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On a Class of Baric Algebras

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ABSTRACT

It is known that baric algebras satisfying the identity $(x^2)^2 = w(x)x^3$ have idempotent elements and every linear form $w: A \to K$ is a multiplicative map. We prove that these algebras are Jordan-Bernstein of order 2 and special train algebras. Moreover, as a corollary we obtain that the train equation of these algebras is $x^4 - w(x)x^3 = 0$, and we give examples of baric algebras satisfying $x^4 - w(x)x^3 = 0$ but not satisfying $(x^2)^2 = w(x)x^3$.

1. INTRODUCTION

In what follows, K is an infinite field of characteristic not 2, and A is a commutative, not necessarily associative algebra over K.

We recall that A is a Jordan algebra if the identity $x^2(yx) = (x^2y)x$ holds in A. If $w: A \to K$ is a nonzero algebra homomorphism, then the ordered pair (A, w) is called a baric algebra and w its weight function. If the baric algebra (A, w) satisfies the identity $x^{[n+2]} = (w(x)x)^{[n+1]}$, it is called a Bernstein algebra of order n, where n is the minimum integer for which the identity holds and $x^{[1]} = x, \ldots, x^{[k+1]} = x^{[k]}x^{[k]}$, $k \geq 1$, are the plenary powers of x. For references, see [2] and [4]. If the baric algebra (A, w) satisfies the equation $x^r + \gamma_1 w(x)x^{r-1} + \cdots + \gamma_{r-1}w(x)^{r-1}x = 0$ (train equation), it is called a train algebra of rank r, where r is the minimum integer for which the above identity holds, $\gamma_1, \ldots, \gamma_{r-1}$ are fixed elements in K, and $x^1 = x, \ldots, x^{k+1} = x^k x$ are the principal powers of x. The baric algebra (A, w) is a special train algebra if $\text{Ker}(w)^k$ is an ideal of A for every $k \in \mathbb{N}$ and $\text{Ker}(w)$ is nilpotent. Moreover, every special train algebra is a train algebra; see [4] for details.

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Let us consider the sets

$$B_1 = \{(A, w) \mid (x^2)^2 = w(x)x^3\},$$

and

$$B_2 = \{(A, w) \mid x^4 = w(x)x^3\}.$$

In this paper we study the elements of B_1, i.e., the class of baric algebras satisfying $(x^2)^2 = w(x)x^3$. We prove that these algebras are Jordan-Bernstein of order 2. Moreover, we prove that $B_1 \subset B_2$ a proper containement and the elements of B_1 are special train algebras.

2. BARIC ALGEBRAS SATISFYING $(x^2)^2 = w(x)x^3$

In [1], it is proved that algebras satisfying the identity

$$(x^2)^2 = w(x)x^3 \tag{1}$$

always have idempotent elements and every linear form $w: A \to K$ is also a multiplicative map. Moreover, A admits a Peirce decomposition $A = Ke \oplus N_{1/2} \oplus N_0$, where $N_i = \{x \in \operatorname{Ker}(w) \mid ex = ix\}$, $i = 0, \frac{1}{2}$ and $N_{1/2}^2 \subseteq N_0$, $N_{1/2}N_0 \subseteq N_{1/2}$, $N_0^2 \subseteq N_0$, and for every $u \in N_{1/2}$, $v \in N_0$ we have

$$u^3 = 0, \qquad v^3 = 0, \tag{2}$$

$$uv^2 = 2(uv)v, \tag{3}$$

$$u^2v = 2u(uv). \tag{4}$$

EXAMPLE. Let $A = \langle x_1, x_2, x_3, x_4 \rangle$ be an algebra with the following multiplication table: $x_1^2 = x_1$, $x_1x_2 = x_2x_1 = \frac{1}{2}x_2$, $x_2^2 = x_3$, $x_4^2 = \lambda x_3$, $\lambda \neq 0$, all other products being zero. Then A is a commutative algebra. Moreover, A is a baric algebra with weight function $w: A \to K$ defined by $w(x_1) = 1$, $w(x_i) = 0$, $i = 2, 3, 4$, and the elements of A satisfy (1).

We observe that (2) implies that Jacobi's identity is valid in $N_{1/2}$ and in N_0. Moreover, (3) and (4) imply that for every u, $u' \in N_{1/2}$, v, $v' \in N_0$, we have

$$u(vv') = (uv)v' + (uv')v, \tag{5}$$

$$(uu')v = u(u'v) + u'(uv). \tag{6}$$

By linearizing the identity (1) we have the following result.

PROPOSITION 2.1. *For every $x, y, z \in A$, we have*

$$4x^2(xy) = w(x)[x^2y + 2x(xy)] + w(y)x^3, \tag{7}$$

$$\begin{aligned} 4x^2(yz) + 8(xy)(xz) &= 2w(x)[x(yz) + y(zx) + z(xy)] \\ &\quad + w(y)[x^2z + 2x(xz)] + w(z)[x^2y + 2x(xy)]. \end{aligned} \tag{8}$$

The relation (7) is equivalent to the identity (1), and if the characteristic of K is also different from 3, the relation (8) is equivalent to the identity (1).

If we denote $N = \mathrm{Ker}(w)$, we have that (7) and (8) imply that for every $x,\ y,\ z \in N$

$$x^2(xy) = 0, \tag{9}$$

$$x^2(yz) + 2(xy)(xz) = 0. \tag{10}$$

Moreover, the relations (3), (5), and (9) imply that

$$(uv^2)v = 0, \qquad u(u^2v) = 0. \tag{11}$$

THEOREM 2.2. *Suppose A satisfies the identity* (1). *Then A is a Jordan-Bernstein algebra of order* 2.

Proof. By straightforward calculations, we prove that $x^2(yx) - (x^2y)x = 0$ for every $x,\ y \in A$. Therefore, A is a Jordan algebra. So A is power-associative and $((x^2)^2)^2 = x^8$. Then, by using the identity (1), we prove that $x^8 = w(x^4)x^4$. Thus, A is a Bernstein algebra of order 2, and Theorem 2.2 follows. ■

REMARK. The converse of Theorem 2.2 is not true. For instance, $A = Ke \oplus V_2$, $V_2 = \langle v_1, v_2, v_3 \rangle$ and multiplication table $e^2 = e$, $v_1^2 = v_2$, $v_1v_2 = v_2v_1 = v_3$, all other products being zero, is a Bernstein algebra of order 2. Moreover, A is a Jordan algebra, and if $x = e + v_1 + v_2 + v_3$ then $(x^2)^2 = e$ and $w(x)x^3 = e + v_3$.

THEOREM 2.3. *For a baric algebra (A, w), the following conditions are equivalent:*

1. *$A = Ke \oplus U \oplus V_2$ is a Jordan-Bernstein algebra of order* 2 *with $v^3 = 0$ for every $v \in V_2$.*
2. *The identity $(x^2)^2 = w(x)x^3$ holds in A.*

Proof. We only need to prove that statement 1 implies statement 2. Let $A = Ke \oplus U \oplus V_2$ be a Jordan-Bernstein algebra of order 2 with $v^3 = 0$ for every $v \in V_2$.

By linearizing the Jordan identity we have

$$(xy)(zt) + (xz)(yt) + (xt)(yz) = [x(yz)]t + [x(zt)]y + [x(yt)]z. \qquad (12)$$

By setting $x = y = z = e$, $t = v \in V_2$ in (12), we have $ev = 0$. Thus, $eV_2 = \{0\}$. Moreover, by Lemma 4.1 of [3] we have $U^2 \subseteq V_2$, $UV_2 \subseteq U$, $V_2^2 \subseteq V_2$, and the elements u, $u' \in U$ and v, $v' \in V_2$ verify the following identities: $u^3 = 0$, $2u(uv) = u^2v$, $2(uv)v = uv^2$, $u(u^2v) = u^2(uv) = 0$, $u(u^2u') = u^2(uu') = 0$, $u^2v^2 + 4(uv)^2 = 2v(u^2v)$, $v(v^2u) = v^2(vu)$, $v^2(vv') = v(v^2v') = 0$, and $v^4 = 0$. By using these relations we prove that $(x^2)^2 - w(x)x^3 = 4(uv)v^2 + [u^2v^2 + 2v(u^2v)]$.

By setting $x = e$, $y = u$, $z = v^2$, and $t = v$ in (12) and by using $UV_2 \subseteq U$, $V_2^2 \subseteq V_2$, we have $uv^3 = (uv^2)v + (uv)v^2$. But $(uv^2)v = (uv)v^2$; hence, $uv^3 = 2(uv)v^2$. Since $v^3 = 0$, we have $(uv)v^2 = 0$. Finally, Jacobi's identity in V_2 and $U^2 \subseteq V_2$ imply that $2v(vu^2) = -v^2u^2$. Therefore, $(x^2)^2 - w(x)x^3 = 0$. ■

COROLLARY 2.4. *Every baric algebra satisfying the identity* $(x^2)^2 = w(x)x^3$ *also satisfies* $x^4 - w(x)x^3 = 0$.

Proof. Since from Theorem 2.2 we have power associativity, Equation (1) immediately gives $x^4 - w(x)x^3 = 0$. ■

Recall the sets

$$B_1 = \{(A, w) \mid (x^2)^2 = w(x)x^3\}$$

and

$$B_2 = \{(A, w) \mid x^4 = w(x)x^3\}.$$

From Corollary 2.4, we have

$$B_1 \subseteq B_2,$$

but in general, we have a proper containement, as we can see in the following examples:

EXAMPLE 1. Let $A = \langle e, x_1, x_2, x_3 \rangle$ be an algebra with the following multiplication table: $e^2 = e$, $ex_1 = x_1e = \frac{1}{2}x_1$, $ex_3 = x_3e = x_2$, $x_3^2 = x_2$, all other products being zero. Then A is a commutative algebra. Moreover, $w\colon A \to K$ defined by $w(e) = 1$, $w(x_i) = 0$, $i = 1, 2, 3$, is its weight function. So A is a baric algebra satisfying $x^4 - w(x)x^3 = 0$, and if $a = e + x_3$ then $(a^2)^2 = e$ and $w(a)a^3 = e + x_2$. Therefore, A is an algebra in B_2 which is not in B_1.

EXAMPLE 2. This example was suggested by the referee (see note 4.9 in [3]). Let $A = Z(n, 2)$ be the zygotic algebra of a diploid population with n alleles and $D(A)$ the commutative duplicate of A. It is known that $D(A)$ is not power-associative, so $(x^2)^2 \neq x^4$ for every $x \in D(A)$. On the other hand, $x^3 = w(x)x^2$ for all $x \in A$, and $y^4 = w_d(y)y^3$ for every $y \in D(A)$, where w_d is the weight function of $D(A)$. So $D(A)$ is an algebra in B_2 which is not in B_1.

THEOREM 2.5. *Suppose A satisfies the identity* (1). *Then A is a special train algebra.*

The proof of this theorem follows from Theorem 2.2 and from Theorem 4.6 of [3].

REMARK. In [3, note 4.8] Ouattara gives an example of a power-associative Bernstein algebra of order 2, not Jordan, which is not a special train algebra.

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