_m \left( \frac{v}{m^u} \right); b,b' \right) = \lim_{l \to \infty} \frac{1}{m^l} \#\{ vm^{l-u} \leq i < m^{l+1} | w^{l+1}(b')_i = b \}.
\]

Let \(l\) be sufficiently large. For \(vm^{l-u} \leq i < m^{l+1}\), write \(i = vm^{l-u} + j\), \(0 \leq j < m^{l-k-u}((m^\Delta - vm^{-u})m^{k+u})\). Denote the factor in parentheses by \(m^d\). Then \(\Delta' \in [0,1]\). Thus

\[
w^{l+1}(b')_i = w^{l-u-k+1}(w^{u+k}(b'))_{vm^{l-u}+j} = w^{l-u-k+1}(w^{u+k}(b')_{vm^{l-u}+j})_j
\]

and

\[
w^{u+k}(b')_{vm^{l-u}+j} = w^{k-1}(w^{u+1}(b')_c)_0 = T^{k-1}(w^{u+1}(b')_c).
\]

Putting everything together gives

\[
K(\Delta;b,b') - K\left( \log_m \left( \frac{v}{m^u} \right); b,b' \right) = m^{-k-u+A'} \lim_{l \to \infty} \frac{1}{m^{l-k-u+A'}}
\]

\[
\times \# \{ 0 \leq j < m^{l-k-u+A'} | w^{l-u-k+1}(T^{k-1}(w^{u+1}(b')_c))_j = b \}
\]

\[
= m^{-k-u+A'} H(\Delta'; b, T^{k-1}(w^{u+1}(b')_c)),
\]

which proves the lemma. \(\square\)
For $\Delta \in [0, 1]$, an $m$-adic expansion of $m^{\Delta}$ can be written in the form

$$m^{\Delta} = \sum_{1 \leq i < l_0} v_i m^{-k_i-\cdots-k_{i-1}}, \quad (3.1)$$

where $v_i \in \{1, \ldots, m-1\}$, $k_i \in \mathbb{N}$ for $1 \leq i < l_0$, and $v_i = 0$ for $i \geq l_0$. For $m^{\Delta} \notin \bigcup_{n \geq 0} m^{-n}\mathbb{Z}$ the sum is infinite (i.e. $i_0 = \infty$) and the $v_i$ and $k_i$ are uniquely determined; for $\Delta = 0$, we have $v_i = 1$ and $k_i = 0$ for $1 \leq i < \infty$; for $\Delta = 1$, we have $v_i = m - 1$, $k_i = 1$ for $1 \leq i < \infty$; for $0 < \Delta < 1$, $m^{\Delta} = vm^{-n}$, $v \in \mathbb{N}$, $u \in \mathbb{N}_{0}$, there is a finite expansion ($i_0 < \infty$) and an infinite one ($i_0 = \infty$). The next proposition will show that an $m$-adic expansion of $m^{\Delta}$ gives rise to a series expansion for $K(\Delta; b, b')$.

For $v \in \{1, \ldots, m-1\}$, $k \in \mathbb{N}$, define

$$I_{v,k} := [\log_m(v + m^{-k}), \log_m(v + m^{1-k})] \subseteq [0, 1],$$

$$f_{v,k} : I_{v,k} \to [0, 1], \quad A \mapsto \log_m(m^A - v) + k,$n

$$X_{v,k} : \mathcal{B} \to \mathcal{B}, \quad b \mapsto T_k^{-1}(w(b)_v) = w^k(b)_v m^{k-1}.$$

**Proposition 3.2.** Let $b, b' \in \mathcal{B}$, $\Delta \in [0, 1]$ and (3.1) an $m$-adic expansion of $m^{\Delta}$. For $1 \leq j < l_0$, define $A_j \in [0, 1]$ by $m^{A_j} := \sum_{1 \leq i < l_0} v_i m^{-k_i-\cdots-k_{i-1}}$. Then for $0 \leq j < l_0 - 1$, we have

$$K(\Delta; b, b') = \sum_{1 \leq i \leq j} m^{-k_i-\cdots-k_{i-1}} K(\log_m v_i; b, X_{v_i,k_i,\cdots,k_1}(b'))$$

$$+ m^{-k_j-\cdots-k_1} K(A_{j+1}; b, X_{v_j,k_j,\cdots,k_1}(b')).$$

**Proof.** For $1 \leq j < l_0 - 1$, we have $v_j + m^{-k_j} \leq m^{A_j} \leq v_j + m^{1-k_j}$ and thus $A_j \in I_{v_j,k_j}$. Furthermore, $f_{v_j,k_j}(A_j) = A_{j+1}$. The proposition is now proved by induction on $j$. The case $j = 0$ is obvious since $A_1 = \Delta$. Now let $0 \leq j < l_0 - 2$ and assume that the identity holds for $j$. Since $A_{j+1} \in I_{v_{j+1},k_{j+1}}$ and $f_{v_{j+1},k_{j+1}}(A_{j+1}) = A_{j+2}$, Lemma 3.1 gives

$$K(A_{j+1}; b, X_{v_j,k_j,\cdots,k_1}(b'))$$

$$= K(\log_m v_{j+1}; b, X_{v_j,k_j,\cdots,k_1}(b'))$$

$$+ m^{-k_{j+1}} K(A_{j+2}; b, X_{v_{j+1},k_{j+1},\cdots,k_1}(b')).$$

Plugging this into the identity for $j$ gives the identity for $j + 1$. \qed

**Corollary 3.3.** Let $b, b' \in \mathcal{B}$, $\Delta \in [0, 1]$ and (3.1) an $m$-adic expansion of $m^{\Delta}$. Then

$$K(\Delta; b, b')$$

$$= \sum_{1 \leq i < l_0} m^{-k_i-\cdots-k_{i-1}} K(\log_m v_i; b, X_{v_i,k_i,\cdots,k_1}(b')).$$
Proof. If \( i_0 = \infty \), use \( 0 \leq K(\cdot; b, b') \leq m \) uniformly in \( b' \in \mathcal{B} \) in Proposition 3.2. If \( i_0 < \infty \), the result follows from \( m^{d_{i_0 - 1}} = \nu_{i_0 - 1} \) for \( j = i_0 - 2 \). \( \square \)

Now we can prove Theorem 1.3 for arbitrary \( H(\cdot; b, b') \) instead of \( H(\cdot; b) \). Let \( b, b' \in \mathcal{B} \) and \( \mathcal{A}(\infty) \in (0, 1) \). Let \( m^{d_{\infty}} = \sum_{i \geq 1} v_i m^{-k_i - \cdots - k_i} \) be the infinite \( m \)-adic expansion of \( m^{d_{\infty}} \). For \( r \in \mathbb{N} \), define \( \mathcal{A}(r) \in [0, 1] \) by

\[
m^{d^{(r)}} := \sum_{i=1}^r v_i m^{-k_i - \cdots - k_i - 1}
\]

and set

\[
I_r := (\log_m(m^{d^{(r)}}) + m^{-k_i - \cdots - k_i}), \log_m(m^{d^{(r)}}) + m^{-k_i - \cdots - k_i + 1}) \subseteq (0, 1).
\]

Then \( I_r \supset I_{r+1} \) for all \( r \geq 1 \). Let \( \mathcal{A} \in I_r \) and define \( m^{d_{r+1}} := (m^d - m^{d^{(r)}})m_{k_1 + \cdots + k_1} \in (1, m) \). Let \( m^{d_r} = \sum_{i \geq 1} v_i m^{-k_i - \cdots - k_i} \) be the infinite \( m \)-adic expansion of \( m^d \). Since \( m^d = m^{d^{(r)}} + m^{d_{r+1}} \), we see that \( \tilde{v}_i = v_i, \tilde{k}_i = k_i \) for \( 1 \leq i \leq r \). Now Proposition 3.2 and Corollary 3.3 show that

\[
K(\mathcal{A}; b, b') = \sum_{i=1}^r m^{-k_i - \cdots - k_i} K(\log_m \tilde{v}_i; b, X_{\tilde{v}_i-1, \tilde{k}_i} \circ \cdots \circ X_{\tilde{v}_1, \tilde{k}_1}(b'))
\]

\[
+ m^{-k_i - \cdots - k_i} K(\mathcal{A}_{r+1}; b, X_{\tilde{v}_i, \tilde{k}_i} \circ \cdots \circ X_{\tilde{v}_1, \tilde{k}_1}(b'))
\]

\[
= \sum_{i=1}^r m^{-k_i - \cdots - k_i} K(\log_m v_i; b, X_{v_i-1, k_i} \circ \cdots \circ X_{v_1, k_i}(b'))
\]

\[
+ m^{-k_i - \cdots - k_i} K(\mathcal{A}_{r+1}; b, X_{v_i, k_i} \circ \cdots \circ X_{v_1, k_i}(b'))
\]

\[
= K(\mathcal{A}(r); b, b') + m^{-k_i - \cdots - k_i} K(\log_m (m^d - m^{d^{(r)}})
\]

\[
+ k_1 + \cdots + k_s; b, X_{v_i, k_i} \circ \cdots \circ X_{v_1, k_i}(b')). \tag{3.2}
\]

For \( 1 \leq r < s \), define the map \( F_{r,s}: I_r^* \to I_s^* \) by

\[
F_{r,s}(\mathcal{A}) = \log_m(m^d + m^{d^{(s)}} + \cdots + k_s - m^{d^{(r)}}) - k_{r+1} - \cdots - k_s.
\]

Assume that \( 1 \leq r < s \) are such that

\[
X_{v_i, k_i} \circ \cdots \circ X_{v_1, k_i}(b') = X_{v_i, k_i} \circ \cdots \circ X_{v_1, k_i}(b'). \tag{3.3}
\]

Then it follows for \( \mathcal{A} \in I_r^* \) from (3.2) that

\[
K(F_{r,s}(\mathcal{A}); b, b')
\]

\[
= K(\mathcal{A}(r); b, b') + m^{-k_i - \cdots - k_i} K(\log_m (m^{F_{r,s}(\mathcal{A})}) - m^{d^{(r)})}
\]

\[
+ k_1 + \cdots + k_s; b, X_{v_i, k_i} \circ \cdots \circ X_{v_1, k_i}(b')).
\]
\[ = K(A^{(s)}; b, b') + m^{-k_1 - \cdots - k_r} K(\log_m (m^A - m^{A'}) )
\]
\[ + k_1 + \cdots + k_r; b, X_{i', k'} \circ \cdots \circ X_{i, k_i} (b')) 
\]
\[ = m^{-k_1 - \cdots - k_r} K(A; b, b') - m^{-k_1 - \cdots - k_r} K(A^{(s)}; b, b') + K(A^{(s)}; b, b'). \quad (3.4) \]

In the case \( m^{A^{(\infty)}} \notin \bigcup_{u \geq 0} m^{-u} \mathbb{Z} \), this is enough to prove Theorem 1.3. To this end, observe that under this assumption there are infinitely many \( i \geq 1 \) with \( v_i < m - 1 \) or \( k_i > 1 \). Thus for all \( r \geq 1 \),
\[
m^{A^{(r)}} + m^{-k_1 - \cdots - k_r} < m^{A^{(\infty)}} < m^{A^{(r)}} + (m - 1)m^{-k_1 - \cdots - k_r} \sum_{i \geq 0} m^{-i} 
\]
\[ = m^{A^{(r)}} + m^{1-k_1 - \cdots - k_r} \]
and consequently \( A^{(\infty)} \in I^*_r \). Since \((X_{i, k}; X_{i', k'} \circ \cdots \circ X_{i, k_i} (b'))_{i \geq 1}\) is a sequence in the finite set \( \mathcal{B} \), there is \( r \geq 1 \) such that for infinitely many \( s \gg r \), condition (3.3) holds. For these \( s \) Eq. (3.4) shows that the graph of \( K(\log_m x; b, b') \) on \( \{m^A | A \in I^*_r \} \) can be transformed into the graph of the same function on \( \{m^A | A \in I^*_r \} \) by an affine linear transformation.

Now let \( m^{A^{(\infty)}} = v m^{-u}, v \in \mathbb{N}, u \in \mathbb{N}_0, m^u < v < m^{u+1} \). Then
\[
v_i = m - 1, \quad k_i = 1 \quad \text{for} \quad i \geq u + 2. \quad (3.5) \]
Thus for \( r \geq u + 1 \), we have
\[ m^{A^{(\infty)}} = m^{A^{(r)}} + m^{1-k_1 - \cdots - k_r} \quad (3.6) \]
and consequently \( A^{(\infty)} \) is the right endpoint of \( I^*_r \).

For \( k \in \mathbb{N} \), define
\[ I'_k := [\log_m (m^{A^{(\infty)}} + m^{-u-k}), \log_m (m^{A^{(\infty)}} + m^{1-u-k})] \subseteq (0,1). \]

From Lemma 3.1 it follows that for \( A \in I'_k \),
\[ K(A; b, b') = K(A^{(\infty)}; b, b') + m^{-u-k} K(\log_m (m^A - m^{A^{(\infty)}}) 
\]
\[ + u + k; b, T^{k-1}(w^{u+1}(b'), V)). \quad (3.7) \]

Since \( |\mathcal{B}| = t \), the sequence \((T^{k-1}(w^{u+1}(b'), v))_{k \geq 1}\) in \( \mathcal{B} \) will be periodic for \( k \geq t \); let \( p \) be its period. For \( l \in \mathbb{N}_0 \), define the map
\[ F'_l : (A^{(\infty)}, \log_m (m^{A^{(\infty)}} + m^{1-u-l})) \rightarrow (A^{(\infty)}, \log_m (m^{A^{(\infty)}} + m^{1-u-l-p})) \]
by
\[ F'_l (A) := \log_m ((m^A - m^{A^{(\infty)}})m^{-l} + m^{A^{(\infty)}}). \]
For \( k \geq t, l \geq 0 \), we have \( F'_l(I'_k) = I'_{k+l,p} \).
Now let \( l \geq 0 \) and \( \Delta \in (A^{(\infty)}, \log_m(m^{d^{(\infty)} + m^{1-u-l}})) =: I^* \). Choose \( k \geq t \) with \( \Delta \in I^*_k \). Then \( F'_l(\Delta) \in I^*_{k+lp} \). From (3.7) together with the definition of \( p \) it follows that

\[
K(F'_l(\Delta); b, b') = K(A^{(\infty)}; b, b') + m^{-u-k-lp}K(\log_m(m^{F'_l(\Delta)} - m^{d^{(\infty)}}))
\]

\[
+ u + k + lp; b, T^{k+lp-1}(w^{u+1}(b))
\]

\[
= K(A^{(\infty)}; b, b') + m^{-u-k-lp}K(\log_m(m^d - m^{d^{(\infty)}}))
\]

\[
+ u + k; b, T^{k-1}(w^{u+1}(b))
\]

\[
= m^{-lp}K(A; b, b') - m^{-lp}K(A^{(\infty)}; b, b') + K(A^{(\infty)}; b, b').
\]

(3.8)

Now we compare this identity with (3.4). From (3.5) and (3.6) it follows that for \( s > r \geq u + 1, \Delta \in I^*_r \), we have

\[
F_{r,s}(\Delta) = \log_m((m^{d} - m^{d^{(\infty)}})^{-u-r} + m^{d^{(\infty)}}).
\]

(3.9)

Define \( U := X_{m-1,1}, b'' := X_{m+1, k_{m+1}} \circ \cdots \circ X_{m+1, k_1}(b') \). Then for \( r \geq u + 1 \), we have

\[
X_{m+1, k_0} \circ \cdots \circ X_{m+1, k_1}(b') = U^{r-u+1}(b'').
\]

(3.10)

Let \( p^* \) be the period of \((U^k(b''))_{k \geq 1} \). Then it follows from (3.10) that (3.3) holds for \( r := u + t, s := r + lp^*, l \in \mathbb{N} \). Thus (3.4) holds for these \( r, s \) and \( \Delta \in I^*_r \). Together with (3.8) and (3.9) it follows that for \( l \in \mathbb{N}_0 \), the function

\[
K(\log_m((m^{d} - m^{d^{(\infty)}})^{-lp^{p^*}} + m^{d^{(\infty)}}); b, b') - m^{lp^{p^*}}K(A; b, b')
\]

is constant on \( I^*_u \) and on \( I^* \). Since \( A^{(\infty)} \) is the right endpoint of \( I^*_u \) and the left endpoint of \( I^* \) it follows by continuity that this function is constant on \( I^*_u \cup I^* \). This interval has \( A^{(\infty)} \) as an inner point. Thus Theorem 1.3 is proved also in the case \( m^{d^{(\infty)}} = v m^{-u} \). \( \square \)

4. Continuous differentiability

In order to prove Theorem 1.4 we need two preparatory results.

**Lemma 4.1.** Let \( f : (a, b) \to \mathbb{R} \) be continuous and \( N \) a dense subset of \((a, b)\). If \( M \subseteq \mathbb{R} \) is finite and \( f(N) \subseteq M \) then \( f \) is constant.

**Proof.** For \( z \in M \) define the open subset \( U(z) := (\mathbb{R} \setminus M) \cup \{z\} \) of \( \mathbb{R} \). Then \( f \) is constant equal to \( z \) on \( N \cap f^{-1}(U(z)) \). Since this set is dense in \( f^{-1}(U(z)) \), the function \( f \) is constant equal to \( z \) on \( f^{-1}(U(z)) \). In particular, the sets \( f^{-1}(U(z)), z \in M \), are disjoint. Since \( \mathbb{R} \) is the union of the open sets \( U(z), z \in M \), the interval \((a, b)\) is the disjoint
union of the relatively open sets \( f^{-1}(U(z)), z \in M \). Consequently, there is some \( z \in M \) with \( f^{-1}(U(z))=(a,b) \) and thus \( f \) is constant equal to \( z \) on \((a,b)\). \( \square \)

**Proposition 4.2.** Let \( b,b' \in \mathcal{B}, v \in \mathbb{N}, u \in \mathbb{N}_0, m^u \leq v < m^{u+1} \). If \( H(\Delta; b, b') \) is differentiable at \( \log_m(vm^{-u}) \) from the right then there are constants \( c, c' \in \mathbb{R} \) such that

\[
H(\Delta; b, b') = c + c'm^{-\Delta} \quad \text{for} \quad \Delta \in \left[ \log_m \left( \frac{v}{m^u} \right), \log_m \left( \frac{v}{m^u} + \frac{1}{m^{u+t-1}} \right) \right],
\]

\[
H(\cdot; b, T^{k-1}(w^{u+1}(b'), u)) \equiv c \quad \text{for} \quad k \geq t.
\]

The constant \( c \) comes from a finite set independent of \( u \) and \( v \).

**Proof.** For \( k \in \mathbb{N} \), define

\[
I_k := \left[ \log_m \left( \frac{v}{m^u} + m^{-u-k} \right), \log_m \left( \frac{v}{m^u} + m^{1-u-k} \right) \right].
\]

Set \( \Delta_0 := \log_m(vm^{-u}) \). Then it follows from Lemma 3.1 that for \( \Delta \in I_k \), we have

\[
\frac{K(\Delta; b, b') - K(\Delta_0; b, b')}{\Delta - \Delta_0} = \frac{m^\Delta - m^{\Delta_0}}{\Delta - \Delta_0} \cdot H \left( \log_m \left( \left( \frac{m^\Delta}{m^{\Delta_0}} \right) m^{u+1} \right); b, T^{k-1}(w^{u+1}(b'), u) \right).
\]

(4.1)

When \( \Delta \) runs through \( I_k \) from right to left the argument of \( H \) on the right-hand side runs through \([0,1]\) from right to left. The recurrent sequence \( (T^{k-1}(w^{u+1}(b'), u))_{k \geq 1} \) lies in the set \( \mathcal{B} \) of cardinality \( t \) and therefore is periodic at least from \( k=t \) on. When \( \Delta \) runs through

\[
I := \left( \Delta_0, \log_m \left( \frac{v}{m^u} + \frac{1}{m^{u+t-1}} \right) \right) = \bigcup_{k \geq t} I_k
\]

from right to left the left-hand side of (4.1) converges to the right side derivative of \( K(\cdot; b, b') \) at \( \Delta_0 \); the quotient on the right-hand side of (4.1) converges to \( m^{\Delta_0} \cdot \log m \). From the above remarks it follows that there must be a constant \( c \in \mathbb{R} \) such that for \( k \geq t \), we have \( H(\cdot; b, T^{k-1}(w^{u+1}(b'), u)) \equiv c \) on \([0,1]\). This proves one conclusion and we can see in particular that \( c \) comes from a finite set independent of \( u \) and \( v \). Plugging the last equation in (4.1) gives

\[
K(\Delta; b, b') = K(\Delta_0; b, b') + (m^\Delta - m^{\Delta_0})c, \quad \Delta \in I.
\]

For continuity reasons this holds for \( \Delta \in \overline{I} \) and gives the remaining conclusion. \( \square \)

Now we can prove Theorem 1.4 for \( H(\cdot; b, b') \) instead of \( H(\cdot; b) \). The function \( f(\Delta):=m^{-\Delta}(d/d\Delta)K(\Delta; b, b') \) is continuous on \((\alpha, \beta)\). By Proposition 4.2 there is a finite set \( M \subseteq \mathbb{R} \) such that for

\[
N := \left\{ \log_m \left( \frac{v}{m^u} \right) \mid u \in \mathbb{N}_0, v \in \mathbb{N}, m^{u+u} < v < m^{u+u+1} \right\},
\]
we have \( f(N) \subseteq M \). Since \( N \) is dense in \((\alpha, \beta)\), Lemma 4.1 shows that \( f \) is constant which proves the theorem. \( \square \)

5. Effectivity considerations

In general it seems impossible to compute \( H(\cdot; b, b') \) effectively in terms of simpler functions from the data describing \( \mathcal{F} \). This function’s behaviour can be too complicated to allow a reduction to elementary functions (see, examples in Section 7). But it is possible to decide effectively whether \( H(\cdot; b, b') \) is piecewise continuously differentiable. In this case the function is piecewise of the form \( c + c'm^{-d} \) and the involved constants can be computed effectively.

The next proposition gives an effective necessary and sufficient criterion for \( H(\cdot; b, b') \) to be continuously differentiable on certain intervals. For \( b, b' \in \mathcal{B} \), define \( f_b(b') := \mathcal{A}(\mathcal{F})_{|b'} \mathcal{B} \). For \( k \in \mathbb{N} \), define \( \phi_{b'}(k) := \{ w^k(b') \mid 0 \leq i \leq m^k - 1 \} \subseteq \mathcal{B} \). Since \( \mathcal{B} \) is finite and the sequence \( (\phi_{b'}(k))_{k \geq 1} \) is recurrent there are \( k_{b'}, p_{b'} \in \mathbb{N} \) with

\[
\phi_{b'}(k + p_{b'}) = \phi_{b'}(k), \quad k \geq k_{b'}. \tag{5.1}
\]

Define \( k_{\text{max}} := \max \{ k_{b'} + p_{b'} - 1 \mid b' \in \mathcal{B} \} \).

**Proposition 5.1.** Let \( b, b' \in \mathcal{B}, u \in \mathbb{N}_0, v, v' \in \mathbb{N}, m^u \leq v < v' \leq m^{u+1} \). Then \( H(\cdot; b, b') \) is continuously differentiable on \((\log_m(vm^{-u}), \log_m(v'm^{-u}))\) iff \( f_b \) is constant on

\[
\{ w^k(v^{u+1}(b'), v') \mid 1 \leq k \leq k_{\text{max}}, 0 \leq i \leq m^k - 1, v \leq v'' < v' \}.
\]

**Proof** (Necessity). According to Theorem 1.4 for \( H(\cdot; b, b') \) there are constants \( c, c' \in \mathbb{R} \) with \( H(\Delta; b, b') = c + c'm^{-d} \) on \((\log_m(vm^{-u}), \log_m(v'm^{-u}))\). For continuity reasons the equation also holds for the endpoints of the interval. Thus for \( k \in \mathbb{N}, v \leq v' < v'', v''m^k \leq v^* \leq v''(1 + m^k), u^* := u + k \), we have

\[
v^* G(v^*, u^*; b, b') = v^* H \left( \log_m \left( \frac{v^*}{m^{u^*}} \right); b, b' \right) = cv^* + c'm^{u^*}. \tag{5.2}
\]

by Lemma 2.2. Now let \( k \in \mathbb{N}, v \leq v'' < v', 0 \leq i \leq m^k - 1 \). Plug \( u^* := u + k \) and \( v^* := v''m^k + i + 1 \) resp. \( v^* := v''m^k + i \) in (5.2). Subtracting the left- and right-hand sides of these two equations gives

\[
c = f_b(w^{u+1}(b'), v''m^k + i) = f_b(w^k(v''m^k + i)).
\]

**Sufficiency.** From (5.1) and the assumption it follows that there is a constant \( c \in \mathbb{R} \) with \( f_b(w^k(v''m^k + i)) = c \) for \( k \in \mathbb{N}, 0 \leq i \leq m^k - 1, v \leq v'' < v' \). Now let \( k \in \mathbb{N}, vm^k \leq v^* < v'm^k \). Define \( u^* := u + k \). There are \( v \leq v'' < v' \) and \( 0 \leq i \leq m^k - 1 \) with \( v'' = v'm^k + i \). Furthermore, \( m^u \leq v^* < v^* + 1 \leq m^{u+1} \). Thus

\[
(v^* + 1) G(v^* + 1, u^*; b, b') - v^* G(v^*, u^*; b, b')
\]

\[
= f_b(w^{u'+1}(b'), v'') = f_b(w^k(w^{u+1}(b'), v''))) = c.
\]
From this it follows that for $k \in \mathbb{N}$, $vm^k \leq v^* \leq v'm^k$, $u^* := u + k$, we have

$$v^*H \left( \log_m \left( \frac{v^*}{m_{u^*}} \right); b, b' \right)$$

$$= v^*G(v^*, u^*; b, b')$$

$$= c(v^* - vm^k) + vm^kG(vm^k, u^*; b, b') = cv^* + c'(k),$$

(5.3)

where $c'(k) \in \mathbb{R}$ depends only on $k$ and is independent of $v^*$. For $A:(\log_m(vm^{-u}), \log_m(v'm^{-u}))$ there is a sequence $(v^*_k)_{k \geq 1}$ such that $vm^k \leq v^*_k \leq v'm^k$ and $\log_m(v^*_k m_{u^*}) \rightarrow A$ for $k \rightarrow \infty$ where $u^*_k := u + k$. From (5.3) it follows that the limit

$$c' := \lim_{k \rightarrow \infty} \frac{c'(k)}{m^{u^*_k}} = \lim_{k \rightarrow \infty} \left( \frac{v^*_k}{m_{u^*_k}}H \left( \log_m \left( \frac{v^*_k}{m_{u^*_k}} \right); b, b' \right) - c \frac{v^*_k}{m_{u^*_k}} \right)$$

$$= m^AH(A; b, b') - cm^A$$

exists and is independent of $A$. Thus $H(A; b, b') = c + c'm^{-A}$ throughout the interval with constants $c, c'$.  

**Proposition 5.2.** Let $b, b' \in \mathbb{B}$, $u^* \in \mathbb{N}_0$, $v^* \in \mathbb{N}_0$, $m^{u^*} < v^* < m^{u^*+1}$. Then $H(\cdot; b, b')$ is locally continuously differentiable at $\log_m(v^*m^{-u'})$ iff $f_b$ is constant on

$$\{w^k(X_{0,1}^l(w^{u^*+1}(b')c_{u^*}), w^k(X_{m-1,1}^l(w^{u^*+1}(b')c_{u^*}^{-1})), |1 \leq k \leq k_{max}, 0 \leq i \leq m^k - 1\}.$$

**Proof.** A necessary and sufficient condition for $H(\cdot; b, b')$ to be locally continuously differentiable at $\log_m(v^*m^{-u'})$ is that there is some $l \in \mathbb{N}_0$ such that $H(\cdot; b, b')$ is continuously differentiable on $(\log_m(v^*m^{-u'}), \log_m((v^*+2)m^{-u'}))$ where $v^* := v^*m^{-l} - 1$, $u^* := u^* + l$. From Proposition 5.1 it follows that this is equivalent to $f_b$ being constant on

$$M_l := \{w^k(X_{m-1,1}^l(w^{u^*+1}(b')c_{u^*}^{-1})), w^k(X_{0,1}^l(w^{u^*+1}(b')c_{u^*}), |1 \leq k \leq k_{max}, 0 \leq i \leq m^k - 1\}.$$

From the formulation in terms of continuous differentiability it is clear that $f_b$ is constant on $M_{l+1}$ if it is constant on $M_l$. The two recurrent sequences

$$(X_{m-1,1}^l(w^{u^*+1}(b')c_{u^*}^{-1})), (X_{0,1}^l(w^{u^*+1}(b')c_{u^*})), |l \geq 0,$$

in $\mathbb{B}$ are periodic at least for $l \geq t$. Let $p$ be a common period. Now if $f_b$ is constant on $M_l$ for some $l \geq 0$ then there is $l' \in \mathbb{N}_0$ such that $f_b$ is constant on $M_{l'p+t} = M_l$. Together with the above remarks this proves the proposition.  

Now Propositions 5.1 and 5.2 are reformulated. Choose a new symbol $*$ and define $\mathbb{B} := \mathbb{B} \times (\mathbb{B} \cup \{\ast\})$. For $b \in \mathbb{B}$, let $\mathbb{B}^{(1)}_b$ be the set of all $b' \in \mathbb{B}$ such that $f_{b'}$ is not
constant on \( \{w^k(b'), 1 \leq k \leq k_{\text{max}}, 0 \leq i \leq m^k - 1\} \). Let \( B_b^{(2)} \) be the set of all \((b', b'') \in B \times B\) such that \( f_b \) is not constant on

\[
\{w^k(X_{b_0}(b')), w^k(X'_{m-1,1}(b'')) | 1 \leq k \leq k_{\text{max}}, 0 \leq i \leq m^k - 1\}.
\]

Define \( \widetilde{B}_b := (\widetilde{B}_b^{(1)} \times (B \cup \{\ast\})) \cup B_b^{(2)} \subseteq \widetilde{B} \). For \( v \in \mathbb{N} \) there is a uniquely determined \( u \in \mathbb{N}_0 \) with \( m^u \leq v < m^{u+1} \). Define \( I(v) := [\log_m(vm^{-u}), \log_m((v + 1)m^{-u})] \subseteq [0, 1) \).

For \( b' \in B \), set \( \phi_{b'}(v) := (w^{u+1}(b'), w^{u+1}(b')_{v-1}) \) if \( m^u < v < m^{u+1} \) and \( \phi_{b'}(v) := (w^{u+1}(b'), \ast) \) if \( v = m^u \). Thus a map \( \phi_{b'} : \mathbb{N} \to \widetilde{B} \) is defined.

Let \( V(b, b') \) be the set of all \( A \in (0, 1) \) at which \( H(\cdot; b, b') \) is not locally continuously differentiable. The following proposition follows directly from Propositions 5.1 and 5.2.

**Proposition 5.3.** For all \( b, b' \in B, v \in \mathbb{N} \), we have

\[
V(b, b') \cap I(v) \neq \emptyset \iff \phi_{b'}(v) \in \widetilde{B}_b.
\]

For \( 0 \leq i < m \), define \( Y_i : \mathbb{N} \to \mathbb{N} \) as follows: If \( i = 0 \), let \( Y_0(b', b'') := (w(b'))_0 \), \( w(b''_{m-1}) \) if \( b'' \neq \ast \) and \( Y_0(b', b'') := (w(b'))_0, \ast \) otherwise. If \( i \geq 1 \), let \( Y_i(b', b'') := (w(b')_i, w(b')_{i-1}) \). The proof of the following lemma is straightforward.

**Lemma 5.4.** For \( b' \in B, v \in \mathbb{N}, 0 \leq i < m \), we have \( \phi_{b'}(vm + i) = Y_i(\phi_{b'}(v)) \).

One can look upon \( \phi_{b'}(v) \) as a signature for the interval \( I(v) \) which tells whether this interval is critical with respect to \( V(b, b') \). The set

\[
\widetilde{B}_b^* := \{(b', b'') \in \mathbb{N} | Y_i(b', b'') \in \widetilde{B}_b \text{ for at most one } 0 \leq i < m\}
\]

characterizes all those intervals which have at most one “child” which is critical. It will turn out that \( V(b, b') \) is finite iff all sufficiently small intervals have at most one critical child. The next proposition reduces this problem to a finite condition. Let \( K := |\mathbb{N}| = t(i + 1) \). For \( v \in \mathbb{N}, j \in \mathbb{N}_0 \), define \( F_j(v) := [vm^{-j}] \in \mathbb{N}_0 \).

**Proposition 5.5.** Let \( b, b' \in B \) and for all \( m^K \leq v < m^{2K} \), let \( \phi_{b'}(v) \in \widetilde{B}_b^* \). Then \( \phi_{b'}(v) \in \widetilde{B}_b^* \) for all \( v \geq m^K \).

**Proof.** We use induction. Let \( l \geq K \) and assume that for all \( m^K \leq v < m^{K+1} \), we have \( \phi_{b'}(v) \in \widetilde{B}_b^* \). Let \( m^{K+1} \leq v' < m^{K+1+1} \). Among the \( K + 1 = |\mathbb{N}| + 1 \) elements \( \phi_{b'}(F_j(v')) \), \( 0 \leq j \leq K \), of \( \mathbb{N} \) there must be at least two identical ones, i.e. there are \( 0 \leq j_1 < j_2 \leq K \) with \( \phi_{b'}(F_{j_1}(v')) = \phi_{b'}(F_{j_2}(v')) \). By induction it follows from Lemma 5.4 that for all \( 0 \leq i < m^{j_1} \), we have

\[
\phi_{b'}(F_{j_1}(v')m^{j_1} + i) = \phi_{b'}(F_{j_2}(v')m^{j_1} + i).
\]

Choose \( i := v' - F_{j_1}(v')m^{j_1} \) and define \( v := F_{j_2}(v')m^{j_1} + i \). Then \( \phi_{b'}(v') = \phi_{b'}(v) \) and \( m^K \leq v < m^{K+1} \). By assumption, \( \phi_{b'}(v') = \phi_{b'}(v) \in \widetilde{B}_b^* \).
Now one half of the finiteness criterion for \( V(b, b') \) can be proved.

**Proposition 5.6.** Let \( b, b' \in \mathcal{B} \) and assume \( \phi_{b'}(v) \in \mathcal{B}^*_b \) for all \( v \geq m^K \). Then \( |V(b, b')| \leq m^K(m-1) \).

**Proof.** Let \( m^K \leq v < m^{K+1} \) and assume \( M := V(b, b') \cap I(v) \neq \emptyset \). We will define recursively a sequence \((v_j)_{j \geq 0} \) of natural numbers such that \( m^K + 1 \leq v_j < m^{K+1} \) and \( M \subseteq I(v_j) \). Set \( v_0 := v \). Now let \( v_l \) be already defined. By assumption, we have \( \phi_{b'}(v_l) \in \mathcal{B}^*_b \). From the definition of this set, Proposition 5.3 and Lemma 5.4 it follows that \( V(b, b') \cap I(v_jm + i) \neq \emptyset \) for at most one \( 0 \leq i < m \). Since \( M \neq \emptyset \) and \( \bigcup_{0 \leq i < m} I(v_jm + i) = I(v_l) \), there is exactly one \( 0 \leq i < m \) with \( M \subseteq I(v_jm + i) \). Define \( i_l := i \) and \( v_{l+1} := v_jm + i_l \).

Now for every \( l \in \mathbb{N}_0 \), the diameter of \( M \) is
\[
\leq \log_m \left( \frac{v_l + 1}{v_l} \right) \leq \frac{1}{v_l \log m} \leq \frac{1}{m^{K+1} \log m}.
\]
Thus \( |V(b, b') \cap I(v_l)| \leq 1 \) for all \( m^K \leq v < m^{K+1} \). Therefore
\[
|V(b, b')| = \sum_{m^K \leq v < m^{K+1}} |V(b, b') \cap I(v)| \leq m^K(m-1).
\]

The second half of the finiteness criterion for \( V(b, b') \) is contained in

**Proposition 5.7.** Let \( b, b' \in \mathcal{B} \) and assume \( \phi_{b'}(v) \notin \mathcal{B}^*_b \) for some \( v \geq m^K \). Then \( |V(b, b')| = \infty \).

**Proof.** Choose \( v_0 \geq m^K \) with \( \phi_{b'}(v_0) \notin \mathcal{B}^*_b \). We will define recursively a sequence \((v_l)_{l \geq 0} \) of natural numbers with \( v_l \geq m^{K+l} \) and \( \phi_{b'}(v_l) \notin \mathcal{B}^*_b \). Assume that \( v_l \) has already been defined. Among the \( K + 1 = |\mathcal{B}| + 1 \) elements \((\phi_{b'}(F_j(v_l)))_{0 \leq j \leq K} \) of \( \mathcal{B} \) there must be at least two identical ones, i.e. there are \( 0 \leq j_1 < j_2 \leq K \) with \( \phi_{b'}(F_{j_1}(v_l)) = \phi_{b'}(F_{j_2}(v_l)) \).

By induction it follows from Lemma 5.4 that for all \( 0 \leq i < m^l \), we have
\[
\phi_{b'}(F_{j_2}(v_l)m^l + i) = \phi_{b'}(F_{j_1}(v_l)m^l + i).
\]
Choose \( i := v_l - F_{j_2}(v_l)m^l \) and define \( v_{l+1} := F_{j_1}(v_l)m^l + i \). Then \( \phi_{b'}(v_{l+1}) = \phi_{b'}(v_l) \notin \mathcal{B}^*_b \) and \( v_{l+1} \geq m^{K+l+1} \).

Now for every \( l \in \mathbb{N}_0 \), we have \( \phi_{b'}(v_l) \notin \mathcal{B}^*_b \), i.e. there are \( 0 \leq i_1 < i_2 < m \) with \( Y_{i_1}(\phi_{b'}(v_l)), Y_{i_2}(\phi_{b'}(v_l)) \in \mathcal{B}_b \). Lemma 5.4 gives \( \phi_{b'}(v_lm + i_1), \phi_{b'}(v_lm + i_2) \in \mathcal{B}_b \), and Proposition 5.3 shows that there are \( A_1 \in V(b, b') \cap I(v_lm + i_1), A_2 \in V(b, b') \cap I(v_lm + i_2) \). Consequently,
\[
0 < A_2 - A_1 \leq \log_m \left( \frac{v_lm + i_2 + 1}{v_lm + i_1} \right) \leq \frac{i_2 - i_1 + 1}{(v_lm + i_1) \log m}
\]
\[
\leq \frac{m}{v_lm \log m} \leq \frac{1}{m^{K+l} \log m}.
\]
Since such a pair \( A_1, A_2 \) of elements of \( V(b, b') \) exists for every \( l \in \mathbb{N}_0 \), we have \( |V(b, b')| = \infty \). \( \square \)

Propositions 5.5, 5.6 and 5.7 now give the following effective criterion.

**Theorem 5.8.** Let \( b, b' \in B \). Then \( V(b, b') \) is finite iff for all \( m^K \leq v < m^{2K} \), we have \( \phi_{b'}(v) \in \tilde{B}_b^* \).

**Remark.** In the case \( |V(b, b')| < \infty \), Proposition 5.6 gives an effective upper bound for \( |V(b, b')| \). Furthermore, the function \( H(\cdot; b, b') \) can be effectively computed. In order to do this, it is only necessary to compute this function at the supporting points in \( V(b, b') \cup \{0,1\} \) and interpolate between these points by functions of the form \( c + c'm^{-A} \) (see Theorem 1.4 for \( H(\cdot; b, b') \)). For \( A \in [0,1] \) such that \( m^A \) is rational the \( m \)-adic expansion of \( m^A \) is finite or ultimately periodic. Corollary 3.3 can then be used to compute \( H(\lambda; b, b') \) effectively with the help of geometric series and a reduction of \( H(\log m; b, b'') \) to \( G(v; u; b, b'') \) (see Lemma 2.2). So it remains to show that the elements of \( V(b, b') \) are \( m \)-logarithms of rational numbers which can be computed effectively.

We use the notation of the proof of Proposition 5.6. Let \( m^K \leq v < m^{K+1} \) and assume \( M := V(b, b') \cap I(v) = \{A_0\} \) (Proposition 5.5 gives an effective test). Find \( l \in \mathbb{N}_0, \ p \in \mathbb{N}, \) such that \( \phi_{b'}(v_l) = \phi_{b'}(v_{l+p}) \). By induction, we have \( \phi_{b'}(v_{l'}) = \phi_{b'}(v_{l'+p}) \) and \( i_{l'} = i_{l'+p} \) for all \( l' \geq l \). Thus for all \( k \geq 0 \),

\[
v_{l+k} = v_l m^{kp} + \sum_{j=0}^{k-1} i_{l+j} m^{kp-1-j}
\]

Since \( M \subseteq I(v_{l+k}) \) for all \( k \geq 0 \), we have

\[
m^{A_0} = \lim_{k \to \infty} \frac{v_{l+k}}{m^{K+l+k}} = \frac{v_l}{m^{K+l}} + \sum_{j=0}^{p-1} \frac{i_{l+j}}{m^{K+l+j+1}} \cdot \frac{1}{1 - m^{-p}}.
\]

Since the sequences \( (v_{l'})_{l' \geq 0} \) and \( (i_{l'})_{l' \geq 0} \) can be computed effectively this gives an effective formula for \( m^{A_0} \).

6. Hölder density

This section is devoted to a proof of Theorem 1.5. The only fact about uniform tag sequences that is used is the asymptotic formula from Theorem 1.1. The proof is very close to that of [12], Theorem in Section 2.
First, Theorem 1.1 is generalized to the Hölder means $M_k(z; b)$. For $\alpha, \beta \in [1, m]$, define the kernel

$$u(\alpha, \beta) := \frac{1}{\alpha(m-1)} + \frac{1}{\alpha} I_{[1, \alpha]}(\beta),$$

where $I_X$ is the characteristic function of the set $X \subseteq \mathbb{R}$. For $\alpha \in [1, m]$, $b \in \mathcal{B}$, define

$$H_1(x; b) := H(\log_\alpha x; b),$$

$$H_{k+1}(x; b) := \int_1^m u(\alpha, \beta) H_k(\beta; b) \, d\beta, \quad k \in \mathbb{N}.$$

Choose $0 < \delta < 1$ with $\delta \leq \delta(\mathcal{T})$. Set $c_0 = c_0(\delta) = 7m + m^{1-\delta} - 1$.

**Proposition 6.1.** For $b \in \mathcal{B}$, $k \in \mathbb{N}$, the function $H_k(\cdot; b)$ is Lipschitz continuous with constant $3$ and bounded by $1$. Uniformly in $k \in \mathbb{N}$, $\alpha \in [1, m]$, $u \in \mathbb{N}_0$, we have $M_k(\alpha^u; b) = H_k(\alpha; b) + O((u + 1)^\delta \alpha^{\delta} c_0^k)$.

**Proof.** From the proof of Lemma 2.3(1) it follows that $H_1(\cdot; b)$ is Lipschitz continuous with constant $2$ and bounded by $1$. Now assume that $H_k(\cdot; b)$ is Lipschitz continuous with constant $3$ and bounded by $1$. Then for $1 < \alpha_1 < \alpha_2 < m$, we have

$$|H_{k+1}(\alpha_2; b) - H_{k+1}(\alpha_1; b)| \leq |\frac{1}{\alpha_2} - \frac{1}{\alpha_1}| \int_1^{\alpha_2} H_k(\beta; b) \, d\beta + \frac{1}{\alpha_1} \int_1^{\alpha_1} H_k(\beta; b) \, d\beta$$

$$+ \frac{1}{\alpha_2} \int_{\alpha_1}^{\alpha_2} H_k(\beta; b) \, d\beta + \frac{1}{\alpha_1} \int_{\alpha_1}^{\alpha_2} H_k(\beta; b) \, d\beta$$

$$\leq |\alpha_2 - \alpha_1| \left( \frac{m-1}{m-1} + \frac{\alpha_2 - 1}{\alpha_1} \frac{1}{\alpha_1} \right) \leq 3 |\alpha_2 - \alpha_1|.$$

Furthermore, for $1 < \alpha \leq m$ we have

$$0 \leq H_{k+1}(x; b) \leq \frac{1}{\alpha(m-1)} (m-1) + \frac{1}{\alpha} (\alpha - 1) = 1.$$

Induction on $k$ proves the first part.

In the sequel, let the symbol $O^*$ be defined as in the usual $O$-notation with the additional requirement that the involved constant is $\leq 1$. The second half of the proposition is proved by induction on $k$. For $k=1$, it follows from Theorem 1.1 and the 1-periodicity of $H(\cdot; b)$. Let $C$ be the involved $O$-constant. Now assume that for $\alpha \in [1, m]$, $u \in \mathbb{N}_0$, we have

$$M_k(\alpha^u; b) = H_k(\alpha; b) + O^*(C(u + 1)^\delta \alpha^{\delta} c_0^k).$$
For $x,u$ in the given range,
\[
M_{k+1}(zm^n; b) = \frac{1}{zm^n} \left( \sum_{r=0}^{u-1} \sum_{m' \leq j < m^{r+1}} M_k(j; b) + \sum_{m'' \leq j < zm^n} M_k(j; b) \right). \tag{6.2}
\]

From (6.1) it follows that the first $j$-sum equals
\[
\sum_{0 \leq i < m'(m-1)} M_k \left( m' \left( 1 + \frac{i}{m'} \right); b \right)
\]
\[
= \sum_{0 \leq i < m'(m-1)} H_k \left( 1 + \frac{i}{m'}; b \right) + O^*(Cm'(m-1)(r + 1)m^{-\delta r^k}).
\]

From the Lipschitz continuity of $H_k$ it follows that the sum on the right-hand side equals
\[
\sum_{0 \leq i < m'(m-1)} \left( m' \int_{1+(i+1)m^{-r}}^{1+im^{-r}} H_k(\beta; b) \, d\beta + O^*(m' m^{-r} 3m^{-r}) \right)
\]
\[
= m' \int_1^{m} H_k(\beta; b) \, d\beta + O^*(3(m-1)).
\]

An analogous argument gives for the second $j$-sum in (6.2)
\[
m^n \int_1^{x} H_k(\beta; b) \, d\beta + O^*(2 + 3m + Cm^{n+1-\delta u}(u + 1)' c_0^k).
\]

Thus
\[
M_{k+1}(zm^n; b) = \frac{1}{xm^n} \int_1^{m} H_k(\beta; b) \, d\beta + \frac{1}{x} \int_1^{x} H_k(\beta; b) \, d\beta
\]
\[
+ O^*(C(u + 1)' m^{-\delta u} c_0^{k+1}),
\]
which is (6.1) with $k$ replaced by $k+1$. \qed

Now define the iterated kernels
\[
u^{(1)}(x,\beta) := u(x,\beta),
\]
\[
u^{(k+1)}(x,\beta) := \int_1^{m} u(x,\gamma)u^{(k)}(\gamma, \beta) \, d\gamma; \quad k \in \mathbb{N}, \ x, \beta \in [1, m].
\]

The following two lemmas follow easily by induction on $k$.

**Lemma 6.2.** For all $k \in \mathbb{N}$ there are polynomials $p_k, q_k \in \mathbb{R}[X,Y]$ such that
\[
u^{(k)}(x,\beta) = \frac{1}{x} p_k(\log x, \log \beta) + \frac{1}{x} q_k(\log x, \log \beta) \mathcal{M}_{[1,x]}(\beta).
\]

In particular, every $\nu^{(k)}$ is measurable and bounded on $[1,m]^2$. 

Lemma 6.3. For all \( b \in B, k \geq 2, x \in [1, m] \), we have

\[
H_k(x; b) = \int_1^m u^{(k-1)}(x, \beta) H_1(\beta; b) \, d\beta.
\]

Our goal is to show that for large \( k \), the function \( H_k(\cdot; b) \) varies only slightly. To this end, we use the Fourier series of \( u^{(k)}(m^A, \beta) \).

Lemma 6.4. For all \( k \geq 2, A \in [0, 1], \beta \in [1, m] \), we have

\[
u^{(k)}(m^A, \beta) = \sum_{r \in \mathbb{Z}} \frac{(\log m)^{k-1}}{(\log m + 2\pi ir)^2} e^{2\pi ir(A - \log_m \beta)}.
\]

Proof. Fix \( k \geq 2, \beta \in [1, m] \). From Lemma 6.2 it is clear that \( u^{(k)}(\cdot, \beta) \) is piecewise continuously differentiable. A similar argument to that for \( H_k(\cdot; b) \) in Proposition 6.1 shows that \( u^{(k)}(\cdot, \beta) \) is Lipschitz continuous. A simple calculation shows that \( u^{(k)}(1, \beta) = u^{(k)}(m, \beta) \). Thus \( f_k(A; \beta) := u^{(k)}(m^A, \beta) \) can be extended to a 1-periodic continuous, piecewise continuously differentiable function. Dirichlet’s theorem thus gives for \( k \geq 2, \beta \in [1, m], A \in [0, 1] \),

\[
f_k(A; \beta) = \sum_{r \in \mathbb{Z}} c_{kr}(\beta) e^{2\pi irA}, \tag{6.3}
\]

where the summation is done symmetrically, and

\[
c_{kr}(\beta) = \frac{1}{(\log m + 2\pi ir)\beta} e^{-2\pi ir \log_m \beta}. \tag{6.4}
\]

Eq. (6.3) only holds for \( k \geq 2 \) but the Fourier coefficients (6.4) can be defined also for \( k = 1 \). Now we will evaluate them recursively. For \( r \in \mathbb{Z}, \beta \in [1, m] \), we have

\[
c_{1r}(\beta) = \frac{1}{(\log m + 2\pi ir)\beta} e^{-2\pi ir \log_m \beta}. \tag{6.5}
\]

Furthermore, for \( k \in \mathbb{N}, r \in \mathbb{Z}, \beta \in [1, m] \), we have

\[
c_{k+1r}(\beta) = \int_0^1 e^{-2\pi ir A} u^{(k+1)}(m^A, \beta) \, dA
\]

\[
= \int_0^1 e^{-2\pi ir A} \int_1^m u(m^A, \gamma) u^{(k)}(\gamma, \beta) \, d\gamma \, dA
\]

\[
= \int_1^m u^{(k)}(\gamma, \beta) \int_0^1 e^{-2\pi ir A} u(m^A, \gamma) \, d\gamma \, dA = \int_1^m u^{(k)}(\gamma, \beta) c_{1r}(\gamma) \, d\gamma.
\]

Plugging (6.5) in the last integral and substituting \( \gamma = m^A \) gives

\[
c_{k+1r}(\beta) = \frac{\log m}{\log m + 2\pi ir} c_{kr}(\beta). \tag{6.6}
\]

Eqs. (6.3), (6.5) and (6.6) prove the lemma. \( \square \)
Now Theorem 1.5 can be proved. In Lemma 6.4, we isolate the term for \( r = 0 \) and estimate the others. This gives

\[
\begin{aligned}
    u^{(k)}(m^4, \beta) &= \frac{1}{\beta \log m} + O^* \left( \sum_{r \neq 0} \frac{(\log m)^{k-1}}{(2\pi r)^2 |\log m + 2\pi i r|^k} \right) \\
    &= \frac{1}{\beta \log m} + O^* \left( \frac{(\log m)^{k-1}}{4\pi^2 (\log^2 m + 4\pi^2)^{(k-2)/2}} \sum_{r \neq 0} \frac{1}{r^2} \right) \\
    &= \frac{1}{\beta \log m} + O_m \left( \frac{\log m}{\sqrt{\log^2 m + 4\pi^2}} \right)^k
\end{aligned}
\]

for \( k \geq 2 \), \( A \in [0, 1] \), \( \beta \in [1, m] \). Lemma 6.3 now gives for \( k \geq 3 \), \( z \in [1, m] \), \( b \in \mathcal{B} \),

\[
H_k(z; b) = \int_1^m H_1(\beta; b) \left( \frac{1}{\beta \log m} + O_m \left( \frac{\log m}{\sqrt{\log^2 m + 4\pi^2}} \right)^k \right) \, d\beta
\]

\[
= \int_0^1 H(A; b) \, dA + O_m \left( \frac{\log m}{\sqrt{\log^2 m + 4\pi^2}} \right)^k.
\]

Theorem 1.5 follows from this asymptotics and Proposition 6.1.

7. Examples

7.1. The Thue–Morse sequence

Let \( m := t := 2 \), \( \mathcal{B} := \{0, 1\} \) and \( w(0) := 01 \), \( w(1) := 10 \). Then \( y = \text{intseq} (\mathcal{F}) \) can also be described as follows: Let \( s_2(n) \) be the sum of digits of \( n \) in base 2. Then \( y_n = 0 \) if \( s_2(n) \) is even and \( y_n = 1 \) if \( s_2(n) \) is odd. We have

\[
\mathcal{M}(\mathcal{F}) = \mathcal{M}(\mathcal{F})^\infty = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

and \( H(\cdot; 0) = H(\cdot; 1) \equiv \frac{1}{2} \). The graph of the associated automaton is

```
0  1  0
\downarrow
1
```

7.2. Leading \(m\)-adic digits

Let \(m \geq 2\), \(\mathcal{B} = \{0, \ldots, m - 1\}\) and \(w(0) = 01 \ldots m - 1, w(1) = 11 \ldots , w(m - 1) = m - 1 \ldots m - 1\). For \(n \in \mathbb{N}\), the element \(y_n\) is the leading digit of \(n\) in the \(m\)-adic expansion. We have

\[
\mathcal{M}(\mathcal{T}) = \begin{pmatrix}
\frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \\
0 & 1 & 0 \\
\vdots & \ddots \\
0 & 0 & \cdots & 1
\end{pmatrix}, \quad \mathcal{M}(\mathcal{T})^\infty = \begin{pmatrix}
0 & \frac{1}{m-1} & \cdots & \frac{1}{m-1} \\
0 & 1 & 0 \\
\vdots & \ddots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\]

For \(b \in \{1, \ldots, m - 1\}\), \(A \in [0, 1]\), we have

\[
H(A; b) = \frac{1}{m - 1} m^{-A} + \begin{cases}
0, & 0 \leq A \leq \log_m b, \\
1 - bm^{-A}, & \log_m b < A \leq \log_m (b + 1), \\
m^{-A}, & \log_m (b + 1) < A \leq 1.
\end{cases}
\]

The graph of the associated automaton is

\[\begin{array}{c}
\bigcirc & 0 & \vdots \\
\rightarrow & \rightarrow & \rightarrow \\
\bigcirc & m-1 & \ldots \\
\end{array}\]

7.3. Consecutive 1-digits

Let \(m = 2\), \(t = 4\), \(\mathcal{B} = \{1, 2, 3, 4\}\) and \(w(1) = 12, w(2) = 13, w(3) = 44, w(4) = 33\). Then

\[
\mathcal{M}(\mathcal{T}) = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

has eigenvalues 1 and \(-1\) and thus \(\mathcal{T}\) is not admissible. But \(\mathcal{T}^2\) is so and has

\[
\mathcal{M}(\mathcal{T}^2)^\infty = \begin{pmatrix}
0 & 0 & \frac{3}{5} & \frac{2}{5} \\
0 & 0 & \frac{1}{5} & \frac{4}{5} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
The associated graph is

![Graph Image]

The automaton scans the binary representation of the input $n$ and switches between the inner states 1 and 2 until it finds two consecutive digits 1. Then it changes into state 3 and toggles between states 3 and 4. The graph of $H(\cdot;3)$ is sketched below:

![Graph Image]

7.4. Two 1-digits

Let $m,t,\mathcal{B}$ be as above and $w(1):=12$, $w(2):=23$, $w(3):=44$, $w(4):=33$. Then

$$
\mathcal{M}(\mathcal{T}) = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$
again has eigenvalues $\pm 1$ but $S^2$ is admissible with

$$M(S^2) = \begin{pmatrix}
0 & 0 & \frac{5}{3} & \frac{4}{3} \\
0 & 0 & \frac{1}{3} & \frac{2}{3} \\
0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$  

The associated graph is

[Graph image]

The automaton reads the first digit 1 and changes into state 2 where it remains until it sees another digit 1. Then it changes into state 3 and toggles between states 3 and 4. The graph of $H(\cdot; 3)$ is sketched below:

[Graph image]
References