Extensions of Liouville Theorems

P. Ramankutty

Department of Mathematics, University of New Orleans, New Orleans, Louisiana 70122

Submitted by Alex McNabb

The method of deriving Liouville's theorem for subharmonic functions in the plane from the corresponding Hadamard three-circles theorem is extended to a more general and abstract setting. Two extensions of Liouville's theorem for vector-valued holomorphic functions of several complex variables are also mentioned.

1. INTRODUCTION

Among the basic results of analytic function theory and of the theory of harmonic and subharmonic functions that have found natural extensions to the theory of elliptic partial differential equations and inequalities, the theorem of Liouville on the unboundedness of nonconstant entire functions is a prominent one. Equally prominent is the three-circles theorem of Hadamard and the extensions of this theorem to higher dimensional spaces. Various generalizations of these theorems to partial differential operators especially of the elliptic type have appeared since the pioneering result of Bernstein [1]. Some of these results and extensive references to the literature are available in Miranda [3], Protter and Weinberger [4], and Vyborny [6]. A strong version of Liouville's theorem for functions subharmonic in the plane punctured at one point is obtainable from the Hadamard's three circles theorem; this is demonstrated in [4]. It is also remarked there that such a theorem fails in higher dimensions and an example is given to illustrate this failure in three dimensions.

This paper presents a generalization of the process of deriving Liouville's theorem from the Hadamard three-circles theorem and shows that once a Hadamard-type result is available, the process thereafter is almost totally independent of the convexity and many other indigenous properties of the exponential function. For convenience of generality, the result is formulated in the context of arbitrary-normed linear spaces and is not subject to pathologies arising out of the dimension of the space.

In Section 2 the Hadamard's three-circles theorem for subharmonic functions in Euclidean spaces is quoted from [4, p. 129, 131] and the
resulting Liouville's theorem in two dimensions is also stated. The proof of this derivation of Liouville's theorem from Hadamard's is outlined emphasising why the proof fails in dimensions other than two. Examples are included to show that such Liouville theorems are indeed false except in two dimensions.

Two results analogous to the derivation of Liouville's theorem from Hadamard's are given in Section 3 in a rather general context.

Section 4 contains some mild extensions of Liouville's theorem to an abstract setting in the spirit of Rudin [5, Theorem 3.32]; these are independent of Sections 2 and 3.

2. LIOUVILLE'S THEOREM FROM HADAMARD'S THREE-CIRCLES THEOREM

The following version of Hadamard's three-circles theorem for functions subharmonic in an annulus of Euclidean n-space appears in [4, p. 129, 131]:

2.1 HADAMARD'S THEOREM. Let \( A = \{ x \in \mathbb{R}^n : \rho < \| x \| < R \} \), \( U : A \to \mathbb{R} \) be subharmonic and \( M : (\rho, R) \to \mathbb{R} \) and \( V : (\rho, R) \to \mathbb{R} \) be defined by \( M(r) = \max \{ U(x) : \| x \| = r \} \) and

\[
V(r) = \begin{cases} 
\log r, & \text{if } n = 2, \\
\frac{r^{2-n}}{n}, & \text{if } n > 2. 
\end{cases}
\]

Then \( M(r) \) is a convex function of \( V(r) \), i.e., there holds,

\[
M(r) \leq M(a)(V(b) - V(r))/(V(b) - V(a)) + M(b)(V(r) - V(a))/(V(b) - V(a))
\]

for \( \rho < a < r < b < R \). Moreover, the equality \( M(r) = a + \beta V(r) \) holds for constants \( a \) and \( \beta \) iff \( U(x) = a + \beta V(\| x \|) \) for all \( x \) in \( A \).

This result implies the following strong version of Liouville's theorem if \( n = 2 \), but not if \( n \neq 2 \).

2.2 LIOUVILLE'S THEOREM. If \( U : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \) is a nonconstant subharmonic function, then \( \lim \inf (M(r)/\log r) > 0 \) either as \( r \downarrow 0 \) or as \( r \to \infty \), where, for \( r > 0 \), \( M(r) = \max \{ U(x) : \| x \| = r \} \).

Proof. The proof leans heavily on the fact that (for \( n = 2 \)) \( V(r) = \log r \) so that \( |V(r)| \to \infty \) both as \( r \downarrow 0 \) and as \( r \to \infty \). Consequently, if it is assumed that \( \lim \inf M(r)/\log r \leq 0 \) both as \( r \downarrow 0 \) and as \( r \to \infty \), then taking limits as \( b \to \infty \) and as \( a \downarrow 0 \) in two separate steps, the convexity inequality stated in 2.1 above yields: \( M(r) \leq M(a) \) for \( r \geq a \) and \( M(r) \leq M(b) \) for \( r \leq b \).
This implies that $M$ is constant on $(0, \infty)$ which in turn by the strong maximum principle shows that $u$ is constant too.

2.3 Remarks. (i) Since $U(x) = \lambda \log(\|x\|)$, where $\lambda \neq 0$ satisfies the hypotheses of Theorem 2.2, the conclusion of Theorem 2.2 is, in a sense, the best possible.

(ii) That the analog of the version 2.2 of Liouville's theorem is false in dimensions other than two is demonstrated by the following examples:
For $n = 1$, let
\[ U(x) = e^{x}, \quad \text{for } x < 0, \]
\[ = e^{-x}, \quad \text{for } x > 0. \]

For $n \geq 3$ let
\[ U(x) = -\|x\|^{2-n}, \quad \text{for } \|x\| \geq 1, \]
\[ = -\frac{n(n+2)}{8} + \frac{n^2-4}{4}\|x\|^2 - \frac{n(n-2)}{8}\|x\|^4, \quad \text{for } \|x\| \leq 1. \]

3. EXTENSION OF THE METHOD OF 2.2

The procedure outlined in the proof of Theorem 2.2 of deriving Liouville's theorem in two dimensions from Hadamard's three-circles theorem may be extended to certain more general situations. This is done in Theorems 3.1 and 3.2.

3.1 THEOREM. Let $X, Y$ be normed linear spaces, $0 \leq \rho < R \leq \infty$, $A = \{x \in X : \rho < \|x\| < R\}$, $U: A \to Y$ be a bounded function with $U(rx) \to 0$ as $r \downarrow \rho$ uniformly in $x$ for $\|x\| = 1$, and let $M: (\rho, R) \to [0, \infty)$ be defined by $M(r) = \sup\{\|U(x)\| : \|x\| = r\}$. If there exists a one-to-one function $f: [0, \infty) \to \mathbb{R}$ which is everywhere locally bounded above and a function $g$ from $\mathbb{R}$ onto $(\rho, R)$ such that $f \circ M \circ g$ is convex, then $U = 0$.

Proof: Since $U$ is a bounded function, so is $M$. Also, each point in $[0, \infty)$ has a neighborhood on which $f$ is bounded above; hence by a routine compactness argument it follows that $f(B)$ is bounded above for each bounded set $B \subset [0, \infty)$. Consequently, the function $f \circ M \circ g$ is bounded above; since $f \circ M \circ g$ is convex and defined over all of $\mathbb{R}$, it must therefore be a constant function. But, this implies that $M \circ g$ itself must be a constant function since $f$ is one-to-one. As the range of $g = (\rho, R) = \text{domain of } M$, it now follows that $M$ is a constant function. So then, for any $r \in (\rho, R)$, $M(r) = \lim_{s \downarrow \rho} M(s) = 0$ by the uniform convergence hypothesis on $U$. This gives $U = 0$ as desired.
3.2 Theorem. Let $X$ be a normed linear space, $0 \leq p < R \leq \infty$. $A = \{x \in X: p < \|x\| < R\}$, $U: A \to \mathbb{R}$ be bounded above with $U(rx) \to 0$ as $r \downarrow p$ uniformly in $x$ for $\|x\| = 1$ and let $M: (\rho, R) \to \mathbb{R}$ be defined by $M(r) = \sup\{U(x): \|x\| = r\}$. If there exists a one-to-one function $f: \mathbb{R} \to \mathbb{R}$ with $f(B)$ bounded above for each set $B \subset \mathbb{R}$ bounded above and a function $g$ from $\mathbb{R}$ onto $(\rho, R)$ such that $f \circ M \circ g$ is convex, then $U \leq 0$.

Proof. Just as in the preceding proof, we have $M(r) = 0$ for all $r \in (\rho, R)$ and this gives $U \leq 0$ on all of $A$.

3.3 Remarks. (i) The convergence $r \downarrow p$ in the hypotheses of Theorems 3.1 and 3.2 may be replaced by $r \uparrow R$ or $r \uparrow p$ or $r \downarrow p$ for some fixed $p \in (\rho, R)$.

(ii) The boundedness condition on $f$ in Theorem 3.2 may be stated thus: $\sup\{f(t): t \leq N\} < \infty$ for each positive integer $N$.

(iii) To derive the usual version of Liouville's theorem for entire functions from Theorem 3.1 above, set $X = Y = \mathbb{C}$, $\rho = 0$, $R = \infty$, $f(t) = t$, $g(t) = e^t$, and let $\phi: \mathbb{C} \to \mathbb{C}$ be analytic and bounded. Set $U(z) = \phi(z) - \phi(0)$ and suppose that $U \neq 0$. Then by Hadamard's three-circles theorem, $\log M(r)$ is a convex function of $\log r$; hence also $M(r)$ is a convex function of $\log r$. (This follows from the easy fact that if $F$ and $G$ are convex functions and if $F$ is nondecreasing, then $F \circ G$ is also convex.) That is, $f \circ M \circ g$ is convex for the particular choice of $f$ and $g$. Hence, by Theorem 3.1, $U \to 0$ in contradiction to the assumption that $U \neq 0$.

(iv) In comparing Theorems 3.1 and 3.2 above with the general Hadamard convexity result presented in Theorem 2.1, the suggested choice for $g$ is the inverse of the function $V$. For $n > 2$, the range of $V$ is contained in $(0, \infty)$ while for $n = 2$, $V = \log$ has the whole real line for range iff $\rho = 0$ and $R = \infty$. Thus, $V^{-1}$ can serve for $g$ only in the special case $n = 2$, $\rho = 0$, $R = \infty$.

4. Abstract Versions

The following version of Liouville's theorem in an abstract setting appears in Rudin [5, p. 81] and is readily proved by appealing to the classical version for complex-valued entire functions defined on the complex plane.

4.1 Theorem. Let $X$ be a complex topological vector space on which $X^*$ separates points and let $f: \mathbb{C} \to X$ be weakly holomorphic. If $f(\mathbb{C})$ is a weakly bounded subset of $X$, then $f$ is constant.

A several variables version for vector-valued holomorphic functions that appears in Dieudonné [2, p. 232] may be stated as
4.2 Theorem. Let $X$ be a complex Banach space, $f: \mathbb{C}^p \to X$ be analytic and suppose there exist positive numbers $K$ and $\gamma$ such that 
\[ \|f(z)\| \leq K(1 + \|z\|^\gamma) \]
for all $z \in \mathbb{C}^p$. Then $f(z)$ is a polynomial in the variables $z_1, \ldots, z_p$ of degree $\leq \gamma$.

The following two results are mild generalizations of Theorems 4.1 and 4.2:

4.3 Theorem. Let $X$ be any set, $Y$ a complex Banach space, $\Lambda$ a family of maps from $X$ to $Y$ which separates points of $X$ and let $f: \mathbb{C}^p \to X$ be such that for each $\phi \in \Lambda$:

(i) $\phi \circ f$ is holomorphic, and

(ii) there exist a positive number $\gamma < 1$ and a positive number $K$ such that
\[ \|\phi \circ f(z)\| \leq K(1 + \|z\|^\gamma) \]
for all $z \in \mathbb{C}^p$.

Then $f$ is constant.

[Note that $K$ and $\gamma$ in (ii) could be dependent on $\phi$.]

Proof. Let $\phi \in \Lambda$. Since $\phi \circ f$ is holomorphic and the growth restriction (ii) holds, it follows from Theorem 4.2 that $\phi \circ f(z)$ is a polynomial of degree $\leq \gamma < 1$, i.e., that $\phi \circ f(z)$ is constant for all $z \in \mathbb{C}^p$. Hence, $\phi(f(z)) = \phi(f(0))$ for all $z$ and for all $\phi \in \Lambda$. Since $\Lambda$ separates points of $X$, it follows that $f(z) = f(0)$ for all $z$.

4.4 Theorem. Let $X$ be a locally convex quasicomplete complex topological vector space, $Y$ be a complex Banach space, and $\Lambda$ be a family of continuous linear maps from $X$ to $Y$ which separates points of $X$. Let $f: \mathbb{C}^p \to X$ be holomorphic and suppose there exists a positive number $\gamma$ such that for each $T \in \Lambda$ there exists a number $M > 0$ with $\|T(f(z))\| \leq M(1 + \|z\|^\gamma)$ for all $z \in \mathbb{C}^p$. Then $f(z)$ is a polynomial of degree $\leq \gamma$ in the variables $z_1, \ldots, z_p$. [Note again that $M$ could depend on $T$.]

Proof. We use the standard multi-index notation where for $v = (v_1, \ldots, v_p) \in \mathbb{Z}^p_+$ ($\mathbb{Z}_+$ denoting the set of all nonnegative integers) and $z = (z_1, \ldots, z_p) \in \mathbb{C}^p$, $|v| = v_1 + \cdots + v_p$, $v! = (v_1)! \cdots (v_p)!$, $z^v = z_1^{v_1} \cdots z_p^{v_p}$ and the partial differential operators $D^v$ are defined by
\[ D^v = \left( \frac{\partial}{\partial z_1} \right)^{v_1} \cdots \left( \frac{\partial}{\partial z_p} \right)^{v_p}. \]

Since $f$ is holomorphic, $D^v f(z)$ exists at each $z \in \mathbb{C}^p$ and for all $v \in \mathbb{Z}^p_+$; let
\[ a_v = (1/v!) D^v f(0) \]
and let $P$ be the polynomial defined by
\( P(z) = \sum_{|r| \leq \gamma} a_r z^r \). Now let \( T \in A \). From the linearity of \( T \), it follows that for each \( z \in \mathbb{C}^p \) and for each \( h \in \mathbb{C} \setminus \{0\} \), there holds \( \frac{1}{h}(T \circ f(z + he_k) - T \circ f(z)) = T\left(\frac{1}{h}(f(z + he_k) - f(z))\right) \), where \( e_k \in \mathbb{C}^p \) is the unit vector with \( k \)th component 1 and all other components zero. Taking limits of the above equality as \( h \to 0 \), it follows by the continuity of \( T \) that \( (\partial T \circ f(z)) \) exists and that \( (\partial/\partial z_k)(T \circ f(z)) = T(\partial f(z)/\partial z_k) \). Thus, \( T \circ f(z) \) is holomorphic; the growth estimate \( \|T(f(z))\| \leq M(1 + \|z\|^{\gamma}) \) now implies, by Theorem 4.2, that \( T \circ f(z) \) is a polynomial in \( z \) of degree \( \leq \gamma \). Hence \( T \circ f(z) = \sum_{|r| \leq \gamma} (1/r!)(D^r T \circ f(0)) z^r \). But, repeated application of the rule \( (\partial/\partial z_k)(T \circ f(z)) = T(\partial f(z)/\partial z_k) \) results in \( D^r T \circ f(z) = T(D^r f(z)) \) so that \( (1/r!)(D^r T \circ f(0)) - (1/r!)(T(D^r f(0))) = T((1/r!)(D^r f(0)) - T(a_r)) \); hence \( T \circ f(z) = \sum_{|r| \leq \gamma} T(a_r) z^r = T(P(z)) \). Thus, \( T(f(z)) = T(P(z)) \) for all \( z \in \mathbb{C}^p \) and for all \( T \in A \). Since \( A \) separates points of \( X \), it follows that \( f(z) = P(z) \) for all \( z \in \mathbb{C}^p \).

**References**