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On 3rd and 4th moments of finite upper half plane graphs ☆

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Abstract

Terras [A. Terras, Fourier Analysis on Finite Groups and Applications, Cambridge Univ. Press, 1999] gave a conjecture on the distribution of the eigenvalues of finite upper half plane graphs. This is known as a finite analogue of Sato–Tate conjecture. There are several modified versions of them. In this paper, we show that this conjecture is not correct in its original form (i.e., Conjecture 1.1). This is shown for the calculations of the 3rd and 4th moments of the distribution of the eigenvalues. We remark that a weaker version of the conjecture (i.e., Conjecture 1.2) may still hold.

Keywords: Finite upper half plane graphs; Eigenvalues; Distribution; Sato–Tate conjecture

1. Introduction

We suppose F_q is a finite field with q elements (q odd). For a fixed non-square element $\delta \in F_q$, we define a finite upper half plane as follows:

$$H_q := F_q(\sqrt{\delta}) - F_q = \{x + y\sqrt{\delta} \mid x, y \neq 0 \in F_q\}.$$

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We can define a distance d between points z and $w \in F_q$:

$$d(z,w) = \frac{N(z-w)}{\operatorname{Im} z \cdot \operatorname{Im} w},$$

where $z = x + y\sqrt{\delta} \in H_q$, and we denote $N(z) = x^2 - \delta y^2$ and $\operatorname{Im} z = y$.

Also, we can define a graph $X_q(\delta, a)$ for $a \in F_q$ whose vertices are the points of H_q and whose edges are the pairs of two vertices z, w in H_q such that $d(z,w) = a$. Then we define an adjacency matrix A_a by

$$(A_a)_{z,w} = \begin{cases} 1, & \text{if } z \text{ is adjacent to } w, \\ 0, & \text{otherwise.} \end{cases}$$

Terras [10] gave the following conjecture about the distribution of these eigenvalues. This is known as a finite analogue of the Sato–Tate conjecture.

Conjecture 1.1. *Given q, we fix δ. For $a \neq 0, 4\delta$, the distribution of the eigenvalues of the upper half plane graphs is asymptotically semi-circle or Sato–Tate distribution. That is,*

$$\frac{1}{q-1}\sharp\left\{\lambda \;\middle|\; \frac{\lambda}{\sqrt{q}} \in E\right\} \sim \frac{1}{2\pi}\int\limits_E \sqrt{4-x^2}\,dx, \quad \textit{as } q \to \infty,$$

for any Borel set E of the interval $[-2, 2]$.

Terras gave some modifications of Conjecture 1.1. Conjecture 1.2 is one of them.

Conjecture 1.2. [6] *Given q, we fix δ. Let Λ be the multi-set of all eigenvalues of the $q-2$ graphs $X_q(\delta, a)$, where a runs through F_q^* with $a \neq 4\delta$. Λ has an asymptotic semi-circle distribution as $q \to \infty$.*

Kuang [6] proved that the first moments and the second moments, or average and variance of the distribution asymptotically match those of the semi-circle distribution for Conjectures 1.1 and 1.2. In the next section, we will consider the properties and the facts about finite upper half plane graphs. In Section 3, we will calculate the 3rd and the 4th moments of the distribution of the eigenvalues. This implies that Conjecture 1.1 is not correct although Conjecture 1.2 may still hold.

2. Preliminaries

In this section, we will consider the properties and the facts about the finite upper half plane graphs. See Terras [10] for the proofs.

First, the matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G = GL(2, F_q)$ acts transitively on $z \in H_q$ by fractional linear transformation

$$g \cdot z = \frac{az+b}{cz+d} \in H_q.$$

So we can identify H_q with G/K, where K is a subgroup of G which fixes $\sqrt{\delta}$. That is,

$$K = \left\{ \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \;\middle|\; a, b \in F_q,\ a^2 - \delta b^2 \neq 0 \right\},$$

which is isomorphic to the multiplicative group $F_q(\sqrt{\delta})^*$.

Proposition 2.1. [10] *Assume that $q = p^r$, where p is an odd prime. Suppose that δ is a non-square in F_q. Let $a \in F_q$.*

(1) *The graph $X_q(\delta, a)$ is a $(q+1)$-regular graph provided that $a \neq 0$ or 4δ.*
(2) *The graphs $X_q(\delta, a)$ and $X_q(\delta c^2, ac^2)$ are isomorphic for any $c \in F_q^*$.*
(3) *The graph $X_q(\delta, a)$ is connected, provided that $a \neq 0, 4\delta$. In fact, the graph $X_q(\delta, a)$ is a Cayley graph for the affine group*

$$\mathrm{Aff}(q) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x, y \in F_q,\ y \neq 0 \right\},$$

using the generators

$$S_q(\sqrt{\delta}, a) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x, y \in F_q,\ y \neq 0,\ x^2 = ay + \delta(y-1)^2 \right\}.$$

(4) *The K-double cosets for G are represented by the sets $S_q(\sqrt{\delta}, a)$, for $a \in F_q$.*

And our graphs are Ramanujan graphs, k-regular graphs such that for all non-trivial eigenvalues λ, $|\lambda| \leqslant 2\sqrt{k-1}$. This definition was given by Lubotzky, Phillips, and Sarnak [9]. Katz [5] and Li [7,8] independently proved this positively, using different methods.

The eigenvalues of our graphs are given by R. Evans [4]. We call them $\lambda_j(a)_q$ and let their multiplicities be m_j, $j = 0, \ldots, q-1$, as in [6]. We recall $\lambda_0(a)_q = q+1$, $m_0 = 1$ and $m_j = q-1, q$ or $q+1$, for $j = 1, \ldots, q-1$.

Regarding Conjecture 1.1, Terras [10, p. 358] says: "We neglect the multiplicities and look only at the $q-1$ eigenvalues." But as above, all non-trivial eigenvalues have multiplicity $q-1, q$, or $q+1$. So we do not worry about multiplicity because we may multiply the sum of the all non-trivial eigenvalues or Eq. (3) (see Section 3) by $1/q$.

3. Main results

In this section, we will give the new results and make some preparations. Theorems 3.1 and 3.2 are for Conjecture 1.1. Moreover, in the next section, we will be referring to the proofs of these theorems, considering Conjecture 1.2.

Theorem 3.1. *The* 3*rd moment of the distribution of the set* $\{\lambda_i(a)_q/\sqrt{q} \mid i = 1, \dots, q-1\}$ *asymptotically matches with that of the Sato–Tate semi-circle distribution. That is,*

$$\lim_{q\to\infty} \frac{1}{q-1}\sum_{i=1}^{q-1}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^3 = 0,$$

and the limits are uniform and independent of $a \neq 0$ *and* δ *as long as* $a \neq 4\delta$.

Theorem 3.2. *For* $a \in F_q \neq 0,\ 2\delta,\ 4\delta$, *the* 4*th moment of the distribution of the set* $\{\lambda_i(a)_q/\sqrt{q} \mid i = 1, \dots, q-1\}$ *asymptotically matches with that of the Sato–Tate semi-circle distribution. That is,*

$$\lim_{q\to\infty} \frac{1}{q-1}\sum_{i=1}^{q-1}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^4 = 2,$$

and the limits are uniform and independent of $a \neq 0,\ 2\delta,\ 4\delta$ *and* δ.

For $a = 2\delta$, *the* 4*th moment of the above set does not asymptotically match with that of semi-circle distribution.*

The 3rd and 4th moments of the semi-circle distribution are, respectively, 0 and 2. Theorem 3.1 gives positive evidence to Conjecture 1.1. Theorem 3.2 gives a counterexample. Now we give one definition for the proof.

Definition 3.3. [1,10] A connected graph $X(V, E)$ is highly regular with collapsed adjacency matrix $C = (c_{ij})$ if and only if for every vertex $v \in V$, there is a partition of V into sets V_i, $i = 1, \dots, n$, with $V_1 = \{v\}$, such that each vertex $y \in V_i$ is adjacent to exactly c_{ij} vertices in V_j.

Our graphs are highly regular since we can take $S_q(\sqrt{\delta}, i)$, $i \in F_q$, as the above partition, where $S_q(\sqrt{\delta}, 0) = \{\sqrt{\delta}\}$ and $S_q(\sqrt{\delta}, 4\delta) = \{-\sqrt{\delta}\}$. And it is known that c_{ij} is 0, 1, 2 or $q + 1$. So we have the following proposition. See Angel [1] for the details of the proof.

Proposition 3.4. [1,10] *The graph* $X_q(\delta, a)$ *is highly regular. Also, the entries of the collapsed adjacency matrix of* $X_q(\delta, a)$ *are as follows*:

$$c_{ij} = \begin{cases} q+1, & \text{if } (i,j) = (0,a), (4\delta, 4\delta - a), \\ 2, & \text{if } \Delta_{ij} \text{ is square}, \\ 1, & \text{if } \Delta_{ij} = 0, \\ 0, & \text{if } \Delta_{ij} \text{ is non-square}, \end{cases} \tag{1}$$

where $\Delta_{ij} = \delta(i-j)^2 + a\delta(a - 2i - 2j) + aij$.

In view of association scheme [2], Definition 3.3 is not important, and Proposition 3.4 is well known. For it is known that the upper half plane is a symmetric association scheme with relation of the distance, and the entry of the collapsed adjacency matrix c_{ij} corresponds to the intersection numbers of this association scheme. But here we used the above definition, as well as Terras [10] and Angel [1].

4. Proof of the main result

We will give the proofs of Theorems 3.1 and 3.2. We use the idea in Biggs [3] that the number of closed walks of length l in graph is equal to the sum of all l powers of each eigenvalue of the adjacency matrix. That is, let N_l be the number of the closed walks of the length l in the graph $X_q(\delta, a)$; we have

$$N_l = \sum_{i=0}^{q-1} m_i \big(\lambda_i(a)_q\big)^l. \tag{2}$$

Then we can interpret the lth moments of our distribution as the following limits:

$$\lim_{q\to\infty} \frac{1}{q-1} \sum_{i=1}^{q-1} \frac{m_i}{q} \left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^l = \lim_{q\to\infty} \frac{1}{q(\sqrt{q})^l(q-1)} \big\{N_l - (q+1)^l\big\}. \tag{3}$$

So, to get the lth moments, we may count up all the closed walks of length l. Clearly $N_1 = 0$ and $N_2 = q(q-1)(q+1)$ for $a \neq 0, 4\delta$, we get the first and second moments. We can give another proof of Kuang [6].

Proposition 4.1. *For $a \neq 0, 4\delta$, N_3 is given by*

$$N_3 = \begin{cases} 2q(q+1)(q-1), & \textit{if } a-3\delta \textit{ is square},\\ q(q+1)(q-1), & \textit{if } a-3\delta = 0,\\ 0, & \textit{if } a-3\delta \textit{ is non-square}. \end{cases} \tag{4}$$

Proof. Since G acts transitively on H_q, we consider the closed walks whose origin and terminal are $\sqrt{\delta}$. If two vertices z_1 and z_2 in $S_q(\sqrt{\delta}, a)$ are adjacent, we get such walk. In other words, if the entry in position (a, a) of the collapsed adjacency matrix is one or two, we have one or two triangles for one vertex in $S_q(\sqrt{\delta}, a)$.

Since $\Delta_{a,a} = a^2(a-3\delta)$, when $a - 3\delta = 0$, we have the one walk $\{\sqrt{\delta}, z_1, z_2, \sqrt{\delta}\}$ for all $z_1 \in S_q(\sqrt{\delta}, a)$, where $z_1 \in S_q(\sqrt{\delta}, a)$ is adjacent to z_2. By Proposition 2.1(4), $S_q(\sqrt{\delta}, a) = K z_a$ for $z_a \in S_q(\sqrt{\delta}, a)$. So, we have such $|K|$ walks, then $N_3 = q(q+1)(q-1)$.

For a such that $a - 3\delta$ is a square, as well as above, we have $N_3 = 2q(q+1)(q-1)$. The factor 2 causes from that we have the two closed walks whose origin and terminal are $\sqrt{\delta}$ for all $z_1 \in S_q(\sqrt{\delta}, a)$ because $\Delta_{a,a}$ is a square. □

Before Proposition 4.4 which gives the number of the walks of length 4, we give some preliminary propositions.

Proposition 4.2. *Let Q be the set of squares in F_q^*, and let N be the set of non-squares in F_q^*. For any $c \in F_q^*$, we have*

$$\left|(N+c) \cap Q\right| = \frac{1}{4}\left\{q - 1 - \varepsilon(c) + \varepsilon(-c)\right\},$$

$$\left|(N+c) \cap N\right| = \frac{1}{4}\left\{q - 3 + \varepsilon(c) + \varepsilon(-c)\right\}.$$

Here ε is a non-trivial quadratic character of F_q^* and $N + c = \{y + c \mid y \in F_q^*\}$.

Proof. We can consider ε as a multiplicative character of F_q, that is,

$$\varepsilon(x) = \begin{cases} 1, & \text{if } x \text{ is square,} \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x \text{ is non-square.} \end{cases}$$

We calculate

$$\sum_{x \in F_q^*} \varepsilon(x)\varepsilon(x+c) = \sum_{x \in F_q^*} \varepsilon\bigl(x^2\bigr)\varepsilon\bigl(1 + x^{-1}c\bigr) = \sum_{x \in F_q} \varepsilon\bigl(1 + x^{-1}c\bigr) - \varepsilon(1) = -1.$$

Also, we have

$$\begin{aligned} \sum_{x \in F_q^*} \varepsilon(x)\varepsilon(x+c) &= \sum_{x \in Q} \varepsilon(x)\varepsilon(x+c) + \sum_{x \in N} \varepsilon(x)\varepsilon(x+c) \\ &= \sum_{x \in Q} \varepsilon(x+c) - \sum_{x \in N} \varepsilon(x+c). \end{aligned}$$

So, we have

$$\sum_{x \in Q} \varepsilon(x+c) - \sum_{x \in N} \varepsilon(x+c) = -1. \tag{5}$$

Since the sum of ε over F_q is zero, we have

$$\sum_{x \in Q} \varepsilon(x+c) + \sum_{x \in N} \varepsilon(x+c) = \sum_{x \in F_q^*} \varepsilon(x+c) = -\varepsilon(c). \tag{6}$$

By (5) and (6), we get the equation

$$\sum_{x \in N} \varepsilon(x+c) = \frac{1}{2}\bigl(1 - \varepsilon(c)\bigr). \tag{7}$$

Moreover, we have

$$\sum_{x\in N}\varepsilon(x+c)=\sum_{\substack{x\in N\\ x+c\in Q}}\varepsilon(x+c)+\sum_{\substack{x\in N\\ x+c\in N}}\varepsilon(x+c)=\bigl|(N+c)\cap Q\bigr|-\bigl|(N+c)\cap N\bigr|.$$

So Eq. (7) implies

$$\bigl|(N+c)\cap Q\bigr|-\bigl|(N+c)\cap N\bigr|=\frac{1}{2}\bigl(1-\varepsilon(c)\bigr). \tag{8}$$

Also, we have

$$\begin{aligned}\bigl|(N+c)\cap Q\bigr|+\bigl|(N+c)\cap N\bigr|&=\bigl|(N+c)\cap F_q^*\bigr|\\ &=\begin{cases}\frac{q-1}{2}-1, & \text{if } -c\in N, \text{ that is, } 0\in N+c,\\ \frac{q-1}{2}, & \text{if } -c\in Q.\end{cases}\end{aligned}$$

Therefore, we get the equation

$$\bigl|(N+c)\cap Q\bigr|+\bigl|(N+c)\cap N\bigr|=\frac{q-1}{2}+\frac{1}{2}\bigl(\varepsilon(-c)-1\bigr). \tag{9}$$

By Eqs. (8) and (9), we get the proposition. □

Proposition 4.3. *The number of elements of* $\{n\in F_q^*\mid \Delta_{n,a}$ *is square*$\}$ *is* $\frac{1}{2}(q+\varepsilon(-1))$.

Proof. We have the equation

$$\Delta_{n,a}=n(n\delta+a^2-4a\delta)=\delta\left\{\left(n+\frac{a(a-4\delta)}{2\delta}\right)^2-\left(\frac{a(a-4\delta)}{2\delta}\right)^2\right\}.$$

Suppose that $\Delta_{n,a}=\delta k$, where k is non-square and $k+\bigl(\frac{a(a-4\delta)}{2\delta}\bigr)^2$ is square. For such k, the equation $\Delta_{n,a}=\delta k$ has two solutions for n. And the equation has only one solution for k such that $k+\bigl(\frac{a(a-4\delta)}{2\delta}\bigr)^2=0$.

By Proposition 4.2, the number of choices of such k is

$$\begin{aligned}\left|\left\{k+\left(\frac{a(a-4\delta)}{2\delta}\right)^2\in Q\ \middle|\ k\in N\right\}\right|&=\left|\left(N+\left(\frac{a(a-4\delta)}{2\delta}\right)^2\right)\cap Q\right|\\ &=\frac{1}{4}\bigl\{q-2+\varepsilon(-1)\bigr\}.\end{aligned}$$

Therefore, we have

$$\bigl|\{n\in F_q^*\mid \Delta_{n,a}\text{ is square}\}\bigr|=2\times\frac{1}{4}\bigl\{q-2+\varepsilon(-1)\bigr\}+1=\frac{1}{2}\bigl\{q+\varepsilon(-1)\bigr\}. \qquad\square$$

Now, we are ready to give the number of the walks of length 4. We will prove this in the same way as in the proof of Proposition 4.1.

Proposition 4.4. *For $a \neq 0, 4\delta$, N_4 is given by*

$$N_4 = \begin{cases} q(q+1)(q-1)(4q+2\varepsilon(-1)+2), & \text{if } a = 2\delta, \\ q(q+1)(q-1)(3q+2\varepsilon(-1)+2), & \text{if } a \neq 2\delta, \end{cases} \tag{10}$$

where ε is a quadratic character of F_q^*.

Proof. We consider the closed walks of length 4 whose origin and terminal are $\sqrt{\delta}$. If $\Delta_{n,a}$ is a square, there exist two edges from one vertex $x + y\sqrt{\delta} \in S_q(\sqrt{\delta}, n)$ to two different vertices in $S_q(\sqrt{\delta}, a)$. So, for $x + y\sqrt{\delta} \in S_q(\sqrt{\delta}, n)$, we have a 4-cycle containing $\sqrt{\delta}$ and $x + y\sqrt{\delta}$. Since $S_q(\sqrt{\delta}, 4\delta) = \{-\sqrt{\delta}\}$ and

$$\Delta_{4\delta,a} = 4\delta(a-2\delta)^2 = \begin{cases} 0, & \text{if } a = 2\delta, \\ \text{non-square}, & \text{if } a \neq 2\delta, \end{cases}$$

we have two cases according to whether a is 2δ or not.

When $a \neq 2\delta$, for n such that $\Delta_{n,a}$ is a square, we get the above 4-cycle, and there is a path of length 2 whose origin is $\sqrt{\delta}$ in that cycle. Also, for n such that $\Delta_{n,a} = 0$, that is, $n = \frac{a(4\delta-a)}{\delta}$, we have $q+1$ paths of length 2 whose origin is $\sqrt{\delta}$. Clearly, the path of length 2 is a walk of length 4. So, we have

$$N_4 = \left[\left\{(q+1) \times 2 \times \frac{1}{2}\big(q+\varepsilon(-1)\big)\right\} + \left\{(q+1) \times 2 \times \frac{1}{2}\big(q+\varepsilon(-1)\big) + (q+1) + q(q+1)\right\} + (q+1)\right] \times q(q-1).$$

Similarly, when $a = 2\delta$, we get the above 4-cycles and the paths of length 2 on these cycles, for n such that $\Delta_{n,a}$ is a square. For n such that $\Delta_{n,a} = 0$, that is, $n = 4\delta$, all $q+1$ vertices in $S_q(\sqrt{\delta}, a)$ are adjacent to $-\sqrt{\delta}$ in $S_q(\sqrt{\delta}, 4\delta)$. Taking two different vertices z_1, $z_2 \in S_q(\sqrt{\delta}, a)$, we have a 4-cycle whose vertices are $\sqrt{\delta}$, z_1, z_2 and $-\sqrt{\delta}$. The number of these 4-cycles is $\binom{q+1}{2}$. Therefore, we have

$$N_4 = \left[\left\{(q+1) \times 2 \times \frac{1}{2}\big(q+\varepsilon(-1)\big) + 2 \times \binom{q+1}{2}\right\} + \left\{(q+1) \times 2 \times \frac{1}{2}\big(q+\varepsilon(-1)\big) + (q+1) + q(q+1)\right\} + (q+1)\right] \times q(q-1).$$

Thus we obtain the proposition. $\square$

We finished preparations to prove Theorems 3.1 and 3.2. First, by Proposition 4.1, for $a \neq 0, 4\delta$, we have

$$0 \leqslant N_3 \leqslant 2q(q-1)(q+1).$$

By (3), this inequality implies Theorem 3.1.

Next, by Proposition 4.4, for $a \neq 0, 2\delta, 4\delta$, we have

$$\frac{1}{q-1}\sum_{i=1}^{q-1}\frac{m_i}{q}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^4 = \frac{q+1}{q^3(q-1)}\bigl\{2q^3 + \bigl(2\varepsilon(-1)-4\bigr)q^2 - \bigl(2\varepsilon(-1)+5\bigr) - 1\bigr\}.$$

This equation implies Theorem 3.2. Moreover, for $a = 2\delta$ the coefficient of q^3 in the numerator of the above equation is 3, the result of (3) is 3. But it is not the 4th moment of the semi-circle.

Finally, using Propositions 4.1 and 4.4, we consider about Conjecture 1.2. Using Proposition 4.2, we have

$$\bigl|\bigl\{a - 3\delta \in Q \bigm| a \in F_q^*,\ a \neq 4\delta\bigr\}\bigr| = \frac{1}{2}\bigl(q - 2 + \varepsilon(-3)\bigr).$$

So, we have the equations

$$\sum_{\substack{a\in F_q^*\\ a\neq 4\delta}}\sum_{i=1}^{q-1} m_i\bigl(\lambda_i(a)_q\bigr)^3 = (q+1)\bigl\{\bigl(\varepsilon(-3)-2\bigr)q^2 - \bigl(4+\varepsilon(-3)\bigr)q - 2\bigr\},$$

$$\sum_{\substack{a\in F_q^*\\ a\neq 4\delta}}\sum_{i=1}^{q-1} m_i\bigl(\lambda_i(a)_q\bigr)^4 = (q+1)\bigl\{2q^4 + \bigl(2\varepsilon(-1)-1\bigr)q^3 + \bigl(-2\varepsilon(-1)-4\bigr)q^2 + \bigl(-4\varepsilon(-1)+1\bigr)q + 2\bigr\}.$$

These two equations imply the following corollary.

Corollary 4.5. *Given q, we fix δ. Let Λ be the multi-set of all eigenvalues of the $q-2$ graphs $X_q(\delta, a)$, where a runs through F_q^* with $a \neq 4\delta$. The 3rd and 4th moment of the distribution of the set Λ asymptotically match with those of the semi-circle distribution. That is,*

$$\lim_{q\to\infty}\frac{1}{(q-1)(q-2)}\sum_{\substack{a\in F_q^*\\ a\neq 4\delta}}\sum_{i=1}^{q-1}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^3 = 0$$

and

$$\lim_{q\to\infty}\frac{1}{(q-1)(q-2)}\sum_{\substack{a\in F_q^*\\ a\neq 4\delta}}\sum_{i=1}^{q-1}\left(\frac{\lambda_i(a)_q}{\sqrt{q}}\right)^4 = 2.$$

5. Remarks

By Theorem 3.2, we have a counterexample $a = 2\delta$ for Conjecture 1.1. So we have to modify Conjecture 1.1. One such modification might be Conjecture 1.2. Then Conjecture 1.2 is given positive evidences by Corollary 4.5, and its validity is an open problem.

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