Topology and its Applications 28 (1988) 177-179
North-Holland

A TOPOLOGY ON THE UNION OF THE DOUBLE ARROW SPACE AND THE INTEGERS

Stephen WATSON*
Department of Mathematics, York University, North York, Ontario, Canada M3J 1P3

William WEISS*
Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 1A1

Received 1 July 1986
Revised 15 September 1986

We construct a topology on the union of the double arrow space (Cantor set version) and the integers which is a hereditarily Lindelöf hereditarily separable 0-dimensional compact Hausdorff space but not the continuous image of a closed subspace of the product of the double arrow space and the closed unit interval (answering a question of Fremlin).

AMS (MOS) Subj. Class.: Primary 54G20, 54D20; secondary 54F05, 54D30, 54C05

double arrow space (two arrow space)
hereditarily Lindelöf hereditarily separable
0-dimensional compact

1. Introduction

This paper originates in the observation that there do not seem to be many examples of compact perfectly normal nonmetrizable spaces in ZFC. The fact that compact perfectly normal nonmetrizable spaces are hereditarily Lindelöf and are consistently hereditarily separable seemingly reduces the possibilities to the double arrow space of Alexandroff and some minor variations. In particular some pairs of points in the double arrow space can be identified and so long as there remain uncountably many unidentified pairs, the resulting space remains a compact perfectly normal space which is not metrizable. Another variation is to multiply the double arrow space by the closed unit interval. It seemed to David Fremlin that these might exhaust the possibilities and so Fremlin [2] offered 2 pounds sterling for a solution to the problem: Is there, in ZFC, a compact hereditarily Lindelöf space which is not the continuous image of a closed subspace of the product of the closed unit

* This work has been supported by the Natural Sciences and Engineering Research Council of Canada.

interval and the double arrow space? In this paper, we construct such a space, as the union of the (Cantor set variant of the) double arrow space and the integers:

2. The construction

Let K be the Cantor middle thirds set. Let S be the set of all midpoints of those removed middle third intervals. If $s \in S$, let $|s| \in \omega$ be such that s is the midpoint of a removed interval of length $3^{-|s|}$. For each $p \in K$, there is a canonical sequence $\vec{p} = \{p(n): n > 0\}$ of elements of S which converges to p: define $p(n)$ to be the closest element to p such that $|p(n)| = n$. $p(n)$ is well-defined because if x is a point midway between two elements $s_1, s_2 \in S$ where $|s_1| = |s_2| = n$ then $x \in S$ and $|x| < n$. Let FLIP be a function from K into $\mathscr{P}(\omega)$. We define a topological space $X(\text{FLIP})$ on the set $(K \times \{l, r\}) \cup S$. For each $p \in K$ and $t, u \in S$ where $t < p < u$, let $l(t, p, u) = \{(p, l)\} \cup (\{q \in K: t < q < p\} \times \{l, r\}) \cup \{s \in S: t < s < p \text{ and } s \notin \vec{p}\} \cup \{p(n): n \in \omega \text{ and } t < p(n) < u \text{ and } (p(n) > p \wedge n \in \text{FLIP}(p)) \vee (p(n) < p \wedge n \notin \text{FLIP}(p))\}$ and let $r(t, p, u) = \{(p, r)\} \cup (\{q \in K: p < q < u\} \times \{l, r\}) \cup \{s \in S: p < s < u \text{ and } s \notin \vec{p}\} \cup \{p(n): n \in \omega \text{ and } t < p(n) < u \text{ and } (p(n) < p \wedge n \in \text{FLIP}(p)) \vee (p(n) > p \wedge n \notin \text{FLIP}(p))\}$.

Declare these sets $l(t, p, u)$ and $r(t, p, u)$ to be open and declare the points of S to be isolated. Now some elementary facts about these sets, no matter what the function FLIP turns out to be.

$$l(t, p, u) \subset l(t', p, u') \quad \text{and} \quad r(t, p, u) \subset r(t', p, u') \quad \text{whenever } t' \leqslant t < p < u \leqslant u'; \tag{1}$$

$$l(t, p, u) \cap r(t', p, u') = \emptyset; \tag{2}$$

$$(l(t, p, u) \cup r(t, p, u)) \cap (l(t', p', u') \cup r(t', p', u')) = \emptyset \quad \text{whenever } t \geqslant u'; \tag{3}$$

$$l(t, p, u) \cup r(t, p, u) \subset l(t', p', u') \cup \vec{p}' \quad \text{whenever } t' \leqslant t < p < u < p' < u'$$

and (4)

$$l(t, p, u) \cup r(t, p, u) \subset r(t', p', u')\vec{p}' \quad \text{whenever } t' < p' < t < p < u \leqslant u'.$$

(1) and (4) imply that we have defined a topology for $X(\text{FLIP})$ and (2) and (3) imply that this a Hausdorff topology.

Furthermore, as a subspace, $K \times \{l, r\}$ is homeomorphic to a compact subspace of the double arrow space (see pp. 270 of [4]). The crucial property of S is that any infinite subset A of S has a limit point x in K and thus a limit point (x, r) or (x, l) in $X(\text{FLIP})$ (depending on whether A has a monotone sequence converging to x from the left or the right and inside FLIP(x) or outside FLIP(x).) The topological properties of X(FLIP) can now be deduced: Any infinite subset of $K \times \{l, r\}$ has a limit point as the subspace is compact so the Hausdorff space X(FLIP) is countably compact (see pp. 258 of [4]). X(FLIP) is the union of a hereditarily Lindelöf-hereditarily separable space and a countable space and is therefore a hereditarily Lindelöf hereditarily separable compact Hausdorff space.

3. The problem of Fremlin

We shall show that if $\mathscr{F}$ is any family of $2^{\aleph_0}$-many hereditarily separable hereditarily Lindelöf compact Hausdorff spaces, then X(FLIP) can be defined so that it is not the continuous image of any closed subspace of an element of $\mathscr{F}$.

First, we can assume $\mathscr{F}$ is closed hereditary since any element of $\mathscr{F}$ has at most $2^{\aleph_0}$-many closed subspaces (by inequality 9.1 of [3]). Next, any continuous mapping f from an element of $\mathscr{F}$ onto X(FLIP) contains a bijection Π from a subspace of an element of $\mathscr{F}$ onto S. There are at most $2^{\aleph_0}$-many countable subsets of each element of $\mathscr{F}$ (by inequality 4.10 of [3]) and so at most $2^{\aleph_0}$-many possibilities for Π. List these possibilities with index set K by

$$\Pi_r\colon\ A_r \to S \ (r \in K)$$

where A_r is a subset of an element of $\mathscr{F}$ and Π_r is the restriction of a mapping from an X_r to S.

For each $r \in K$ let $A = \Pi_r^{-1}(\vec{r})$. A is an infinite subset of X_r so let $B \subset A$ be a convergent sequence in X_r (each element of $\mathscr{F}$ is sequentially compact). We can assume $\Pi_r(B)$ lies all on one side of r without loss of generality.

Choose FLIP(r) such that

$$|\Pi_r(B) \cap \{r(n)\colon n \in \mathrm{FLIP}(r)\}| = \omega \quad \text{and} \quad |\Pi_r(B) \setminus \{r(n)\colon n \in \mathrm{FLIP}(r)\}| = \omega.$$

Π_r is not the restriction to A_r of any continuous function from X_r to X(FLIP) since B is a convergent sequence in X_r while $\Pi_r(B)$ has two limit points in X(FLIP).

This paper is in the tradition of V. Filippov [1] who proved that there are 2^c-many nonhomeomorphic perfectly normal compact Hausdorff spaces and thus that there is no universal perfectly normal compact Hausdorff space. In particular we can let K consist of one space, a putative universal space Y, and note that the resulting X(FLIP) is not a (necessarily closed) subspace of Y.

References

[1] V. Filippov, On perfectly normal bicompacta, Soviet Math. Doklady 10 (1969) 1505–1507.
[2] D.H. Fremlin, Consequences of Martin's Axiom, Question List, Preprint.
[3] R. Hodel, Cardinal Function I Handbook of Set Theoretic Topology (North-Holland, Amsterdam, 1984) 1–61.
[4] R. Engelking, General Topology (Polish Scientific Publishing, Warszawa, 1977).