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Some Further Generalizations of Knaster–Kuratowski–Mazurkiewicz Theorem and Minimax Inequalities

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1. INTRODUCTION

Several generalizations of the celebrated Ky Fan minimax inequality [9] make use of a structure of linear spaces [2, 1, 19, 12, 17, 10, 18].

In the years 1983–1985 C. Horvath obtained minimax inequalities by replacing convexity assumptions with merely topological properties: pseudo-convexity in [13] and contractibility in [14, 15].

In this paper, in the Horvath setting, we state generalized minimax inequality for functions taking values in ordered vector spaces.

Let X be a topological space and (E, C) a topological Riesz space, where C is the positive cone. As in [14, 15], X is provided with a topological structure on which the contractible sets replace the convex hulls.

Given two functions f, g defined on $X \times X$ and taking values in (E, C), Theorems 3 and 4 give sufficient conditions to establish generalized minimax inequalities. We observe that the compactness assumption on X is also relaxed.

So our minimax inequalities extend some of the more recent ones, e.g., [1, 19, 12, 10, 15].

In order to prove Theorems 3 and 4 we use the Knaster–Kuratowski–Mazurkiewicz-type theorem approach. The KKM theorem [16], was first generalized to the infinite dimensional case by Fan [5], who gave numerous applications of this generalization [6–10].

Taking into account our abstract setting, it was necessary to state a generalized reformulation of the KKM theorem. Theorems 1 and 2 are well-behaved tools for our purpose. Moreover, these theorems generalize

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the recent results of M. Lassonde [17, Theorem I], Fan [10, Theorem 4], and Horvath [15, Corollary 1 and Theorem 2].

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2. Generalized Knaster–Kuratowski–Mazurkiewicz Theorem

The following definition is suggested by a recent generalization of the KKM theorem obtained by C. Horvath [15].

Definition 1. By *H-space* we mean a pair $(X, \{\Gamma_A\})$, where X is a topological space and $\{\Gamma_A\}$ is a given family of nonempty contractible subsets of X, indexed by the finite subsets of X, such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

Let $(X, \{\Gamma_A\})$ be an H-space. A subset $D \subset X$ is called *H-convex* if, for every finite subset $A \subset D$, it follows that $\Gamma_A \subset D$.

A subset $D \subset X$ is called *weakly H-convex* if, for every finite subset $A \subset D$, it results that $\Gamma_A \cap D$ is nonempty and contractible. This is equivalent to saying that the pair $(D, \{\Gamma_A \cap D\})$ is an H-space.

Finally, a subset $K \subset X$ is called *H-compact* if, for every finite subset $A \subset X$, there exists a compact, weakly H-convex set $D \subset X$ such that $K \cup A \subset D$.

Remarks. (1) Any Hausdorff topological vector space is an H-space: for every finite subset $A = \{x_1, ..., x_n\} \subset X$, we can set $\Gamma_A = \text{conv}\{x_1, ..., x_n\}$; moreover, any convex subset of X is H-convex and any nonempty compact convex subset is H-compact.

(2) Every contractible space X is an H-space: at first we may put $\Gamma_A = X$ for every finite subset $A \subset X$. With this structure, the only H-convex subset of X is X itself.

(3) The definition of H-space generalizes Horvath pseudo-convexity [13] and the concept of convex space due to Lassonde [17].

(4) As Horvath remarked in [13], every contractible space is a pseudo-convex space. This enables us to endow a contractible space X with an H-space structure: for every finite subset $A \subset X$ we put $\Gamma_A = C_h\{A\}$, where $C_h\{A\}$ is the h-convex hull of A (see [13]). In this setting every h-convex subset of X (see [13]) is H-convex.

(5) The definition of H-compactness generalizes Lassonde's c-compactness ([17]).

The following, used by Horvath in [15], is a generalization of the definition of the h-KKM function given in [13].

DEFINITION 2. In a given H-space $(X, \{\Gamma_A\})$, a multifunction $F: X \to X$ is called *H–KKM* if $\Gamma_A \subset \bigcup_{x \in A} F(x)$, for each finite subset $A \subset X$.

We now prove the following.

THEOREM 1. *Let $(X, \{\Gamma_A\})$ be an H-space and $F: X \to X$ an H–KKM multifunction such that:*

(a) *For each $x \in X$, $F(x)$ is compactly closed, that is, $B \cap F(x)$ is closed in B, for every compact $B \subset X$.*

(b) *There is a compact set $L \subset X$ and an H-compact $K \subset X$, such that, for each weakly H-convex set D with $K \subset D \subset X$, we have $\bigcap_{x \in D} (F(x) \cap D) \subset L$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Proof. It suffices to show that $\bigcap_{x \in X}(F(x) \cap L) \neq \emptyset$. By (a), $F(x) \cap L$ is closed in the compact set L; thus we have only to prove that $\bigcap_{x \in A}(F(x) \cap L) \neq \emptyset$, for every finite subset $A \subset X$.

Let $A \subset X$ be a finite set and let $X_0 \subset X$ be a compact, weakly H-convex set such that $K \cup A \subset X_0$. By (b), $\bigcap_{x \in X_0}(F(x) \cap X_0) \subset L$ and thus $\bigcap_{x \in A}(F(x) \cap L) \supset \bigcap_{x \in X_0}(F(x) \cap X_0)$. Hence, it is sufficient to show that $\bigcap_{x \in X_0}(F(x) \cap X_0)$ is nonempty.

Let us consider now the multifunction $G: X_0 \to X_0$ defined by $G(x) = F(x) \cap X_0$. The H-KKM property on F easily implies the same property on G with respect to the H-space $(X_0, \{\Gamma_{A \cap X_0} \cap X_0\})$.

By the closedness of $G(x)$ in the compact set X_0 and using Corollary 1 in [15], we deduce: $\bigcap_{x \in X_0} G(x) = \bigcap_{x \in X_0}(F(x) \cap X_0) \neq \emptyset$, which completes the proof of the theorem.

Remark. The property:

(I) There is an H-compact set $K \subset X$ such that $\bigcap_{x \in K} F(x)$ is compact, implies (b). Hence, if $(X, \{\Gamma_A\})$ is an H-space and X is compact, property (I), and so (b), are immediately fulfilled. Thus Theorem 1 is an extension of Corollary 1 in [15], to the noncompact case. On the other hand, Theorem 1 represents an extension to the nonconvex case of Fan's KKM theorem [10, Theorem 4].

The following theorem, which is a consequence of Theorem 1, serves as the key tool in the proof of generalized minimax inequalities of Section 2.

We premise some notations: given a multifunction $F: X \to X$, we put:

$$F^{-1}(y) = \{x \in X: y \in F(x)\} \qquad \text{and} \qquad F^*(y) = X - F^{-1}(y).$$

THEOREM 2. *Let $(X, \{\Gamma_A\})$ be an H-space, G, $F: X \to X$ two multifunctions such that:*

(a) *for every* $x \in X$, $G(x)$ *is compactly closed and* $F(x) \subset G(x)$;

(b) $x \in F(x)$, *for every* $x \in X$;

(c) *for every* $x \in X$, $F^*(x)$ *is H-convex;*

(d) *the multifunction G verifies property* (b) *of Theorem* 1.

Then $\bigcap_{x \in X} G(x) \neq \varnothing$.

Proof. By virtue of Theorem 1, it suffices to show that the multifunction G is H–KKM; that is, for every finite subset $A \subset X$, $\Gamma_A \subset \bigcup_{x \in A} G(x)$.

Suppose that $\Gamma_A \not\subset \bigcup_{x \in A} G(x)$, for some finite subset $A \subset X$; then there exists $y \in \Gamma_A$ and $y \notin G(x)$, for every $x \in A$ and so $A \subset G^*(y)$. Since $F(y) \subset G(y)$, it follows $G^*(y) \subset F^*(y)$ and by (c), $\Gamma_A \subset F^*(y)$. Hence $y \in \Gamma_A$ implies $y \in F^*(y)$ which is equivalent to $y \notin F(y)$ and this contradicts (b).

Theorem 2 extends a result of Horvath's [15, Theorem 2] to the case of noncompact topological space X.

3. Some Generalizations of Fan's Minimax Inequality

Our aim is to establish minimax inequalities without compactness and convexity hypotheses on X for functions taking values in ordered topological vector spaces.

Let (E, C) be a Riesz space, where C is the positive cone, provided with a linear, order compatible topology. This means that C is closed (see, e.g., [11]). It will be assumed that the interior of the cone C, denoted by $\mathring{C}$, is nonempty.

We remark that, in the case $E = \mathbb{R}$, Theorems 3, 4 below generalize to nonconvex case previous theorems of G. Allen [1] (see also Fan [10, Theorem 6]) and A. Granas [12]; on the other hand, they extend Theorem 5.1 in [13] to neither the compact nor the pseudo-convex case.

From Theorem 2 we deduce the following.

Theorem 3. *Let* $(X, \{\Gamma_A\})$ *be an H-space,* $f, g\colon X \times X \to (E, C)$ *two functions with the following properties:*

(a) $g(x, y) \leqslant f(x, y)$, *for every* $(x, y) \in X \times X$;

(b) *for every* $y \in X$ *and any* $\lambda \in E$ *the set* $\{x \in X\colon f(x, y) \in \lambda + \mathring{C}\}$ *is H-convex*;

(c) *for every* $x \in X$ *and any* $\lambda \in E$ *the set* $\{y \in X\colon g(x, y) \in \lambda + \mathring{C}\}$ *is compactly open*; *that is, for every compact* $B \subset X$, $B \cap \{y \in X\colon g(x, y) \in \lambda + \mathring{C}\}$ *is open in* B;

(d) *there is an H-compact* $X_0 \subset X$ *such that* $\{y \in X: g(x, y) \notin \lambda + \mathring{C}$, *for each* $x \in X_0\}$ *is a compact set, for every* $\lambda \in E$.

Then, for every $\lambda \in E$, *the following alternative holds*:

(1) *there exists* y_0 *such that, for every* $x \in X$, $g(x, y_0) \notin \lambda + \mathring{C}$, *or*

(2) *there exists* $x_0 \in X$ *such that* $f(x_0, x_0) \in \lambda + \mathring{C}$.

Proof. For a fixed $\lambda \in E$, we define: $F(x) = \{y \in X: f(x, y) \notin \lambda + \mathring{C}\}$ and $G(x) = \{y \in X: g(x, y) \notin \lambda + \mathring{C}\}$. By (c) $G(x)$ is compactly closed for every $x \in X$. By (a) it follows that $F(x) \subset G(x)$; indeed if $y \notin G(x)$, then $g(x, y) \in \lambda + \mathring{C}$ and there is a neighborhood V of $0 \in E$ such that $g(x, y) + V \subset \lambda + \mathring{C}$. But $g(x, y) \leqslant f(x, y)$ implies $\lambda < g(x, y) + v \leqslant f(x, y) + v$, for every $v \in V$ and thus $f(x, y) + V \subset \lambda + \mathring{C}$, that is $y \notin F(x)$.

If there exists $x_0 \in X$ with $x_0 \notin F(x_0)$, then $f(x_0, x_0) \in \mathring{C}$ and so we have (2). Otherwise, $x \in F(x)$ for each $x \in X$.

Let $A \subset F^*(y) = \{x \in X: f(x, y) \in \lambda + \mathring{C}\}$ be a finite set. By (b) $\Gamma_A \subset F^*(y)$; moreover condition (d) implies the existence of an H-compact set X_0, such that $\bigcap_{x \in X_0} G(x)$ is compact. So, all of the assumptions of Theorem 2 are fulfilled and hence $\bigcap_{x \in X} G(x) \neq \emptyset$. This implies part (1) of the alternative.

In order to give a second generalization of minimax inequality, we prove

THEOREM 4. *Let* $(X, \{\Gamma_A\})$ *be an H-space,* f, $g: X \times X \to (E, C)$ *two functions with the properties:*

(a) $g(x,y) \leqslant f(x, y)$, *for every* $(x, y) \in X \times X$;

(b) *for every* $y \in X$ *and any* $\lambda \in E$ *the set* $\{x \in X: f(x, y) \nleqslant \lambda\}$ *is H-convex;*

(c) *for every* $x \in X$ *and any* $\lambda \in E$ *the set* $\{y \in X: g(x, y) \leqslant \lambda\}$ *is compactly closed;*

(d) *there is an H-compact subset* $X_0 \subset X$ *such that* $\{y \in X: g(x, y) \leqslant \lambda$, *for each* $x \in X_0\}$ *is compact, for every* $\lambda \in E$.

Then, for every $\lambda \in E$, *the following alternative holds:*

(1) *There exists* $y_0 \in X$ *such that for every* $x \in X$, $g(x, y_0) \leqslant \lambda$, *or*

(2) *There exists* $x_0 \in X$, *such that* $f(x_0, x_0) \nleqslant \lambda$.

Proof. For fixed $\lambda \in E$, we define $F(x) = \{y \in X: f(x, y) \leqslant \lambda\}$, and $G(x) = \{y \in X: g(x, y) \leqslant \lambda\}$. Condition (a) implies $F(x) \subset G(x)$ for every $x \in X$ and (c) ensures that $G(x)$ is compactly closed for each $x \in X$.

If there exists $x_0 \in X$ with $x_0 \notin F(x_0)$, then $f(x_0, x_0) \nleqslant \lambda$ and so we obtain (2). If $x \in F(x)$ for every $x \in X$, the proof is carried out as in Theorem 3.

As a consequence of Theorem 4, we state the following

COROLLARY 1. *Let* $(X, \{\Gamma_A\})$ *be an H-space,* (E, C) *an order complete topological Riesz space. We assume that* $f: X \times X \to (E, C)$ *is bounded above on the set* $\Delta = \{(x, x): x \in X\}$. *Under the assumptions of Theorem* 4 *we have*

$$\inf_{y \in X} \sup_{x \in X} f(x, y) \leqslant \sup_{x \in X} f(x, x),$$

whenever the "inf" in the left-hand side exists.

Proof. Put $\lambda = \sup_{x \in X} f(x, x)$ (which is well defined by order completeness of E), by Theorem 4 there exists $y_0 \in X$ such that:

$$f(x, y_0) \leqslant \sup_{x \in X} f(x, x) \qquad \text{for every} \quad x \in X.$$

Since (E, C) is order-complete it follows that $\sup_{x \in X} f(x, y_0)$ exists and

$$\sup_{x \in X} f(x, y_0) \leqslant \sup_{x \in X} f(x, x),$$

and so the thesis.

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