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On Filling Space with Different Integer Cubes

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In this paper it is shown that 3-space cannot be filled with cubes such that neighbouring cubes have different edge lengths; the result is extended to the 3-torus.

INTRODUCTION

In this paper, we consider the question posed by Daykin [2, 5] in 1964: "Can space be filled with disjoint integer cubes, no two cubes being the same size, and the lengths of the edges of the cubes being integers?" The two-dimensional version of this question has been answered in the affirmative [1, 2]: in fact, it is known that there exist $2^{\aleph_0}$ essentially different dissections of the plane into disjoint integer squares with different edge lengths [3]. The methods which provide solutions to the two-dimensional problem, however, have not proved useful in the three-dimensional case; and we present here a proof that no such dissection of three-dimensional Euclidean space is possible, even with certain relaxations of conditions. This will also be shown to extend to higher-dimensional spaces.

We will also examine the related problem of filling the 3-torus with cubes with different integer edges, which was raised by Brooks, Smith, Stone, and Tutte [1]. It will be shown that this is possible, but only under rather special conditions.

Attempts were made to "arithmetize" this problem, and write all steps in the proofs as numerical equations and inequalities; however, it was determined that this was impractical and unclear. The proofs have therefore been presented geometrically, with frequent diagrams.

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1. Abbreviations and Terminology

In this paper, capital letters will be used to refer to *cubes*; and $l(X)$ will represent the edge length of cube X.

Faces of cubes will denoted by the subscripts u, d, l, r, b, f; thus A_{u}, A_{d}, A_{l}, A_{r}, A_{b}, A_{f} are the upper, downmost, left, right, back, and front faces of cube A, respectively.

Edges and *vertices* will be represented by double and triple subscripts, respectively; thus, A_{rf} is the front right edge of cube A, while A_{drf} is the bottom front right vertex of the same cube.

Corners of faces will be denoted by a subscript representing the face, followed by parenthesized subscripts representing the corner; thus, $A_{\mathrm{r(df)}}$ is the bottom front corner of the right-hand face of cube A.

The *neighbours* of X are those cubes whose faces (as opposed to edges or vertices alone) touch those of X. A cube is *covered* if its surface is a subset of the union of the surfaces of its neighbours. The set of neighbours of a cube X is called its *star*, and is written $\mathrm{St}(X)$.

A collection of cubes is *heterogeneous* if no two cubes are the same size, *locally heterogeneous* if no neighbouring cubes are the same size, and *locally finite* if $\mathrm{St}(X)$ is finite for all X. In a locally finite collection of cubes, we associate with every cube X a cube $X' \in \mathrm{St}(X)$ such that

$$l(X') = \min_{A \in \mathrm{St}(X)} l(A).$$

If, for some cube X,

$$l(X) < l(X'),$$

we write that $\mathrm{St}(X)$ is *singular* and that X is the *core* of a *singular star*.

2. Uniqueness of the Singular Star Configuration

The following result has been known for some time and can perhaps be said to be in the folklore of this problem. For the sake of completeness, the proof is outlined here.

Proposition 1. *Except for the relative sizes of the cubes, all singular stars are identical up to rotation and reflection.*

Proof. A cube X that forms the core of a singular star cannot have the neighbours (or neighbour) that touch(es) one face extending on opposite sides of the cube. Suppose, for example, that a neighbour touching X_{d}

extends to the left and right of the cube beyond the edges X_{dl} and X_{dr}. Then the cubes touching X_l and X_r at the points where the lower neighbour extends past X cannot extend below the plane of X_d. Being larger than X, they must extend beyond X_{ul} and X_{ur}. This, however, leaves a gap on top of X that cannot be filled by a cube larger than X, contradicting our assumption that X is the core of a singular star.

Therefore, each neighbour must touch the core in the manner of Fig. 1a; for if two or more cubes, each larger than X, cover the same face of X, they must extend beyond at least one pair of opposite sides. Starting with cube F, as in Fig. 1a, another cube must be positioned either as B is in Fig. 1b, or in a mirror-image position touching F_u and X_b. After the first two cubes have been positioned, each subsequent step in the construction of $\mathrm{St}(X)$ follows necessarily, in order to complete the packing at points where seven octants have already been filled. In this way, after F and B, cubes C, A, D and E take the positions around X in which the are shown in Fig. 1b, in that order. ∎

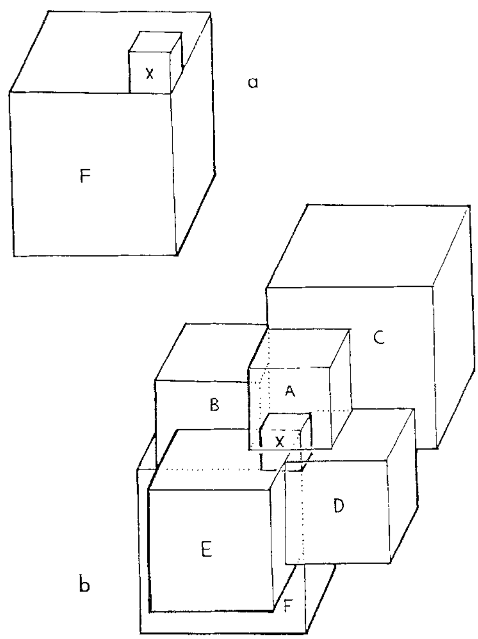

FIGURE 1

3. Properties of the Singular Star Configuration

Proposition 2. *In a locally heterogeneous collection of cubes, let A be a minimal element of a singular star with core X, oriented as in Fig.* 1b, *and let A be covered. Then there exists a cube $K(A) \in (\mathrm{St}(A) \setminus \{X\})$ such that*

$$l(K(A)) < l(A)$$

and such that one of the following holds:

(i) *$K(A)$ touches A at $A_{\mathrm{l(df)}}$, $(K(A))_{\mathrm{d}}$ is flush with A_{d}, and $(K(A))_{\mathrm{f}}$ is flush with A_{f}; or,*

(ii) *$K(A)$ touches A at $A_{\mathrm{f(dl)}}$, $(K(A))_{\mathrm{d}}$ is flush with A_{d}, and $(K(A))_{\mathrm{l}}$ is flush with A_{l}; or,*

(iii) *$K(A)$ touches A at $A_{\mathrm{r(df)}}$, $(K(A))_{\mathrm{d}}$ is flush with A_{d}, and $(K(A))_{\mathrm{f}}$ is flush with A_{f}.*

Proof. Let G be the cube that touches A at $A_{\mathrm{l(df)}}$. G_{d} is flush with A_{d}. If G_{f} is also flush with A_{f},

$$l(G) = l(A) - l(X) < l(A)$$

and we take $K(A) = G$ (Fig. 2).

If G_{f} is not flush with A_{f}, then let H be the cube that touches A at $A_{\mathrm{f(dl)}}$. H_{l} is flush with A_{l}, and H_{d} is flush with A_{d}. Either H extends to the right of A_{r}, or H is smaller than A in which latter case we take $K(A) = H$ (Fig. 3).

If H extends to the right of A_{r}, the cube that touches A at $A_{\mathrm{r(df)}}$ must be smaller than A, and have its front and bottom faces flush with those of A; we then take this to be $K(A)$ (Fig. 4). ∎

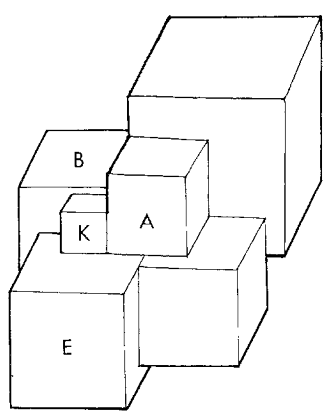

Figure 2

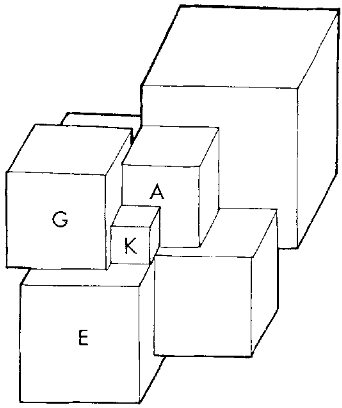

FIGURE 3

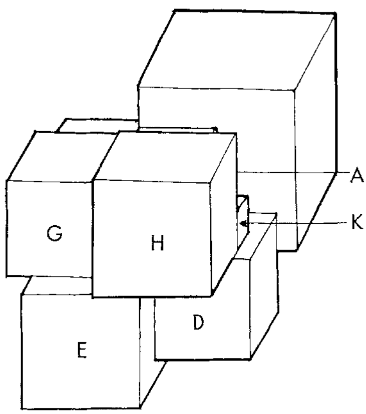

FIGURE 4

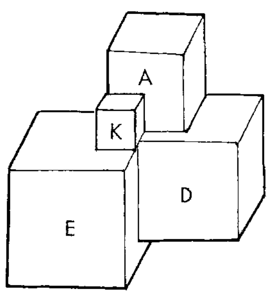

FIGURE 5

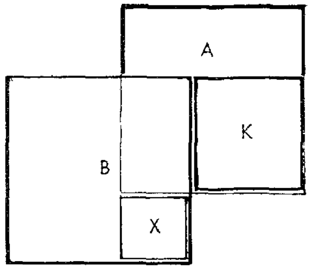

FIGURE 6

PROPOSITION 3. *In a locally heterogeneous collection of cubes, let A be a minimal element of a singular star, and let A be covered. Then $K(A)$, as defined in Proposition 2, is not the core of a singular star.*

Proof. If $K(A)$ touches A at $A_{\mathrm{f(dl)}}$ or at $A_{\mathrm{r(df)}}$, D_{u} and E_u extend to the left and right of $(K(A))_{\mathrm{d}}$, preventing a singular star configuration (Fig. 5).

If $K(A)$ touches A at $A_{\mathrm{l(df)}}$, and $(K(A))_{\mathrm{b}}$ does not touch B_{f}, then E_{u} extends to the front and back of $(\mathrm{K}(A))_{\mathrm{d}}$, preventing a singular star configuration.

If $K(A)$ touches A at $A_{\mathrm{l(df)}}$, and $(K(A))_{\mathrm{b}}$ touches B_{f}, then the singular star configuration requires that B_{u} and $(K(A))_{\mathrm{u}}$ be flush. But (Fig. 6) this would imply

$$\begin{aligned} l(B) &= l(X) + l(K(A)) \\ &= l(A), \end{aligned}$$

contradicting the assumption that the collection of cubes is locally heterogeneous. ∎

4. NONEXISTENCE OF A PACKING

We are now ready to derive the theorem that will yield the main result of this paper.

THEOREM 1. *Any locally finite and locally heterogeneous collection of cubes, each of which is covered, must contain an infinite sequence of cubes with strictly decreasing edge lengths.*

Proof. We define a sequence of cubes A_i as follows. A_0 is chosen arbitrarily from those cubes which are not cores of singular stars, and succeeding A_i are chosen inductively according to the following rules:

Case 1. If A_i' is not the core of a singular star, $A_{i+1} = A_i'$.

Case 2. If A_i' is the core of a singular star, but $l(A_i')' < l(A_i)$, then $A_{i+1} = (A_i')'$.

Case 3. If A_i' is the core of a singular star, and $l(A_i) = l(A_i')'$, then $A_{i+1} = K(A_i)$.

These rules ensure that no A_i is the core of a singular star—in the first two cases, explicitly, and in the third case, as a result of Proposition 3. Therefore, in case 1, $l(A_{i+1}) < l(A_i)$.

In case 2, A_{i+1} is explicitly smaller than A_i; and in Case 3, Proposition 2 makes the same inequality hold. Thus, the sequence $\{l(A_i)\}$ is strictly decreasing, and is defined by induction for all non-negative integers i. ∎

This theorem allows us to prove various results, including a slightly stronger form of the negative answer to Daykin's question.

THEOREM 2. *There does not exist any locally heterogeneous collection of cubes with integer edge lengths that fills space.*

Proof. Any collection of cubes with integer edge lengths must be locally finite. Therefore, if such a collection existed, Theorem 1 would show the existence of an infinite strictly decreasing sequence of integers. ∎

COROLLARY (Daykin's question). *There does not exist any heterogeneous collection of cubes with integer edge lengths that fills space.*

Observation. Application of the algorithm of Theorem 1 to a heterogeneous collection of cubes with integer edges produces a decreasing sequence of cubes that can only terminate at a cube that is not covered. Therefore, given any element A_0 of such a collection that is not the core of a singular star, the distance from the center of A_0 to the complement of the union of the collection is less than

$$\frac{\sqrt{3}}{2}(l(A_0))^2.$$

To prove this, we note that the cubes A_i generated by the algorithm and those A_i' which are cores of singular stars contain a connected chain of cubes, all of edge length less than A_0, joining A_0 to a cube that is not covered. The total diameter of this chain must be

$$D\left(\bigcup_{i>0} A_i\right) < \sum_{j=1}^{l(A_0)-1} \sqrt{3}\,j) = \sqrt{3}\,\frac{l(A_0)(l(A_0)-1)}{2}.$$

Adding to this the half-diagonal of A_0, we get the desired bound.

5. Other Spaces

Some results applicable to other spaces follow from the theorems of the previous section. We can, for example, immediately generalize the theorem to higher-dimensional Euclidean spaces in the following.

Corollary (to Theorem 2). *For $n \geqslant 3$, there does not exist any locally heterogeneous collection of hypercubes that fills Euclidean n-space.*

Proof. Suppose this to be done. Then any three-dimensional orthogonal cross section through the space would yield a counterexample to Theorem 2. ∎

We can now also answer the question raised by Brooks, Smith, Stone, and Tutte in their 1940 paper [1], as follows:

The 3-torus can be filled by a heterogeneous collection of cubes with integer edges, but not by a locally heterogeneous collection of cubes with integer edges. This may at first seem paradoxical; the explanation is that a cube in a heterogeneous collection may have contact with itself, while a cube in a locally heterogeneous collection may not.

A trivial example of a 3-torus filled with a heterogeneous collection of cubes with integer edges is the torus obtained by identifying opposite faces of the $n \times n \times n$ cube, which may be considered to be filled by the same cube. A non-trivial example may be derived from the space filling (due to Rogers [4]) with cubes of two sizes shown in Fig. 7, by identifying points whose positions differ by an integral linear combination of the three period vectors.

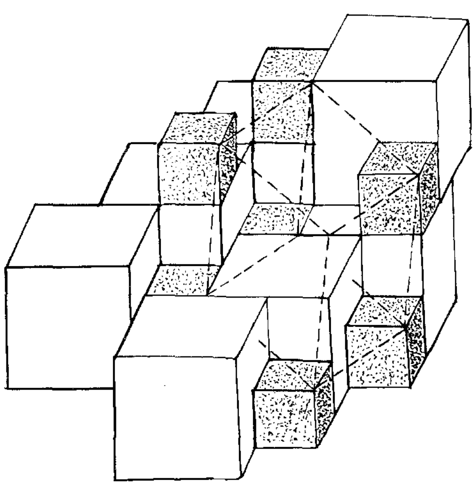

Figure 7

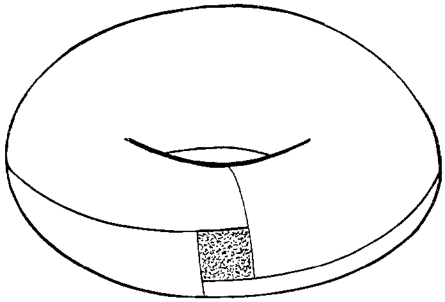

FIGURE 8

This yields a torus packed with two cubes, in a way analogous to that shown in Fig. 8 for two dimensions. (It should be noted that the edges are not parallel to the principal coordinates of the 3-torus.)

Conversely, a locally heterogeneous packing of a 3-torus may be repeated periodically to yield a locally heterogeneous packing of space. Therefore, by Theorem 2, no locally heterogeneous packing of a 3-torus with cubes of integer edge is possible. Similar results may be obtained for other spaces that can be defined as quotient spaces of $\mathbb{R}^3$.

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