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Quartically hyponormal weighted shifts need not be 3-hyponormal

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Abstract

We give the first example of a quartically hyponormal unilateral weighted shift which is not 3-hyponormal.

Keywords: k-Hyponormal; Weakly k-hyponormal; Weighted shift

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geqslant TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal, then T is also hyponormal.

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The Bram–Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j}\left(T^i x_j, T^j x_i\right) \geqslant 0$$

for all finite collections $x_0, x_1, \ldots, x_k \in \mathcal{H}$ ([2], [3, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \cdots & T^{*k} \\ T & T^*T & \cdots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geqslant 0 \quad (\text{all } k \geqslant 1). \tag{1}$$

Condition (1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity of (1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (1) for all k.

Let $[A, B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := \left(\left[T^{*j}, T^i\right]\right)_{i,j=1}^k \tag{2}$$

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (1); the Bram–Halmos criterion can then be rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geqslant 1$ [11].

Recall [1,4,11] that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS\left(T, T^2, \ldots, T^k\right) := \left\{\sum_{j=1}^k \alpha_j T^j \colon \alpha_1, \ldots, \alpha_k \in \mathbb{C}\right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive* (cf. [11]), i.e.,

$$\left\langle M_k(T)\begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix}\right\rangle \geqslant 0 \quad \text{for all } x \in \mathcal{H} \text{ and } \lambda_1, \ldots, \lambda_k \in \mathbb{C}. \tag{3}$$

The operator T is said to be *quadratically hyponormal* when (3) holds for $k = 2$, and *cubically hyponormal* (respectively *quartically hyponormal*) when (3) holds for $k = 3$ (respectively $k = 4$). Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is straightforward to verify that k-hyponormality implies weak k-hyponormality, but the converse is not true in general. For unilateral weighted shifts, quadratic hyponormality is detected through the analysis of an associated tridiagonal matrix, while cubic hyponormality requires a pentadiagonal matrix [6,16]. The associated nested determinants satisfy either a two-step recurring relation (in the tridiagonal case) or a six-step recurring relation (in the pentadiagonal case). The concrete calculation of the above mentioned nested determinants has helped shed light on quadratic and cubic hyponormality. On the other hand, quartic hyponormality requires heptadiagonal matrices, and a similar multi-step recurring relation for the nested determinants is not known. As a result, there is very little information available about quartic hyponormality, and the notion has remained highly inscrutable.

In this paper, we present the first concrete example of a quartically hyponormal operator which is not 3-hyponormal. Although the result is somewhat expected, and consistent with previous results in this area (e.g., for general operators polynomial hyponormality does not imply 2-hyponormality [12], and for weighted shifts cubic hyponormality does not imply 2-hyponormality [16]), the techniques needed to prove it are new. For instance, the proof of Theorem 4(i) includes a new trick to compute a key determinant, while the proof of Theorem 4(ii) is based on a special rearrangement of the terms in the quadratic form Δ whose positivity ensures that the weighted shift is quartically hyponormal.

Recall that given a bounded sequence of positive numbers α: $\alpha_0, \alpha_1, \dots$ (called *weights*), the (*unilateral*) weighted shift W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geqslant 0$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leqslant \alpha_{n+1}$ for all $n \geqslant 0$. The moments of α are given as

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0, \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0. \end{cases}$$

We now recall a well known characterization of subnormality for single-variable weighted shifts, due to C. Berger (cf. [3, III.8.16]), and independently established by R. Gellar and L.J. Wallen [15]: W_α is subnormal if and only if there exists a probability measure ξ (called the *Berger measure* of W_α) supported in $[0, \|W_\alpha\|^2]$ such that $\gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int t^k \, d\xi(t)$ $(k \geqslant 1)$. If W_α is subnormal, and if for $h \geqslant 1$ we let $\mathcal{M}_h := \bigvee \{e_n \colon n \geqslant h\}$ denote the invariant subspace obtained by removing the first h vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$, then the Berger measure of $W_\alpha|_{\mathcal{M}_h}$ is $\frac{1}{\gamma_h} t^h \, d\xi(t)$.

The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality [5–11,14,16,17]. The study of this gap has been mostly successful at the level of k-hyponormality; for example, for Toeplitz operators on the Hardy space of the unit circle, the gap is described in [10]. For weighted shifts, on the other hand, positive results appear in [6] and [10], although no concrete example of a weighted shift which is polynomially hyponormal and not subnormal has yet been found (the existence of such weighted shifts was established in [12] and [13]). For weak k-hyponormality there nevertheless exist some partial results. For example, in [6], the gap between 2-hyponormality and quadratic hyponormality was established, and in [16] weighted shifts which are cubically hyponormal and not 2-hyponormal were found. In this paper, we give an example of a weighted shift which is weakly 4-hyponormal but not 3-hyponormal.

2. Main results

We begin with an observation about quadratic hyponormality.

Proposition 1. *W_α is quadratically hyponormal if and only if $W_\alpha + sW_\alpha^2$ is hyponormal for all $s \geqslant 0$.*

Proof. ($\Rightarrow$) This implication is trivial.

($\Leftarrow$) Suppose $W_\alpha + sW_\alpha^2$ is hyponormal for all $s \geqslant 0$. We must show that $W_\alpha + cW_\alpha^2$ is hyponormal for all $c \in \mathbb{C}$. For $c \equiv se^{i\theta}$ ($s > 0$), there exists a unitary operator U such that $UTU^* = e^{-i\theta}T$. Then

$$\begin{aligned} U\big(T + cT^2\big)U^* &= UTU^* + cUT^2U^* = UTU^* + c(UTU^*)^2 \\ &= e^{-i\theta}T + se^{i\theta} \cdot e^{-2i\theta}T^2 = e^{-i\theta}\big(T + sT^2\big) \end{aligned}$$

is hyponormal. Therefore, $T + cT^2$ is hyponormal. □

Lemma 2. *The following statements are equivalent*:

(i) *W_α is quartically hyponormal*;
(ii) *for each $x \equiv \{x_n\}_{n=0}^{\infty} \in \ell^2$, we have $(\langle [W_\alpha^{*j}, W_\alpha^i]x, x\rangle)_{i,j=1}^4 \geqslant 0$;*
(iii) *for each $a, b, c \in \mathbb{C}$ and $x \equiv \{x_n\}_{n=0}^{\infty} \in \ell^2$,*

$$\begin{aligned} \Delta := |c|^2 r_0 |x_0|^2 &+ \left\langle \Theta_1 \begin{pmatrix} \bar{b}x_0 \\ \bar{c}x_1 \end{pmatrix}, \begin{pmatrix} \bar{b}x_0 \\ \bar{c}x_1 \end{pmatrix} \right\rangle + \left\langle \Theta_2 \begin{pmatrix} \bar{a}x_0 \\ \bar{b}x_1 \\ \bar{c}x_2 \end{pmatrix}, \begin{pmatrix} \bar{a}x_0 \\ \bar{b}x_1 \\ \bar{c}x_2 \end{pmatrix} \right\rangle \\ &+ \sum_{i=0}^{\infty} \left\langle \Delta_i \begin{pmatrix} x_i \\ \bar{a}x_{i+1} \\ \bar{b}x_{i+2} \\ \bar{c}x_{i+3} \end{pmatrix}, \begin{pmatrix} x_i \\ \bar{a}x_{i+1} \\ \bar{b}x_{i+2} \\ \bar{c}x_{i+3} \end{pmatrix} \right\rangle \geqslant 0. \end{aligned}$$

Here

$$\Theta_1 := \begin{pmatrix} p_0 & \sqrt{g_0} \\ \sqrt{g_0} & r_1 \end{pmatrix}, \qquad \Theta_2 := \begin{pmatrix} v_0 & \sqrt{t_0} & \sqrt{f_0} \\ \sqrt{t_0} & p_1 & \sqrt{g_1} \\ \sqrt{f_0} & \sqrt{g_1} & r_2 \end{pmatrix},$$

$$\Delta_i := \begin{pmatrix} u_i & \sqrt{w_i} & \sqrt{s_i} & \sqrt{q_i} \\ \sqrt{w_i} & v_{i+1} & \sqrt{t_{i+1}} & \sqrt{f_{i+1}} \\ \sqrt{s_i} & \sqrt{t_{i+1}} & p_{i+2} & \sqrt{g_{i+2}} \\ \sqrt{q_i} & \sqrt{f_{i+1}} & \sqrt{g_{i+2}} & r_{i+3} \end{pmatrix} \quad (i \geqslant 0),$$

where

$$\begin{aligned} u_i &:= \alpha_i^2 - \alpha_{i-1}^2, \\ w_i &:= \alpha_i^2\big(\alpha_{i+1}^2 - \alpha_{i-1}^2\big)^2, \\ v_i &:= \alpha_i^2\alpha_{i+1}^2 - \alpha_{i-1}^2\alpha_{i-2}^2, \\ s_i &:= \alpha_i^2\alpha_{i+1}^2\big(\alpha_{i+2}^2 - \alpha_{i-1}^2\big)^2, \\ t_i &:= \alpha_i^2\big(\alpha_{i+1}^2\alpha_{i+2}^2 - \alpha_{i-1}^2\alpha_{i-2}^2\big)^2, \end{aligned}$$

$$p_i := \alpha_i^2\alpha_{i+1}^2\alpha_{i+2}^2 - \alpha_{i-1}^2\alpha_{i-2}^2\alpha_{i-3}^2,$$
$$q_i := \alpha_i^2\alpha_{i+1}^2\alpha_{i+2}^2\left(\alpha_{i+3}^2 - \alpha_{i-1}^2\right)^2,$$
$$f_i := \alpha_i^2\alpha_{i+1}^2\left(\alpha_{i+2}^2\alpha_{i+3}^2 - \alpha_{i-1}^2\alpha_{i-2}^2\right)^2,$$
$$g_i := \alpha_i^2\left(\alpha_{i+1}^2\alpha_{i+2}^2\alpha_{i+3}^2 - \alpha_{i-1}^2\alpha_{i-2}^2\alpha_{i-3}^2\right)^2,$$
$$r_i := \alpha_i^2\alpha_{i+1}^2\alpha_{i+2}^2\alpha_{i+3}^2 - \alpha_{i-1}^2\alpha_{i-2}^2\alpha_{i-3}^2\alpha_{i-4}^2.$$

(*As usual, we let* $\alpha_{-1} = \alpha_{-2} = \alpha_{-3} = \alpha_{-4} = 0$.)

Proof. This is a straightforward computation. □

Remark 3. Observe that W_α is 4-hyponormal if and only if $\Theta_2 \geqslant 0$ and $\Delta_i \geqslant 0$ for all $i \geqslant 0$.

We now give an example of a weighted shifts which is quartically hyponormal but not 3-hyponormal.

Theorem 4. *For* $x > 0$, *let* $W_{\alpha(x)}$ *be the unilateral weighted shift with weight sequence given by* $\alpha_0 := \sqrt{x},\ \alpha_n := \sqrt{\frac{n+2}{n+3}}$ $(n \geqslant 1)$. *Then*

(i) $W_{\alpha(x)}$ *is* k*-hyponormal if and only if* $0 < x \leqslant \frac{2(k+1)^2(k+2)^2}{3k(k+3)(k^2+3k+4)} =: H_k$;
(ii) *if* $0 < x \leqslant \frac{667}{990}$, *then* $W_{\alpha(x)}$ *is quartically hyponormal.*

Proof. (i) By [6, Theorem 4(d)], we know that $W_{\alpha(x)}$ is k-hyponormal if and only if

$$A(n;k) := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \geqslant 0 \quad (\text{all } n \geqslant 0).$$

Since $W_{\alpha(x)}$ has a Bergman tail, it is enough to check at $n = 0$. In this case, $A(0;k) \geqslant 0$ is equivalent to

$$\det \begin{pmatrix} \frac{1}{3x} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k+2} & \frac{1}{k+3} & \frac{1}{k+4} & \cdots & \frac{1}{2k+2} \end{pmatrix} \geqslant 0.$$

Let

$$A := \begin{pmatrix} \frac{1}{3x} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k+2} & \frac{1}{k+3} & \frac{1}{k+4} & \cdots & \frac{1}{2k+2} \end{pmatrix},$$

$$B := \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{k+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k+2} & \frac{1}{k+3} & \frac{1}{k+4} & \cdots & \frac{1}{2k+2} \end{pmatrix}$$

and

$$C := \begin{pmatrix} \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{k+3} \\ \frac{1}{5} & \frac{1}{6} & \cdots & \frac{1}{k+4} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{k+3} & \frac{1}{k+4} & \cdots & \frac{1}{2k+2} \end{pmatrix}.$$

Expanding the determinants of A and B by the first row, we have

$$\det A = \frac{1}{3x}\det C + Q, \qquad \det B = \frac{1}{2}\det C + Q,$$

so that

$$\det A = \frac{1}{3x}\det C + \det B - \frac{1}{2}\det C = \frac{2-3x}{6x}\det C + \det B.$$

Now, for $H := (h_{ij})_{i,j=1}^{n}, h_{ij} := (p+i+j-1)^{-1}$ and $p \geqslant 0$, recall that

$$\det H = \bigl(1!2!\cdots(n-1)!\bigr)^2 \frac{\Gamma(p+1)\Gamma(p+2)\cdots\Gamma(p+n)}{\Gamma(n+p+1)\Gamma(n+p+2)\cdots\Gamma(2n+p)}.$$

Thus,

$$\det A = \frac{(1!2!\cdots(k-1)!)^2\Gamma(4)\Gamma(5)\cdots\Gamma(k+2)}{\Gamma(k+4)\Gamma(k+5)\cdots\Gamma(2k+3)} \times \left[\left(\frac{2-3x}{6x}\right)\Gamma(k+3) + \frac{(k!)^2\Gamma(2)\Gamma(3)}{\Gamma(k+3)}\right].$$

Therefore, $\det A \geqslant 0$ if and only if $0 < x \leqslant \frac{2(k+1)^2(k+2)^2}{3k(k+3)(k^2+3k+4)}$, as desired.

(ii) By a direct computation, we have

$$\Theta_1 = \begin{pmatrix} \frac{3}{5}x & \frac{\sqrt{x}}{2} \\ \frac{\sqrt{x}}{2} & \frac{3}{7} \end{pmatrix}, \qquad \Theta_2 = \begin{pmatrix} \frac{3}{4}x & \frac{3}{5}\sqrt{x} & \sqrt{\frac{x}{3}} \\ \frac{3}{5}\sqrt{x} & \frac{1}{2} & \frac{2\sqrt{3}}{7} \\ \sqrt{\frac{x}{3}} & \frac{2\sqrt{3}}{7} & \frac{1}{2} \end{pmatrix},$$

$$\Delta_0 = \begin{pmatrix} x & \frac{3}{4}\sqrt{x} & \frac{2}{5}\sqrt{3x} & \frac{1}{2}\sqrt{\frac{5}{3}x} \\ \frac{3}{4}\sqrt{x} & \frac{3}{5} & \frac{1}{\sqrt{3}} & \frac{\sqrt{15}}{7} \\ \frac{2}{5}\sqrt{3x} & \frac{1}{\sqrt{3}} & \frac{4}{7} & \frac{\sqrt{5}}{4} \\ \frac{1}{2}\sqrt{\frac{5}{3}x} & \frac{\sqrt{15}}{7} & \frac{\sqrt{5}}{4} & \frac{5}{9} \end{pmatrix},$$

$$\Delta_1=\begin{pmatrix} \frac{3}{4}-x & \frac{\sqrt{3}}{2}(\frac{4}{5}-x) & \sqrt{\frac{3}{5}}(\frac{5}{6}-x) & \frac{1}{\sqrt{2}}(\frac{6}{7}-x) \\ \frac{\sqrt{3}}{2}(\frac{4}{5}-x) & \frac{2}{3}-\frac{3}{4}x & \frac{2}{\sqrt{5}}(\frac{5}{7}-\frac{3}{4}x) & \frac{\sqrt{3}}{2\sqrt{2}}(1-x) \\ \sqrt{\frac{3}{5}}(\frac{5}{6}-x) & \frac{2}{\sqrt{5}}(\frac{5}{7}-\frac{3}{4}x) & \frac{5}{8}-\frac{3}{5}x & \sqrt{\frac{5}{6}}(\frac{2}{3}-\frac{3}{5}x) \\ \frac{1}{\sqrt{2}}(\frac{6}{7}-x) & \frac{\sqrt{3}}{2\sqrt{2}}(1-x) & \sqrt{\frac{5}{6}}(\frac{2}{3}-\frac{3}{5}x) & \frac{3}{5}-\frac{x}{2} \end{pmatrix},$$

$$\Delta_2=\begin{pmatrix} \frac{1}{20} & \frac{1}{6\sqrt{5}} & \frac{\sqrt{\frac{3}{2}}}{14} & \frac{1}{4\sqrt{7}} \\ \frac{1}{6\sqrt{5}} & \frac{4}{35} & \frac{\sqrt{\frac{3}{10}}}{4} & \frac{8}{9\sqrt{35}} \\ \frac{\sqrt{\frac{3}{2}}}{14} & \frac{\sqrt{\frac{3}{10}}}{4} & \frac{1}{6} & \frac{\sqrt{\frac{6}{7}}}{5} \\ \frac{1}{4\sqrt{7}} & \frac{8}{9\sqrt{35}} & \frac{\sqrt{\frac{6}{7}}}{5} & \frac{16}{77} \end{pmatrix}, \qquad \Delta_3=\begin{pmatrix} \frac{1}{30} & \frac{\sqrt{\frac{2}{15}}}{7} & \frac{3}{8\sqrt{35}} & \frac{\sqrt{\frac{2}{5}}}{9} \\ \frac{\sqrt{\frac{2}{15}}}{7} & \frac{1}{12} & \frac{\sqrt{\frac{2}{21}}}{3} & \frac{1}{5\sqrt{3}} \\ \frac{3}{8\sqrt{35}} & \frac{\sqrt{\frac{2}{21}}}{3} & \frac{9}{70} & \frac{3\sqrt{\frac{2}{7}}}{11} \\ \frac{\sqrt{\frac{2}{5}}}{9} & \frac{1}{5\sqrt{3}} & \frac{3\sqrt{\frac{2}{7}}}{11} & \frac{1}{6} \end{pmatrix},$$

and

$$\Delta_4=\begin{pmatrix} \frac{1}{42} & \frac{1}{4\sqrt{42}} & \frac{1}{12\sqrt{3}} & \frac{\sqrt{\frac{2}{3}}}{15} \\ \frac{1}{4\sqrt{42}} & \frac{4}{63} & \frac{3}{10\sqrt{14}} & \frac{8}{33\sqrt{7}} \\ \frac{1}{12\sqrt{3}} & \frac{3}{10\sqrt{14}} & \frac{9}{88} & \frac{1}{6\sqrt{2}} \\ \frac{\sqrt{\frac{2}{3}}}{15} & \frac{8}{33\sqrt{7}} & \frac{1}{6\sqrt{2}} & \frac{16}{117} \end{pmatrix}.$$

Note that $\Delta_n \geqslant 0$ for all $n \geqslant 5$, and that all above matrices are positive except possibly for Δ_1. For the positivity of Δ, we minimize the positivity of $\Theta_1, \Theta_2, \Delta_0, \Delta_2, \Delta_3, \Delta_4$, that is, we replace $\Theta_1, \Theta_2, \Delta_0, \Delta_2, \Delta_3, \Delta_4$ by $\Theta_1', \Theta_2', \Delta_0', \Delta_2', \Delta_3', \Delta_4'$, with $\operatorname{rank}\Theta_j' = j$ for $j = 1, 2$, and $\operatorname{rank}\Delta_j' = 3$ for $j = 0, 2, 3, 4$, respectively, where

$$\Theta_1' := \begin{pmatrix} \cdot & \cdot \\ \cdot & \frac{5}{12} \end{pmatrix}, \qquad \Theta_2' := \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \frac{612}{1225} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix},$$

$$\Delta_0' := \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1411}{2352} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad \Delta_2' := \begin{pmatrix} \frac{627}{12544} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$\Delta_3' := \begin{pmatrix} \frac{1411}{42336} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \text{and} \quad \Delta_4' := \begin{pmatrix} \frac{2057}{86400} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

(In all of the above expressions, the symbol "$\cdot$" denotes an entry that remains unchanged.) Let

$$\tilde{\Delta}(a,b,c) := \begin{pmatrix} A & \frac{\sqrt{3}}{2}(\frac{4}{5}-x)\bar{a} & \sqrt{\frac{3}{5}}(\frac{5}{6}-x)\bar{b} & \frac{1}{\sqrt{2}}(\frac{6}{7}-x)\bar{c} \\ \frac{\sqrt{3}}{2}(\frac{4}{5}-x)a & B & \frac{2}{\sqrt{5}}(\frac{5}{7}-\frac{3}{4}x)a\bar{b} & \frac{\sqrt{3}}{2\sqrt{2}}(1-x)a\bar{c} \\ \sqrt{\frac{3}{5}}(\frac{5}{6}-x)b & \frac{2}{\sqrt{5}}(\frac{5}{7}-\frac{3}{4}x)\bar{a}b & (\frac{5}{8}-\frac{3}{5}x)|b|^2+\frac{1}{211680} & \sqrt{\frac{5}{6}}(\frac{2}{3}-\frac{3}{5}x)b\bar{c} \\ \frac{1}{\sqrt{2}}(\frac{6}{7}-x)c & \frac{\sqrt{3}}{2\sqrt{2}}(1-x)\bar{a}c & \sqrt{\frac{5}{6}}(\frac{2}{3}-\frac{3}{5}x)\bar{b}c & (\frac{3}{5}-\frac{x}{2})|c|^2+\frac{1}{604800} \end{pmatrix}$$

where $A := \frac{3}{4} - x + \frac{|a|^2}{11760} + \frac{|b|^2}{2450} + \frac{|c|^2}{84}$ and $B := (\frac{2}{3} - \frac{3}{4}x)|a|^2 + \frac{1}{62720}$. If $\tilde{\Delta}(a,b,c) \geqslant 0$ for all $a, b, c \in \mathbb{C}$, then $\Delta \geqslant 0$ and hence, by Lemma 2, $W_{\alpha(x)}$ is quartically hyponormal. Note that in each nested determinant of $\tilde{\Delta}(a,b,c)$ the parameters a, b, c occur in modulus square form. So, using the Nested Determinants Test, we can easily see that if $x \leqslant \frac{22580899}{33531912} =: \xi$ then every coefficient in $\det \tilde{\Delta}(a,b,c)$ is positive. Therefore, $\tilde{\Delta}(a,b,c) \geqslant 0$ for all $a, b, c \in \mathbb{C}$. Note that $H_2 = \frac{24}{35}$ and $H_3 = \frac{200}{297}$. Observe that $H_3 < \xi < H_2$. Moreover, we can show that $\tilde{\Delta}(a,b,c) \geqslant 0$ for $x = \frac{667}{990} > \xi$, again using the Nested Determinants Test. □

Corollary 5.

(i) *If* $\frac{200}{297} < x \leqslant \frac{667}{990}$, *then* $W_{\alpha(x)}$ *is quartically hyponormal but not* 3*-hyponormal.*
(ii) *If* $W_{\alpha(x)}$ *is* 3*-hyponormal then it is also quartically hyponormal.*

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