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Best Approximation in the Space of Continuous Vector-Valued Functions

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Using a well-known characterization theorem for best approximations, direct proofs are given of some (generalizations of) recent results of Tanimoto who deduced them from a general minimax theorem that he first established. © 1988 Academic Press, Inc.

1. INTRODUCTION

In a recent paper in this *Journal*, Tanimoto [6] has given some characterization theorems for best approximation by finite dimensional subspaces in the space of continuous vector-valued functions. He deduced these results from an abstract "minimax theorem" which he first proved and which generalized a similar one of Fan [3].

The purpose of this note is to point out that these approximation theorems (and even more general ones) can be deduced *directly* from a well-known characterization theorem for best approximations in any normed linear space. Moreover, this shows—at least in the present setting—that the "optimization theory" approach to approximation theory yields no surprises.

2. THE CHARACTERIZATION THEOREM

Let K be a subset of the (real) normed linear space X, $x \in X$, and $y_0 \in K$. Then y_0 is called a *best approximation* to x from K if

$$\|x - y_0\| = \inf\{\|x - y\| \mid y \in K\}.$$

The set of all extreme points in the unit ball of the dual space X^* of X will be denoted by ext $B(X^*)$.

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The following characterization theorem was established in 1967, independently, by Deutsch and Maserick [2] and Havinson [4].

THEOREM A. *Let K be a convex subset of an n-dimensional linear subspace of the normed linear space X, $x \in X \backslash K$, and $y_0 \in K$. Then y_0 is a best approximation to x from K if and only if there exist $m \leqslant n+1$ linear functionals $x_i^* \in \operatorname{ext} B(X^*)$* and m scalars $\lambda_i > 0$ with $\sum_1^m \lambda_i = 1$ *such that*

(i) $\sum_1^m \lambda_i x_i^*(y - y_0) \leqslant 0$ *for all* $y \in K$

(ii) $x_i^*(x - y_0) = \|x - y_0\|$ *for* $i = 1, 2, ..., m$.

If, in addition, K is a linear subspace, then condition (i) *may be replaced by*

(i′) $\sum_1^m \lambda_i x_i^*(y) = 0$ *for all* $y \in K$.

This characterization theorem, in the particular case when K is an n dimensional linear subspace, was given earlier by Singer [5].

3. Applications in the Space of Vector-Valued Continuous Functions

Throughout this section, the setting will be as follows. Let T be a locally compact Hausdorff space, $(Y, \|\cdot\|_1)$ a (real) normed linear space, and let $X = C_0(T, Y)$ denote the Banach space of all continuous functions $x: T \to Y$ which "vanish at infinity," i.e., the set $\{t \in T \mid \|x(t)\|_1 \geqslant \varepsilon\}$ is compact for every $\varepsilon > 0$. X is endowed with the norm

$$\|x\| = \sup_{t \in T} \|x(t)\|_1.$$

When T is actually compact, then every continuous function $x: T \to Y$ automatically vanishes at infinity and $C_0(T, Y)$ is often denoted $C(T, Y)$. Further, if $Y = \mathbb{R}$ (the set of all real numbers), then $C_0(T, Y)$ (resp. $C(T, Y)$) is usually denoted by $C_0(T)$ (resp. $C(T)$).

It is known [1, Lemma 3.3] that $x^* \in \operatorname{ext} B(X^*)$ if and only if $x^* = y^* \circ \delta_t$, where $y^* \in \operatorname{ext} B(Y^*)$ and δ_t denotes "point evaluation" at t. That is,

$$x^*(x) = y^*(x(t)) \qquad \text{for every } x \in X.$$

Finally, let K be a convex subset of an n-dimensional linear subspace of $X = C_0(T, Y)$. Then we immediately obtain from Theorem A the following characterization theorem.

THEOREM 1. *Let $x \in X \backslash K$ and $y_0 \in K$. Then y_0 is a best approximation to x from K if and only if there exist $m \leqslant n+1$ linear functionals $y_i^* \in \operatorname{ext} B(Y^*)$, m points $t_i \in T$, and m scalars $\lambda_i > 0$ with $\sum_1^m \lambda_i = 1$ such that*

(i) $\sum_1^m \lambda_i y_1^* [y(t_i) - y_0(t_i)] \leqslant 0$ *for all* $y \in K$

(ii) $y_i^* [x(t_i) - y_0(t_i)] = \|x - y_0\|$ *for* $i = 1, 2, ..., m$.

If, in addition, K is a linear subspace, then condition (i) *may be replaced by*

(i′) $\sum_1^m \lambda_i y_i^* [y(t_i)] = 0$ *for all* $y \in K$.

COROLLARY 1. *Let $x \in X \backslash K$ and $y_0 \in K$. Then y_0 is a best approximation to x from K if and only if there exist $m \leqslant n+1$ points $t_i \in T$ and m scalars $\lambda_i > 0$ with $\sum_1^m \lambda_t = 1$ such that*

(i) $\sum_1^m \lambda_i \|x(t_i) - y_0(t_i)\|_1 \leqslant \sum_1^m \lambda_i \|x(t_i) - y(t_i)\|_1$ *for all* $y \in K$

(ii) $\|x(t_i) - y_0(t_i)\|_1 = \|x - y_0\|$ *for* $i = 1, 2, ..., m$.

Proof. Suppose y_0 is a best approximation to x. By Theorem 1 there exist $m \leqslant n+1$ functionals $y_i^* \in \operatorname{ext} B(Y^*)$, m points $t_i \in T$, and m scalars $\lambda_i > 0$ with $\sum_1^m \lambda_i = 1$ such that

$$\sum_1^m \lambda_i y_i^* [y(t_i) - y_0(t_i)] \leqslant 0 \qquad \text{for all} \quad y \in K \tag{1.1}$$

and

$$y_i^* [x(t_i) - y_0(t_i)] = \|x - y_0\| \qquad \text{for} \quad i = 1, 2, ..., m. \tag{1.2}$$

From (1.2) we obtain

$$\|x - y_0\| \leqslant \|y_i^*\| \; \|x(t_i) - y_0(t_i)\|_1 = \|x(t_i) - y_0(t_i)\|_1 \leqslant \|x - y_0\|$$

which implies (ii). Using (1.1) and (1.2), we obtain that for each $y \in K$

$$\begin{aligned} \sum_1^m \lambda_i \|x(t_i) - y_0(t_i)\|_1 &= \sum_1^m \lambda_i y_i^* [x(t_i) - y_0(t_i)] \\ &\leqslant \sum_1^m \lambda_i y_i^* [x(t_i) - y(t_i)] \leqslant \sum_1^m \lambda_i \|x(t_i) - y(t_i)\|_1. \end{aligned}$$

This proves (i).

Conversely, suppose (i) and (ii) hold. Then for each $y \in K$,

$$\|x-y_0\| = \sum_1^m \lambda_i \|x-y_0\| = \sum_1^m \lambda_i \|x(t_i)-y_0(t_i)\|_1$$
$$\leqslant \sum_1^m \lambda_i \|x(t_i)-y(t_i)\|_1 \leqslant \sum_1^m \lambda_i \|x-y\| = \|x-y\|.$$

Thus y_0 is a best approximation to x. ∎

In the special case when K is an n-dimensional linear subspace and T is compact, Corollary 1 reduces to Theorem 4.1 of Tanimoto [6].

If Y is an inner product space, then condition (ii) of Theorem 1 is equivalent to the statement that y_i^* has the "representer" $[x(t_i)-y_0(t_i)]\,\|x-y_0\|^{-1}$, i.e.,

$$y_i^*(y) = \left\langle y, \frac{x(t_i)-y_0(t_i)}{\|x-y_0\|} \right\rangle \qquad \text{for all} \quad y \in Y.$$

Substituting this into (i) of Theorem 1 yields

COROLLARY 2. *Let Y be an inner product space, $x \in X \backslash K$, and $y_0 \in K$. Then y_0 is a best approximation to x if and only if there exist $m \leqslant n+1$ points $t_i \in T$ and m scalars $\lambda_i > 0$ with $\sum_1^m \lambda_i = 1$ such that*

(i) $\sum_1^m \lambda_i \langle y(t_i)-y_0(t_i), x(t_i)-y_0(t_i)\rangle \leqslant 0$ *for all* $y \in K$,

(ii) $\|x(t_i)-y_0(t_i)\|_1 = \|x-y_0\|$ *for* $i = 1, 2, ..., m$.

If, in addition, K is a linear subspace, then condition (i) *may be replaced by*

(i′) $\sum_1^m \lambda_i \langle y(t_i), x(t_i)-y_0(t_i)\rangle = 0$ *for all* $y \in K$.

In the special case when T is compact and K is an n-dimensional linear subspace, Corollary 2 reduces to Corollary 4.1 of Tanimoto [6].

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