The reaction–diffusion equation with Lewis function and critical Sobolev exponent

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Abstract

In this paper we consider the existence and asymptotic estimates of global solutions and finite time blowup of reaction–diffusion equation with Lewis function and critical Sobolev exponent.

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1. Introduction

The chemical reaction processes accompanied by diffusion is described; see, for example, [13]. By the principle of conservation, we have

\[ au_t - \nabla \cdot (D\nabla u) = q, \]

where \( u(x, t) \) is called the density function at position \( x \) in a diffusion medium \( \Omega \) in \( \mathbb{R}^N \) and in time \( t \). \( a \) is called Lewis number if \( a \) is a constant. \( D \) is called the diffusion coefficient in chemical diffusion processes. The term \( \nabla \cdot (D\nabla u) \) represents the rate of change due to diffusion, and \( q \) is the rate of change due to reaction. The reaction term \( q \) is the density per unit volume per unit time formed through the process of reaction. In many reaction–diffusion-type problems, \( q \) depends on the density function \( u \) and possibly on \( (x, t) \) explicitly. Writing \( D = 1 \) and \( q = u^p \) (\( 1 < p < +\infty \)) in (1.1) leads to the reaction–diffusion equation
\[ au_t - \Delta u = u^p. \quad (1.2) \]

If \( a \) is a constant, e.g., \( a = 1 \), lots of effort has been devoted, since the pioneering work of Fujita [3] in the 1960s, to the study of (1.2). In [12] Ni et al. proved that any global (classical) solution of (1.2) with Dirichlet boundary condition assuming that \( \Omega \) is bounded convex and \( p < 1 + 2/N \). If \( p \geq (N + 2)/(N - 2) \), \( N > 2 \), then there are global unbounded \( L^1 \)-solutions. The very interesting questions whether these global weak solutions are actually classical was left open for a long time. Recently, Galaktionov and Vazquez [4] proved that for \( p = (N + 2)/(N - 2) \) this is true provided \( \Omega \) is a ball and \( u_0 \) is radially symmetric. An improvement of the result from [12] on boundedness of global solutions was given by Cazenave and Lions in [1]. They removed the assumptions on convexity of \( \Omega \) and nonnegativity of \( u_0 \) and showed that global solutions are uniformly bounded in \( \Omega \times (t_0, \infty) \) for every \( t_0 > 0 \), provided that \( p > 1 \), for \( N = 1, 2 \), or \( 1 < p < (N + 2)/(N - 2) \), for \( N \geq 3 \). Later, Giga [5] derived an a priori bound for any nonnegative global solution in terms of the sup-norm of \( u_0 \) for Dirichlet boundary condition on bounded domain and \( p < (N + 2)/(N - 2) \). On the other hand, Tsutsumi [17] and Ishii [7] have studied the existence, nonexistence and asymptotic behavior of global solutions, and finite time blowup of quasilinear parabolic equation involving \( p \)-Laplacian.

In this paper we consider the reaction–diffusion equations of the following form with Lewis function \( a(x) \), i.e., \( a = a(x) \) is a function:

\[
\begin{aligned}
& a(x)u_t - \Delta u = u^p, \quad (x, t) \in \Omega \times (0, T), \\
& u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
& u(x, 0) = u_0(x) \quad u_0(x) \neq 0,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( R^N (N \geq 3) \), with smooth boundary \( \partial \Omega \), and \( p = 2^* - 1 = (N + 2)/(N - 2) \), where \( 2^* \) is the critical Sobolev exponent. We study the existence and time-asymptotic estimates of global solutions, finite time blow up of (1.3) with \( u_0 \in \Sigma_1 \) or \( u_0 \in \Sigma_2 \). We can prove that, for any global solution \( u(x, t; u_0) \), there exists a subsequence \( \{t_n\} \) such that the asymptotic behavior of \( u(x, t_n; u_0) \) as \( t_n \to \infty \) is similar to the Palais–Smale sequence of semilinear elliptic equation of the following form:

\[
\begin{aligned}
& -\Delta u = u^p, \quad x \in \Omega, \\
& u(x) > 0, \quad x \in \Omega, \\
& u(x) = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

To state the main results, we first suppose that \( a(x) \) satisfies the following condition:

(A) \( a(x) \geq 0, a(x) \in L^\infty(\Omega) \), and the eigenvalue problem

\[
\begin{aligned}
& -\Delta u = \lambda a(x)u, \quad x \in \Omega, \\
& u = 0, \quad x \in \partial \Omega,
\end{aligned}
\]

has the first eigenvalue \( \lambda_1 > 0 \).
Now we give some useful definitions and notations.

Denote the usual Sobolev space by $H^1_0(\Omega)$ (denote the dual by $H^{-1}$), endowed with the norm $\|\nabla u\|_2 = (\int_\Omega |\nabla u|^2 \, dx)^{1/2}$, denote the norm of $L^q(\Omega)$ by $\| \cdot \|_q$. $Q_T = \Omega \times (0, T)$. Define the weight function space and $L^2(\Omega, a(x))$:

$$L^2(\Omega, a(x)) = \left\{ u(x); \left( \int_\Omega a(x)|u(x)|^2 \, dx \right)^{1/2} < \infty \right\};$$

denote the norm by $\| \cdot \|_{\Omega, a}$, respectively.

**Definition 1.1.** We say that a function $u$ is a solution of (1.3) in $Q_T$ iff

$$u \in L^\infty(0, T; H^1_0(\Omega)), \quad \int_0^T \int_\Omega a(x)|u_t|^2 \, dx \, dt < \infty,$$

and it satisfies (1.3) in the sense of distribution.

We always denote by $u(x, t; u_0)$ the solution with initial value $u_0(x)$.

Define the $\omega$-limit set

$$\omega(u_0) = \{ w \in H^1_0(\Omega) \mid \exists t_n \to +\infty, \; u(x, t_n; u_0) \rightharpoonup w \mbox{ in } H^1_0(\Omega) \}.$$  

Let $S$ be the best constant for the Sobolev embedding $H^1_0(\Omega) \subset L^{2^*}(\Omega)$, defined as

$$S = \inf_{u \in H^1_0(\Omega), \| u \|_{2^*} = 1} \| \nabla u \|_2^2.$$

**Remark 1.1.** Let $S$ be the best constant for the Sobolev embedding of $W^{1,p}_0(\Omega) \subset L^{p^*}(\Omega)$. Then:

(a) $S$ is independent of $\Omega$ and depends only on $N$.

(b) The infimum $S$ is never achieved when $\Omega$ is a bounded domain.

The proof can be found in [11] or [16].

Denote the energy function of (1.3) by

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx.$$  

Now we can state the main results. First we have the existence and asymptotic estimates of global solutions.
Theorem 1.1. Define

$$\Sigma_1 = \left\{ u \mid u \in H_0^1(\Omega), \ u \geq 0, \ u \neq 0, \ E(u) < \frac{1}{N} S^{N/2}, \right. \left. \int_{\Omega} |u|^{2^*} \, dx < S^{N/2} \right\}. $$

If $u_0 \in \Sigma_1$, then (1.3) has a global solution $u(x, t; u_0)$. Moreover, there exists an $\alpha > 0$ such that

$$\|\nabla u(t)\|_2^2 = O(e^{-\alpha t}), \quad \text{as } t \to \infty, \quad (1.5)$$

and

$$\int_{\Omega} a(x)|u(t)|^2 \, dx = O(e^{-\alpha t}), \quad \text{as } t \to \infty. \quad (1.6)$$

Now we give two sufficient conditions of finite time blowup.

Theorem 1.2. Let $u_0(x) \neq 0$ and $E(u_0) \leq 0$. Then $u(x, t; u_0)$ blows up in finite time.

Theorem 1.3. Define

$$\Sigma_2 = \left\{ u \mid u \in H_0^1(\Omega), \ u \geq 0, \ u \neq 0, \ E(u) < \frac{1}{N} S^{N/2}, \right. \left. \int_{\Omega} |u|^{2^*} \, dx \geq S^{N/2} \right\}. $$

If $u_0 \in \Sigma_2$, then the local solution blows up in finite time.

Finally, we state the general results about the asymptotic behavior of any global solutions.

Theorem 1.4. Suppose that $a(x)$ satisfies the assumption (A) and there exists a number $\eta > 0$ such that $a(x) \geq \eta$. Let $u(x, t; u_0)$ is a global solution of (1.3), and uniformly bounded in $H_0^1(\Omega)$ with respect to $t$. Then, for any subsequence $u(x, t_n; u_0)$ ($t_n \to \infty$), there exists a solution $w$ of (1.4) such that $u(x, t_n; u_0) \rightharpoonup w$ in $H_0^1(\Omega)$.

Theorem 1.5. If $u(x, t; u_0)$ is a global solution of (1.3), then the $\omega$-limit set of $u$ contains a solution $w$ of (1.4).
The rest of this paper is organized as follows: Section 2 contains the proof of Theorem 1.1. Section 3 contains the proof of Theorems 1.2 and 1.3. Section 4 deals with the proof of Theorems 1.4 and 1.5.

2. The global existence and asymptotic behavior

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into several steps.

1. Proof of existence. From [6], [9] and [18], we have the local existence. Let $u_0 \in \Sigma$. Multiplying the equation by $u_t$ and integrating, we obtain

$$
\int_0^t \int_{\Omega} a(x)|u_s(s)|^2 \, dx \, ds + E(u(x, t)) = E(u_0) < \frac{1}{N} S^{N/2}.
$$

Thus $E(u(x, t)) < (1/N) S^{N/2}$ for any $t > 0$. Now we prove $u(x, t) \in \Sigma$ for any $t > 0$. Indeed, if there exists a $t^*$, s.t. $u(x, t) \in \partial \Sigma$, then we have $E(u(x, t)) \geq (1/N) S^{N/2}$, which is a contradiction. Hence

$$
\int_0^t \int_{\Omega} a(x)|u_s(s)|^2 \, dx \, ds + \frac{1}{N} \int_{\Omega} |\nabla u|^2 \, dx \leq E(u_0) < \frac{1}{N} S^{N/2},
$$

which implies

$$
\int_{\Omega} |\nabla u|^2 \, dx < S^{N/2}, \quad \int_0^t \int_{\Omega} a(x)|u_s(s)|^2 \, dx \, ds < \frac{1}{N} S^{N/2},
$$

for any $t > 0$. Thus $u(x, t)$ is a global solution of (1.3).

2. Proof of (1.5). Let

$$
H(u(t)) = \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} u^{2^*} \, dx.
$$

Then by the step 1, we have

$$
H(u(t)) > 0, \quad \text{for all } t \geq 0.
$$

By Sobolev inequality

$$
\int_{\Omega} |u|^{2^*} \, dx < \frac{1}{S^{2^*/2}} \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{2^*/2},
$$
and

\[ E(u_0) > \frac{1}{N} \int_{\Omega} |\nabla u|^2 \, dx \]

implies

\[ \int_{\Omega} |u|^2^* \, dx < \frac{1}{S^{2^*/2}} (NE(u_0))^{2^*/2-1} \int_{\Omega} |\nabla u|^2 \, dx. \quad (2.3) \]

For simplicity, denote \((1/S^{2^*/2})(NE(u_0))^{2^*/2-1}\) by \(0 < \delta < 1\). Let \(\gamma = 1 - \delta\). We have

\[ \int_{\Omega} |u(t)|^2^* \, dx \leq (1 - \gamma) \int_{\Omega} |\nabla u(t)|^2 \, dx. \quad (2.4) \]

Let \(T > t_0\) be a fixed number. Then from

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} a(x)|u(t)|^2 \, dx = -H(u(t)) \]

and Poincare’s inequality, we have

\[ \int_{t}^{T} H(u(s)) \, ds = \frac{1}{2} \int_{\Omega} a(x)|u(t)|^2 \, dx - \frac{1}{2} \int_{\Omega} a(x)|u(T)|^2 \, dx \]

\[ \leq \frac{1}{2} \int_{\Omega} a(x)|u(t)|^2 \, dx \leq \frac{1}{2\lambda_1} \int_{\Omega} |\nabla u(t)|^2 \, dx, \quad (2.5) \]

where \(\lambda_1\) is defined in the assumption (A). Furthermore, (2.4) implies

\[ E(u(t)) = \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 \, dx - \frac{1}{2^*} \int_{\Omega} |u(t)|^{2^*} \, dx \]

\[ = \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 \, dx + \frac{1}{2^*} \left[ H(u(t)) - \int_{\Omega} |\nabla u(t)|^2 \, dx \right] \]

\[ = \frac{1}{N} \int_{\Omega} |\nabla u(t)|^2 \, dx + \frac{1}{2^*} H(u(t)) \geq \frac{1}{N} \int_{\Omega} |\nabla u(t)|^2 \, dx, \quad (2.6) \]

on \([t_0, \infty)\). Therefore, by (2.4) and (2.5) we obtain

\[ \int_{t}^{T} H(u(s)) \, ds \leq C(\Omega) E(u(t)), \quad (2.7) \]

on \([t_0, T]\). On the other hand, (2.4) implies
\[ \gamma \int_{\Omega} |\nabla u(t)|^2 \, dx \leq H(u(t)), \quad (2.8) \]
on \left[ t_0, \infty \right). By (2.6) and (2.8), we have
\[ E(u(t)) \leq \left( \frac{1}{N\gamma} + \frac{1}{2^{*}} \right) H(u(t)). \quad (2.9) \]

Further (2.7) and (2.9) give
\[ C_1 \int_{t}^{T} E(u(s)) \, ds \leq E(u(t)), \]
on \left[ t_0, T \right], where
\[ C_1 = (C(\Omega)(1/N\gamma + 1/2^{*}))^{-1}. \]
Then, from the arbitrariness of \( T > t_0 \), we have
\[ C_1 \int_{t}^{\infty} E(u(s)) \, ds \leq E(u(t)). \]

Performing standard manipulations, by taking \( T > t_0 \) sufficiently large such that \( M \leq T \), it follows that
\[ \int_{t}^{\infty} E(u(s)) \, ds \leq T E(u(t)), \quad (2.10) \]
on \left[ t_0, \infty \right). Setting \( y(t) = \int_{t}^{\infty} E(u(s)) \, ds \), it follows from (2.10) and the monotonicity of \( E(u(t)) \) that
\[ y(t) \leq y(T)e^{1-1/T} \leq T E(u(T))e^{1-t/T} \leq T E(u(t_0))e^{1-t/T}, \quad (2.11) \]
for all \( t > T \). On the other hand, we get
\[ \int_{t}^{\infty} E(u(t)) \, ds \geq \int_{t}^{T+t} E(u(s)) \, ds \geq T E(u(T + t)). \]

By the inequality together with (2.11), we have
\[ E(u(T + t)) \leq E(u(t_0))e^{1-t/T}, \]
for all \( t > T \). By (2.10), we obtain
\[ \|\nabla u(T + t)\|_{2}^2 \leq N E(u(t_0))e^{1-t/T}, \]
which implies the exponential decay of solutions
\[ \|\nabla u(t)\|_{2} \leq C e^{-t/2T}, \]
with some constant \( C > 0 \) for large \( t > T \). Which completes the proof of (1.5).
3. Proof of (1.6). Obviously
\[ \| \nabla u(x, t; u_0) \|_2^2 < S^{N/2} \]
and
\[ \frac{d}{dt} \int_\Omega a(x)|u(t)|^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \leq \int_\Omega |u|^{2^*} \, dx, \quad \text{for all } t > 0. \quad (2.12) \]

By the same argument as in step 2, we have
\[ \frac{d}{dt} \int_\Omega a(x)|u|^2 \, dx < -(1 - \delta) \int_\Omega |\nabla u|^2 \, dx \leq -(1 - \delta) \lambda_1 \int_\Omega a(x)|u|^2 \, dx, \]
where \( \lambda_1 \) is defined in the assumption (A). We see that the estimate
\[ \int_\Omega a(x)|u|^2 \, dx = O(e^{-\alpha t}) \quad \text{as } t \to \infty \]
holds with \( \alpha = (1 - \delta) \lambda_1 / 2 \). This completes the proof of (1.6). \( \square \)

3. Finite time blowup

In this section we prove Theorems 1.2 and 1.3. First we prove Theorem 1.2.

Proof of Theorem 1.2. In fact, we can prove a more general result:

If there exists some \( t_0 \) such that \( E(u(t_0)) \leq 0 \), then \( u(x, t; u_0) \) blows up in finite time.

We shall employ the classical concavity method (see [8,10,14,15]). Suppose that \( t_{\text{max}} = \infty \) and denote \( f(t) = (1/2) \int_0^t \int_\Omega a(x)u^2 \, dx \, ds \). Performing standard manipulations:

\[ \int_0^t \int_\Omega a(x)u_t^2 \, dx \, ds + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx = E(u(t_0)), \quad (3.1) \]

\[ f'(t) = \frac{1}{2} \int_\Omega a(x)|u_0(x)|^2 \, dx + \int_0^t \int_\Omega (-|\nabla u|^2 + |u|^{2^*}) \, dx \, ds, \quad (3.2) \]

\[ f''(t) = - \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |u|^{2^*} \, dx. \quad (3.3) \]

By (3.1), (3.2) and (3.3), we have
\[ f''(t) \geq -\int_{\Omega} |\nabla u|^2 \, dx + \frac{2^*}{2} \int_{\Omega} |\nabla u|^2 \, dx - 2^* E(u(t_0)) + 2^* \int_{t_0}^{t} a(x)u_t^2 \, dx \, ds \]

\[ = \left( \frac{2^*}{2} - 1 \right) \int_{\Omega} |\nabla u|^2 \, dx - 2^* E(u(t_0)) + 2^* \int_{t_0}^{t} a(x)u_t^2 \, dx \, ds. \tag{3.4} \]

Since \( E(u(t_0)) \leq 0 \), we have

\[ \left( \frac{2^*}{2} - 1 \right) \int_{\Omega} |\nabla u|^2 \, dx - 2^* E(u(t_0)) > 0, \tag{3.5} \]

for all \( t \geq t_0 \). If we had \( t_{\max} = \infty \), then this inequality would yield

\[ \lim_{t \to \infty} f'(t) = \lim_{t \to \infty} f(t) = \infty. \]

On the other hand, (3.4) and (3.5) imply

\[ f''(t) \geq 2^* \int_{0}^{t} \int_{\Omega} a(x)u_t^2 \, dx \, ds \]

and

\[ f(t)f''(t) \geq \frac{2^*}{2} \left( \int_{0}^{t} \int_{\Omega} a(x)|u(s)|^2 \, dx \, ds \right) \left( \int_{0}^{t} \int_{\Omega} a(x)|u_t(s)|^2 \, dx \, ds \right) \]

\[ \geq \frac{2^*}{2} \left( \int_{0}^{t} \int_{\Omega} a(x)u_t u_t \, dx \, ds \right)^2 = \frac{2^*}{2} (f'(t) - f'(0))^2. \]

and as \( t \to \infty \) we have for some \( \alpha > 0 \) and \( \forall t \geq t_0 \) that

\[ f(t)f''(t) \geq (1 + \alpha)(f'(t))^2. \]

Hence \( f(t)^{-\alpha} \) is concave on \([t_0, \infty]\), \( f(t)^{-\alpha} > 0 \) and \( \lim_{t \to \infty} f(t)^{-\alpha} = 0 \). This contradiction proves that \( t_{\max} < \infty \), which completes the proof of Theorem 1.2. \( \square \)

**Proof of Theorem 1.3.** We divide the proof into two steps.

(i) First of all, we define a set that consists of the functions that satisfy the following conditions:
\[ E(u_0) < \frac{1}{N} S^{N/2}, \quad (3.6) \]
\[ \int_{\Omega} |u_0|^{2^*} \, dx = S^{N/2}. \quad (3.7) \]

We claim that the set is an empty set.

In fact, let \( u_0 \) belong to the set. If \( u_0 \) satisfies
\[ \int_{\Omega} |\nabla u_0|^2 \, dx \leq \int_{\Omega} |u_0|^{2^*} \, dx, \]
then
\[ S^{N/2} = \int_{\Omega} |u_0|^{2^*} \, dx \geq \int_{\Omega} |\nabla u_0|^2 \, dx \geq S \left( \int_{\Omega} |u_0|^{2^*} \, dx \right)^{2/2^*} = S^{N/2}, \]
and hence
\[ \int_{\Omega} |\nabla u_0|^2 \, dx = \int_{\Omega} |u_0|^{2^*} \, dx = S^{N/2}, \]
\[ E(u_0) = \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 \, dx - \frac{1}{2^*} \int_{\Omega} |u_0|^{2^*} \, dx = \frac{1}{N} S^{N/2}, \]
which is contradictory to (3.6).

If \( u_0 \) satisfies
\[ \int_{\Omega} |\nabla u_0|^2 \, dx > \int_{\Omega} |u_0|^{2^*} \, dx, \]
then from (3.6) we see that
\[ \frac{1}{N} S^{N/2} > E(u_0) = \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 \, dx - \frac{1}{2^*} \int_{\Omega} |u_0|^{2^*} \, dx > \frac{1}{N} \int_{\Omega} |u_0|^{2^*} \, dx \]
implies
\[ \int_{\Omega} |u_0|^{2^*} \, dx < S^{N/2}, \]
which is a contradiction because of (3.7). Therefore, that set is an empty set.

(ii) Thus we consider only the following case:
\[ E(u) < \frac{1}{N} S^{N/2}, \quad \int_{\Omega} |u|^{2^*} \, dx > S^{N/2}. \quad (3.8) \]
Obviously, in this case we have

\[ S^{N/2} < \int_{\Omega} |\nabla u_0|^2 \, dx < \int_{\Omega} |u_0|^{2^*} \, dx. \]

If \( u(x, t) \) is a global solution, then we can deduce that \( u(x, t) \) does not converge strongly to 0 in \( H_0^1(\Omega) \). Otherwise, \( \exists t^*, 0 < t^* < \infty \), such that

\[ E(u(t^*)) < \frac{1}{N} S^{N/2}, \quad \int_{\Omega} |u(t^*)|^{2^*} \, dx = S^{N/2}, \]

which is a contradiction from the step (i). To complete the proof of Theorem 1.3, we first prove the following claim:

**Claim.** If \( u_0 \) satisfies (3.8) and \( u(x, t; u_0) \) is a global solution. Then \( \forall t \in [0, \infty] \) the following inequalities hold:

\[ S^{N/2} < \int_{\Omega} |\nabla u(x, t)|^2 \, dx < \int_{\Omega} |u(x, t)|^{2^*} \, dx. \]

(3.9)

Indeed, if there exists a \( t^* \) such that \( \int_{\Omega} |\nabla u(x, t^*)|^2 \, dx = \int_{\Omega} |u(x, t^*)|^{2^*} \, dx \), then we have

\[ \int_{\Omega} |\nabla u(x, t^*)|^2 \, dx = \int_{\Omega} |u(x, t^*)|^{2^*} \, dx \geq S^{N/2}. \]

But

\[ \frac{1}{N} S^{N/2} > E(u(x, t^*)) = \frac{1}{N} \int_{\Omega} |\nabla u(x, t^*)|^2 \, dx, \]

a contradiction. Therefore there exists a constant \( \eta > 0 \) sufficiently small and independent of \( t \), relying on \( u_0 \), such that

\[ \int_{\Omega} |u(x, t)|^{2^*} \, dx \geq (1 + \eta) \int_{\Omega} |\nabla u(x, t)|^2 \, dx, \]

(3.10)

for any \( t \in [0, \infty] \), which completes the proof of the claim.

Now we can complete the proof of Theorem 1.3. We shall employ the same argument as in the proof of Theorem 1.2. Suppose that \( t_{\text{max}} = \infty \) and denote \( f(t) = (1/2) \int_0^t \int_{\Omega} a(x)|u|^2 \, dx \, ds \). We have the equalities (3.1), (3.2) and (3.3); by (3.10) we obtain

\[ f''(t) \geq -\int_{\Omega} |\nabla u|^2 \, dx + (1 + \eta) \int_{\Omega} |\nabla u(x, t)|^2 \, dx = \eta \int_{\Omega} |\nabla u(x, t)|^2 \, dx. \]
If we have $t_{\text{max}} = \infty$, then this inequality would yield
\[
\lim_{t \to \infty} f'(t) = \lim_{t \to \infty} f(t) = \infty.
\]
By (3.2) and (3.3) we have
\[
f''(t) \geq - \int_{\Omega} |\nabla u|^2 \, dx + \frac{2^*}{2} \int_{\Omega} |\nabla u|^2 \, dx - 2^* J(u_0) \\
+ 2^* \int_{0}^{t} \int_{\Omega} a(x) u_{t}^2 \, dx \, ds
\]
\[
= \left( \frac{2^*}{2} - 1 \right) \int_{\Omega} |\nabla u|^2 \, dx - 2^* J(u_0) + 2^* \int_{0}^{t} \int_{\Omega} a(x) u_{t}^2 \, dx \, ds.
\]
Now we using (3.8), we have that
\[
\left( \frac{2^*}{2} - 1 \right) \int_{\Omega} |\nabla u|^2 \, dx - 2^* J(u_0) \geq 0
\]
implies
\[
f''(t) \geq 2^* \int_{0}^{t} \int_{\Omega} a(x) u_{t}^2 \, dx \, ds
\]
and
\[
f(t) f''(t) \geq \frac{2^*}{2} \left( f'(t) - f'(0) \right)^2.
\]
By the argument in the proof of Theorem 1.2, we obtain a contradiction, which completes the proof of Theorem 1.3. \qed

4. The general case

First of all, we prove Theorem 1.4.

Proof of Theorem 1.4. For any $t_n \to \infty$, let $u_n = u(x, t_n; u_0)$. From the uniform boundedness of $u_n$ in $H^1_0(\Omega)$, we know that there exists a subsequence that we still denote $\{u_n\}$ and a function $w$ such that
\[
u_n \rightharpoonup w \quad \text{in} \quad H^1_0(\Omega),
\]
\[
u_n \rightharpoonup w \quad \text{in} \quad \left( L^{2^*}(\Omega) \right)^{\ast}.
\]
In order to pass to the limit in (1.3), we first introduce suitable test functions similar to Fila [2]. Take

$$\psi \in H^1_0(\Omega), \quad \rho \in C^2_0(0, T), \quad \rho \geq 0, \quad \int_0^T \rho(s) \, ds = 1.$$ 

Put

$$\varphi(x, t) = \begin{cases} \rho(t - t_n)\psi(x) & \text{for } t > t_n, \ x \in \overline{\Omega}, \\ 0 & \text{for } 0 \leq t \leq t_n, \ x \in \overline{\Omega}. \end{cases}$$

From the Definition 1.1 we have

$$T \int_0^T \left[ a(x)u(t_n + s)\rho'(s)\psi - \rho \nabla u(t_n + s) \nabla \psi + u(t_n + s)^{2s-1}\rho(s)\psi \right] \, dx \, ds = 0.$$ 

The transformation $s = t - t_n$ leads to

$$T \int_0^T \left[ a(x)u(t_n + s)\rho'(s)\psi - \rho \nabla u(t_n + s) \nabla \psi + u(t_n + s)^{2s-1}\rho(s)\psi \right] \, dx \, ds = 0. \quad (4.1)$$

Note the uniform boundedness of $u(t_n + s)$ in $H^1_0(\Omega)$ for $0 \leq s \leq T$. Therefore, we can choose the same subsequence of $\{t_n\}$ (not relabeled) and functions $w_s$ and $w$ such that

$$u(t_n + s) \to w_s, \quad \text{strongly in } L^2(\Omega),$$

and

$$u(t_n) \to w, \quad \text{strongly in } L^2(\Omega).$$

Now we claim: $w_s = w$. Indeed, by the energy inequality

$$\int_0^\infty \int_{\Omega} a(x)|u_t|^2 \, dx \, dt + E(u) = E(u_0(x)) < \infty,$$

we have

$$\int_{\Omega} a(x)|u(t_n + s) - u(t_n)|^2 \, dx = s \int_{t_n}^{t_n+s} \int_{\Omega} a(x) \left| \frac{\partial u}{\partial \tau} \right|^2 \, dx \, d\tau \to 0,$$

as $t_n \to \infty$,

for $0 \leq s \leq T$ for any fixed $T < \infty$. Thus

$$u(t_n + s) - u(t_n) \to 0, \quad \text{strongly in } L^2(\Omega, a(x)), \quad \text{as } t_n \to \infty.$$
By the assumption of Theorem 1.4, we have
\[ u(t_n + s) - u(t_n) \to 0, \quad \text{strongly in } L^2(\Omega), \text{ as } t_n \to \infty, \]
for \(0 \leq s \leq T\) for any fixed \(T < \infty\). Hence
\[ w_s = w, \]
which proves the claim.

Now we rewrite (4.1) as follows:
\[
\int_0^T \int_\Omega \left[ a(x)u(t_n)\rho'(s)\psi - \rho \nabla u(t_n) \nabla \psi + u(t_n)^{2^* - 1} \rho(s)\psi \right] dx \, ds \\
+ \int_0^T \int_\Omega a(x) \left[ u(t_n + s) - u(t_n) \right] \rho'(s) \psi \, dx \, ds \\
- \int_0^T \int_\Omega \left[ \nabla u(t_n + s) - \nabla u(t_n) \right] \nabla \psi \, dx \, ds \\
+ \int_0^T \int_\Omega \left[ u(t_n + s)^{2^* - 1} - u(t_n)^{2^* - 1} \right] \rho(s)\psi \, dx \, ds = 0. \tag{4.2}
\]

By the dominated convergence theorem and the choice of \(\rho\), and since \(u(t_n) \to w\) strongly in \(L^2(\Omega)\), we have
\[
\int_0^T \rho \left[ \int_\Omega \nabla u(t_n) \nabla \psi \, dx - \int_\Omega u(t_n)^{2^* - 1} \psi \, dx \right] \, ds = o(1), \quad \text{as } n \to \infty.
\]

Denote \(u(t_n)\) by \(u_n\). From the choice of \(\rho\), we obtain
\[
\int_\Omega \nabla u(t_n) \nabla \psi \, dx - \int_\Omega u(t_n)^{2^* - 1} \psi \, dx = o(1), \quad \text{as } n \to \infty.
\]

This completes the proof of Theorem 1.4. \(\square\)

**Proof of Theorem 1.5.** From now on, denote \((x, t; u_0)\) by \(u\). We have
\[
\int_0^\infty \int_\Omega a(x)u_t^2 \, dx \, ds \leq C < \infty.
\]

Then there exists a sequence \(\{t_n\}\) satisfying \(t_n \to \infty\) as \(n \to \infty\) such that
\[
\int_\Omega a(x)|u_t(x, t_n; u_0)|^2 \, dx \to 0, \quad \text{as } n \to \infty. \tag{4.3}
\]
For the sake of convenience, denote \( u(x, t_n; u_0) \) by \( u_n \). From Theorem 1.1, \( E(u(t)) > 0 \) for all \( t \geq 0 \), and
\[
0 < E(u(t)) \leq E(u_0). \tag{4.4}
\]
Consider the time sequence \( \{t_n\} \) such that
\[
0 < E(u(t_n)) \leq E(u_0). \tag{4.5}
\]
By (4.3) and assumption (A)
\[
\int_{\Omega} |a(x)|^2 |u_t(x, t_n; u_0)|^2 \, dx \leq \|a(x)\|_{L^\infty} \int_{\Omega} a(x)|u_t(x, t_n; u_0)|^2 \, dx \to 0.
\]
Thus, we have
\[
a(x)u_t(x, t_n; u_0) \to 0, \quad \text{in } L^2(\Omega), \quad \text{as } t_n \to \infty. \tag{4.6}
\]
The statement of (4.3) and (4.6) says that \( u_n = u(t_n), \ t_n \to \infty, \) is a Palais–Smale sequence related to the stationary problem of (1.3). Such a situation has been well studied in the theory of nonlinear elliptic equations. It is easy to prove that there exists a constant \( C < +\infty \) such that
\[
\int_{\Omega} |\nabla u_n|^2 \, dx \leq C.
\]
Thus, there exists a subsequence (not relabeled) and a function \( w \) such that
\[
\begin{align*}
  u_n &\to w, \quad \text{weakly in } H^1_0(\Omega), \\
  u_n &\to w, \quad \text{strongly in } L^q(\Omega) \quad (2 \leq q < 2^*).
\end{align*}
\]
From the theory of elliptic equations we can obtain that \( w \) is a solution of (1.4), which completes then proof of Theorem 1.5. \( \square \)

References