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# Positivity properties in noncommutative convolution algebras with applications in pseudo-differential calculus

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**Abstract**

We study $\mathscr{S}'_+$, of all $a \in \mathscr{D}'$ such that $(a *_\sigma \varphi, \varphi) \geqslant 0$ for every $\varphi \in C_0^\infty$, where $*_\sigma$ denotes the twisted convolution. We prove that certain boundedness for $a \in \mathscr{S}'_+$ are completely determined of the behaviour for $a$ at origin, for example that $a \in \mathscr{S}'$, and that if $a(0) < \infty$, then $a \in L^2 \cap L^\infty$. We use the results in order to determine wether positive pseudo-differential operators belong to certain Schatten-casses or not.


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## 0. Introduction

The aim of the paper is to consider general positivity properties for noncommutative convolution algebras, especially in the algebra of twisted convolution (the $*_\sigma$-algebra), from a somewhat abstract point of view. A motivation for this is the close relation between positivity in the $*_\sigma$-algebra, positivity in operator theory and positivity in pseudo-differential calculus. In fact, the ideas for handling these questions were originally raised when discussing positivity in the Weyl calculus of pseudo-differential operators, and the results presented here also apply immediately to this calculus. In particular we may express the well-known lower bound results due to Gårding, Melin, Hörmander

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and Fefferman/Phong (cf. [6], Theorem 6.2 in [3] and [1]) in terms of the $*_\sigma$-algebra. However, we emphasize that one does not need to know anything about Weyl calculus in order to appreciate the results, since the theory is formulated within the framework of noncommutative convolution algebras.

Another motivation is that positivity results in the $*_\sigma$-algebra might be a source of inspiration for positivity results in other situations. An example of this is the extension of the Bochner–Schwartz theorem, which asserts that any distribution, which is positive in the usual convolution algebra, must be a tempered distribution (cf. Theorem IX.10 in [7]). This was first proved in the $*_\sigma$-algebra, but the results presented here are valid for a large class of noncommutative convolution algebras. We also give an example where the technique in the proof is applied to obtain a general positivity result for linear operators in distribution theory (cf. Theorem 2.8).

In order to describe our results in more detail, we shall now give some necessary definitions. Let $V$ be a linear vector space of dimension $n < \infty$, and let $W = T^*V = V \oplus V'$, where $V'$ is the dual for $V$. Then $W$ is a symplectic vector space with symplectic form $\sigma(X, Y) = \langle y, \xi\rangle - \langle x, \eta\rangle$, where $X = (x, \xi) \in W$ and $Y = (y, \eta) \in W$.

The twisted convolution $*_\sigma$ is then defined by the formula

$$(a *_\sigma b)(X) \equiv (2/\pi)^{n/2} \int a(X - Y)b(Y)\mathrm{e}^{2i\sigma(X,Y)}\, dY \tag{0.1}$$

when $a, b \in L^1(W)$. (Cf. [2] or [9–12].) Here and in what follows we use the same notation for the usual functions and distribution spaces as in [4]. The definition of $*_\sigma$ extends in different ways. It extends for example to a continuous bilinear mapping from $\mathscr{D}'(W) \times C_0^\infty(W)$ to $\mathscr{D}'(W)$. We are then concerned with the set $\mathscr{S}'_+(W)$ of positive elements in the $*_\sigma$-algebra, i.e. the set of all $a \in \mathscr{D}'(W)$ such that $(a *_\sigma \varphi, \varphi) \geqslant 0$ for every $\varphi \in C_0^\infty(W)$. Here $(a, \varphi) \equiv \langle a, \bar\varphi\rangle$, where $\langle\cdot\,,\cdot\rangle$ denotes the duality between elements in $C_0^\infty$ and $\mathscr{D}'$. (A motivation for using $\mathscr{S}'_+$ instead of $\mathscr{D}'_+$ is given by Theorem 2.6 below.) We note also that the term $*_\sigma$-algebra is not very proper since $\mathscr{D}'$ is not an algebra under $*_\sigma$. (It is however a $C_0^\infty$-module under $*_\sigma$, where $(C_0^\infty, *_\sigma)$ is an algebra.)

It might seem hard to find common structures for the positivity results in the $*_\sigma$-algebra, but there are indeed such ones. In fact, in most of our results, the following principle holds:

*Assume that $a \in \mathscr{S}'_+$. If $a$ satisfies a certain regularity or boundedness property at the origin, then $a$ and its Fourier transform $\hat a = \mathscr{F}a$ satisfy the same regularity or boundedness property everywhere.*

We prove for example the following results:

(i) (*Growth properties at infinity.*) If $a \in \mathscr{S}'_+$, then $a \in \mathscr{S}'$ (cf. Proposition 2.8).

(ii) (*Local boundedness with respect to Fourier $L^p$-spaces.*) If $a \in \mathscr{S}'_+$ and $\chi a \in \mathscr{F}L^p$ for some $\chi \in \mathscr{S}$ which is nonzero at the origin, then $\psi a \in \mathscr{F}L^p$ and $\psi \hat a \in \mathscr{F}L^p$, for any $\psi \in \mathscr{S}$. A generalization to weighted Fourier $L^p$-spaces is also presented (cf. Proposition 4.9 and Corollary 4.12).

(iii) (*Classical regularity questions.*) If $a \in \mathscr{S}'_+$ is a $C^{2N}$-function near the origin for some integer $N \geqslant 0$, then $a$ and $\hat a$ belong to $C^{2N}$. Moreover, $X^\alpha D^\beta a \in L^2 \cap C_B$ when $|\alpha + \beta| \leqslant N$, and similarly for $\hat a$ (cf. Theorem 3.13). Here and in what follows we let $C_B(W)$ be the set of continuous functions on $W$, vanishing at the infinity.

(iv) (*Regularity in the framework of micro-local analysis*.) If $a \in \mathscr{S}'_+$ and $(0, Y) \notin WF(a)$, then $(X, Y) \notin WF(a)$ and $(X, Y) \notin WF(\mathscr{F}_\sigma a)$ for every $X \in W$. (Cf. Theorem 4.15.) Here $WF(a) \subset W \times (W \setminus \{0\})$ denotes the wave-front set for $a$ and $\mathscr{F}_\sigma$ is the symplectic Fourier transform given by (0.6) below.

The assertion (i) is based on our generalization of the Bochner–Schwartz theorem to weighted convolutions of the type

$$(a *_B \varphi)(x) = \int a(x-y)\varphi(y)B(x,y)\,dy, \tag{0.2}$$

when $a, \varphi \in C_0^\infty(\mathbf{R}^m)$. Here we require that $B \in C^\infty(\mathbf{R}^m \oplus \mathbf{R}^m)$ is nonzero, and that $B(x,y)$, $B(x,y)^{-1}$ and their derivatives do not grow faster than polynomials. It is then clear that the definition of $*_B$ extends in similar ways as $*_\sigma$ above, and we prove that if $a \in \mathscr{D}'(\mathbf{R}^m)$ is positive in the sense that $(a *_B \varphi, \varphi) \geqslant 0$ for every $\varphi \in C_0^\infty(\mathbf{R}^m)$, then $a \in \mathscr{S}'(\mathbf{R}^m)$ (cf. Theorem 2.6).

We note that if $B(x,y) = 1$ everywhere, then we obtain the Bochner–Schwartz theorem in its original shape.

In order to discuss connections between positivity in the $*_\sigma$-algebra and positivity in operator algebras on $\mathscr{S}(V)$ we recall the operator $A$ in [9–12], which is a mapping from $\mathscr{S}'(W)$ to the set of continuous operators from $\mathscr{S}(V)$ to $\mathscr{S}'(V)$. Assume first that $a \in \mathscr{S}(W)$. Then $Aa$ is the operator with Schwartz kernel given by

$$(Aa)(x,y) = (2\pi)^{-n/2}\int a\big((y-x)/2, \xi\big)\mathrm{e}^{-i\langle x+y,\xi\rangle}\,d\xi. \tag{0.3}$$

For general $a \in \mathscr{S}'(W)$, $Aa$ is defined by continuous extension, i.e. $Aa = (T \circ \mathscr{F}_2^{-1})a$, where $TU(x,y) = U((y-x)/2, -(x+y))$ and $\mathscr{F}_2$ denotes the partial Fourier transform of

$$\hat f(\xi) = \mathscr{F} f(\xi) \equiv (2\pi)^{-n/2}\int \mathrm{e}^{-i\langle x,\xi\rangle} f(x)\,dx \tag{0.4}$$

with respect to the second variable. Here and in what follows we identify operators with their Schwartz kernels.

The main relation between positivity in $*_\sigma$-algebra and positivity in operator theory is then that $a \in \mathscr{S}'_+(W)$ if and only if $((Aa)f, f) \geqslant 0$ for every $f \in \mathscr{S}(V)$ (i.e. $Aa$ is a positive semi-definite operator on $\mathscr{S}(V)$). (Cf. [9–12], or Proposition 1.10 below.)

For the reader familiar with the Weyl calculus we observe also the similarities between the $Aa$ and the Weyl quantization $a^w(x, D)$, given by

$$a^w(x,D)f(x) = (2\pi)^{-n}\iint a\big((x+y)/2, \xi\big) f(y)\mathrm{e}^{i\langle x-y,\xi\rangle}\,dy\,d\xi, \tag{0.5}$$

when $a \in \mathscr{S}'(W)$ and $f \in \mathscr{S}(V)$. (Cf. [4,5] or [9–12].) It follows from (0.3) and (0.5) that the integral kernel of $a^w(x,D)$ is $(2\pi)^{-n/2}(Aa)(-x,y)$. A somewhat more "symplectic elegant" way to express this connection might be done by using the symplectic Fourier transform on $\mathscr{S}'(W)$, whose restriction to $\mathscr{S}(W)$ is defined by

$$\hat a(X) = (\mathscr{F}_\sigma a)(X) \equiv \pi^{-n}\int a(Y)\mathrm{e}^{2i\sigma(X,Y)}\,dY. \tag{0.6}$$

Then we have that $a^w(x,D) = (2\pi)^{-n/2} A(\mathscr{F}_\sigma a)$, and it follows from the above that $a^w(x,D) \geqslant 0$, if and only if $(\mathscr{F}_\sigma a) \in \mathscr{S}'_+(W)$. This implies also that $a^w(x,D)$ is bounded from below (i.e. for some constant $C > 0$ then $(a^w(x,D)f, f) \geqslant -C\|f\|^2_{L^2}$ holds for any $f \in \mathscr{S}(V)$), if and only if $\mathscr{F}_\sigma a + C\delta_0 \in \mathscr{S}'_+$, for some constant $C > 0$. In the lower bound results due to Melin (cf. [6]), Hörmander (cf. Theorem 6.2 in [3]), Feffermann/Phong (cf. [1]) and the author (cf. Theorem 4.5 in [10]), one gives sufficient and sometimes necessary conditions on $a$, in order to that $a^w(x,D)$ should be lower bounded. In the last section we explicitely write down some consequences of our investigations in pseudo-differential calculus.

Next we shall discuss the notion of $\sigma$-positivity, an analogue to the definition of positive definite functions, and which is also deeply related to positivity in the $*_\sigma$-algebra. Assume that $a \in C(W)$. Then we say that $a$ is *$\sigma$-positive* if

$$\sum_{j,k=1}^{M} a(X_j - X_k) e^{2i\sigma(X_j,X_k)} c_j \overline{c_k} \geqslant 0, \tag{0.7}$$

for all pairs of sequences $X_1, \ldots, X_M \in W$ and $c_1, \ldots, c_M \in \mathbf{C}$. We let $C_+(W)$ be the set of $\sigma$-positive functions. By a slight reformulation one obtains that $C_+(W) = \mathscr{S}'_+(W) \cap C(W)$. We will prove that $C_+(W) \subset s_1(W)$, where $s_1(W)$ is the set of all $a \in \mathscr{S}'(W)$ such that $Aa$ (or alternatively $a^w(x,D)$) is a trace class operator on $L^2(V)$. More generally, we prove that if $a \in \mathscr{S}'_+(W)$ satisfies a vague boundary condition at the origin, then $a \in s_1(W)$ is $\sigma$-positive (cf. Theorem 3.3). We obtain (iii) in case $N = 0$ on p. 2 from this since $s_1 \subset L^2 \cap C_B$ and $\mathscr{F}_\sigma s_1 = s_1$. (See [9–12] or Section 1 below.) This gives a complete characterization of the $\sigma$-positive functions as the set of all $a \in \mathscr{S}'(W)$ such that $Aa$ (or alternatively $(\mathscr{F}_\sigma a)^w(x,D)$) is a positive semi-definite trace class operator on $L^2(V)$. In the usual convolution algebra, the result corresponds to Bochner's theorem which completely characterizes the set of positive definite functions.

According to the last considerations, we discuss also necessary and sufficient conditions on elements in $\mathscr{S}'_+(W)$ in order to belong to $s_p(W)$, the set of all $a \in \mathscr{S}'(W)$ such that $Aa$ (or alternatively $a^w(x,D)$) is a Schatten–von Neumann operator of order $p \in [1,\infty]$ on $L^2(V)$. We prove for general $p$ that if $a \in \mathscr{S}'_+(W)$ and $\chi, \psi \in \mathscr{S}(W)$ such that $\chi(0) \neq 0$, then $a \in s_p$ if and only if $\chi a \in s_p$ and $(1-\psi)a \in s_p$ (cf. Corollary 4.12). From the same proof one also obtains (ii) above.

An important ingredient in the proof of the last result is the existence of nonzero elements in $C_+(W)$ with small supports at origin. This gives rise to questions concerning support properties for elements in $\mathscr{S}'_+(W)$, and in the last section we prove that there are no nontrivial $a \in \mathscr{S}'_+(W)$ such that $\hat{a}$ has compact support. In this section we also discuss some other properties for elements in $\mathscr{S}'_+$, and complete some discussions from [12]. We prove, for example, that $C_+$, $\mathscr{S}'_+$, $s_1$ and $s_\infty$ are not invariant under dilations, and that elements in $C_+(W)$ might be negative (as functions on $W$) on quite large sets.

We finally remark that the paper is in many sense a completion of [12], where one discusses some general continuity questions for the twisted convolution and the $s_p$-spaces.

## 1. Preliminaries

In this section we review some continuity results concerning the twisted convolution, the operator $A$ and the $s_p$-spaces which we did encounter in the introduction. We omit the proofs since they can be found in Section 1.4 in [9], Section 1.1 in [10] and in Section 1 in [12].

In order to formulate our problems in a coordinate invariant way, we shall, as in [12], consider the usual function and distribution spaces on the $n$-dimensional vector space $V$, as densities with values in $\Omega^{1/2}(V)$, the set of all mappings $\mu : \wedge^n V \to \mathbf{C}$ such that $\mu(t\omega) = |t|^{1/2}\mu(\omega)$, when $t \in \mathbf{R} \setminus \{0\}$ and $\omega \in \wedge^n(V)$. (The reader not interested in this coordinate invariant view may consider $V$ as $\mathbf{R}^n$.) This means that $C_0^\infty(V) = C_0^\infty(V; \Omega^{1/2}(V))$ and similarly for other function and distribution spaces, and that if $f, g \in \mathcal{S}(V)$ and $\omega = e_1 \wedge \cdots \wedge e_n$, then

$$\langle f, g\rangle \equiv \int_{\mathbf{R}^n} f(x_1e_1 + \cdots + x_ne_n; \omega)g(x_1e_1 + \cdots + x_ne_n; \omega)\, dx_1 \cdots dx_n$$

is independent on the choice of basis $e_1, \ldots, e_n$ for $V$. After extending the form $\langle \cdot, \cdot \rangle$ in usual ways, it follows that we may identify the dual spaces of $C_0^\infty(V)$, $\mathcal{S}(V)$ or $L^2(V)$ with $\mathcal{D}'(V)$, $\mathcal{S}'(V)$ and $L^2(V)$ respectively, as usual. We also let $(f, g) = \langle f, \bar{g}\rangle$ for admissible $f$ and $g$. The restriction of $(\cdot, \cdot)$ to $L^2$ is the then usual scalar product.

Assume next that $W$ is a $2n$-dimensional symplectic vector space, with symplectic form $\sigma$. Then $\Omega^{1/2}(W) \simeq \mathbf{C}$, and the usual functions and distributions may be considered as scalar valued. If $(x, \xi) = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ are symplectic coordinates such that $\sigma((x, \xi), (y, \eta)) = \langle y, \xi\rangle - \langle x, \eta\rangle$ then $\int_W f(X)\, dX = \iint_{\mathbf{R}^n \times \mathbf{R}^n} f(x, \xi)\, dx\, d\xi$. This shows also that $W$ may be represented as $W = T^*V = V \oplus V'$, where $V$ is a vector space of dimension $n$, and $V'$ is the dual of $V$ with duality inherited from the symplectic form.

We recall that the symplectic Fourier transform $\mathcal{F}_\sigma$ in (0.6) is a homeomorphism on $\mathcal{S}(W)$ which extends to a homeomorphism on $\mathcal{S}'(W)$, which is unitary on $L^2$. One has that $\mathcal{F}_\sigma^2$ is the identity map on $\mathcal{S}'(W)$, and that

$$\mathcal{F}_\sigma(a_1a_2) = \pi^{-n}\mathcal{F}_\sigma a_1 * \mathcal{F}_\sigma a_2, \qquad \mathcal{F}_\sigma(a_1 * a_2) = \pi^n \mathcal{F}_\sigma a_1 \mathcal{F}_\sigma a_2, \tag{1.1}$$

when $a_1, a_2 \in \mathcal{S}(W)$.

Next we shall discuss the operator $A$ and the twisted convolution. One has that $A$ is a homeomorphism from $\mathcal{S}(W)$ to $\mathcal{S}(V \oplus V)$, which extends to a homeomorphism from $\mathcal{S}'(W)$ to $\mathcal{S}'(V \oplus V)$, and to a unitary map on $L^2$, since similar facts are true for $T$ and $\mathcal{F}_2$ after (0.3).

An important relation between the operator $A$, the twisted convolution product and operator algebras on $\mathcal{S}(V)$ is given by

$$A(a *_\sigma b) = (Aa) \circ (Ab), \tag{1.2}$$

when $a, b \in \mathcal{S}(W)$. (By duality it follows that (1.2) holds also when $a, b \in \mathcal{S}'(W)$ such that $a \in \mathcal{S}(W)$ or $b \in \mathcal{S}(W)$.) The homomorphism (1.2) together with the simple character of the twisted convolution and the similarities between $Aa$ and $a^w(x, D)$, give motivations for using the twisted convolution and the quantization $Aa$ in the Weyl calculus. We note for example that the Weyl product #, defined by $(a\#b)^w(x,D) = a^w(x,D)b^w(x,D)$, is in many situations more technical than the twisted convolution.

In the following lemma we list some important properties of the operator $A$ and the twisted convolution.

**Lemma 1.1.** *Let $A$ be the operator in* (0.3) *and let $U = Aa$ where $a \in \mathscr{S}'(T^*V)$. Then the following are true*:

(i) $\check{U} = A\check{a}$, *where* $\check{a}(X) = a(-X)$, *for every* $X \in T^*V$;
(ii) $J_{\mathscr{F}}U = A\mathscr{F}_\sigma a = (2\pi)^{n/2}a^w(x,D)$, *where* $J_{\mathscr{F}}U(x,y) = \overline{U(-x,y)}$;
(iii) *the Hilbert space adjoint of $Aa$ equals $A\tilde{a}$, where* $\tilde{a}(X) = \overline{a(-X)}$;
(iv) *if* $a_1, a_2, a_3 \in \mathscr{S}(W)$, *then* $(a_1 *_\sigma a_2) *_\sigma a_3 = a_1 *_\sigma (a_2 *_\sigma a_3)$ *and* $(a_1 *_\sigma a_2, a_3) = (a_1, a_3 *_\sigma \tilde{a}_2)$;
(v) $(A^{-1}U)(x,\xi) = (2\pi)^{-n/2}\int U(y/2 - x, y/2 + x)e^{i\langle y,\xi\rangle}\,dy$.

Next we discuss elements of rank one. From [9,10,12] we recall that $u \in L^2(W)$ is called *simple* if $u = \tilde{u} *_\sigma u$ and $\|u\|_{L^2} = 1$.

**Proposition 1.3.** *Assume that $u \in \mathscr{S}'(W)$. Then the following conditions are equivalent.*

(1) *$u$ is simple*;
(2) $Au = (2\pi)^{n/2}(\mathscr{F}_\sigma u)^w(x,D)$ *is an orthonormal projection of rank one*;
(3) $u = A^{-1}(f \otimes \bar{f})$, *for some unit vector* $f \in L^2(V)$.

**Remark 1.4.** We note that the set of simple elements is invariant under compositions by linear symplectic mappings. (Recall that the map $T$ on $W$ is called symplectic if $\sigma(TX, TY) = \sigma(X, Y)$ for every $X, Y \in W$.) In fact, it follows easily that if $u$ is simple, $T$ is a linear symplectic map and $u_T = u \circ T$, then $u_T = \tilde{u}_T *_\sigma u_T$ and $\|u_T\|_{L^2} = 1$, which proves the assertion.

We shall also consider the set $\mathscr{B}(W)$ of all sequences $(v_j)$ in $L^2(W)$ such that $\|v_j\|_{L^2} = 1$, $\tilde{v}_j *_\sigma v_j$ is simple and $\tilde{v}_j *_\sigma v_k = 0$ when $j \neq k$. Then $(v_j) \in \mathscr{B}(W)$, if and only if $Av_j = f_j \otimes g_j$ for every $j$, where $(f_j), (g_j) \in \mathrm{ON}(V)$. Here $\mathrm{ON}(V)$ is the family of orthonormal sets in $L^2(V)$. We let also $\mathscr{B}^0(W)$ be the set of all finite sequences $(v_j) \in \mathscr{B}(W)$ such that $v_j \in \mathscr{S}(W)$ for every $j$. Then $(v_j) \in \mathscr{B}^0(W)$, if and only if $Av_j = f_j \otimes g_j$, for some $(f_j), (g_j) \in \mathrm{ON}_0(V)$, where $\mathrm{ON}_0(V)$ is the set of all finite sequences $(f_j) \in \mathrm{ON}(V)$ such that $f_j \in \mathscr{S}(V)$ for every $j$.

We shall now discuss the Schatten–von Neumann classes, and the $s_p$-spaces from the introduction. Assume that $p \in [1,\infty]$. Then $\mathscr{I}_p$, the set of Schatten–von Neumann operators of order $p$ on $L^2(V)$, consists of all operators $T$ on $L^2(V)$ such that

$$\|T\|_{\mathscr{I}_p} \equiv \sup\big\|\big((Tf_j, g_j)\big)_{j=1}^\infty\big\|_{l^p} \tag{1.3}$$

is finite. (Cf. [8].) Here the supremum is taken over all $(f_j)_{j=1}^\infty$ and $(g_j)_{j=1}^\infty$ in $\mathrm{ON}(V)$ (or alternatively in $\mathrm{ON}_0(V)$). We note that $\mathscr{I}_1$, $\mathscr{I}_2$ and $\mathscr{I}_\infty$ are the sets of trace-class, Hilbert–Schmidt and continuous operators on $L^2$ respectively.

We recall that $s_p(W)$ consists of all $a \in \mathcal{S}'(W)$ such that $Aa \in \mathcal{I}_p$, which is the same as $a^w(x,D) \in I_p$. The topology on $s_p(W)$ is then defined by the norm $\|a\|_{s_p} \equiv \|Aa\|_{\mathcal{I}_p}$, and it follows that the map $a \mapsto Aa$ from $s_p(W)$ to $\mathcal{I}_p$ is an isometric homeomorphism. We also let $s_\infty^0(W)$ be the set of all $a \in \mathcal{S}'(W)$ such that $Aa$ is compact on $L^2(V)$.

**Proposition 1.5.** *The following holds for the $s_p$-spaces*:

(1) *the set $s_p(W)$, $p \in [1,\infty]$, is a Banach space. If $p_1 \leqslant p_2 < \infty$, then $\mathcal{S} \subset s_{p_1} \subset s_{p_2} \subset s_\infty^0 \subset s_\infty$. One has $\|a\|_{s_\infty} \leqslant \|a\|_{s_{p_2}} \leqslant \|a\|_{s_{p_1}}$ and $\|a\|_{L^\infty} \leqslant (2/\pi)^{n/2}\|a\|_{s_1}$. It holds that $s_2 = L^2$ with equality in norms, and that $s_1 \subset C_B$;*

(2) *assume that $p, p' \in [1,\infty]$ satisfies $1/p + 1/p' = 1$. Then the products $(\cdot\,,\cdot)$ and $\langle\cdot\,,\cdot\rangle$ on $\mathcal{S}(W)$ extends uniquely to dualities between $s_p(W)$ and $s_{p'}(W)$, and one has for every $a \in s_p(W)$ and $b \in s_{p'}(W)$ that*

$$\max\bigl(|\langle a,b\rangle|, |(a,b)|\bigr) \leqslant \|a\|_{s_p}\|b\|_{s_{p'}}, \quad \|a\|_{s_p} = \sup\bigl|(a,c)\bigr| = \sup\bigl|\langle a,c\rangle\bigr|,$$

*where the supremums should be taken over all $c \in s_{p'}(W)$ such that $\|c\|_{s_{p'}} = 1$;*

(3) *the set $\mathcal{S}(W)$ is dense in $s_p(W)$, when $p < \infty$, and in $s_\infty^0(W)$. It is dense in $s_\infty(W)$ with respect to the weak* topology.*

The trace of an element $a \in s_1(W)$ is defined by $\mathrm{tr}(a) \equiv \mathrm{Tr}(Aa)$, where the right-hand side is the usual trace defined on the operator class $\mathcal{I}_1$, i.e. if $T \in \mathcal{I}_1$, then $\mathrm{Tr}(T) = \sum(Tf_j, f_j)$, where $(f_j)$ is an orthonormal basis for $L^2(V)$. It is often convenient to use the trace when dealing with positivity questions in operator calculus. Indeed, for any $T \in \mathcal{I}_1$, we have that $|\mathrm{Tr}(T)| \leqslant \|T\|_{\mathcal{I}_1}$, and that $\mathrm{Tr}(T) = \|T\|_{\mathcal{I}_1}$ if and only if $T$ is a positive semi-definite operator. This in turn implies that $a \in s_1(W)$ and $\mathrm{tr}(a) = \|a\|_{s_1}$ if and only if $a \in \mathcal{S}'(W)$ and $Aa$ is a positive semi-definite trace-class operator.

**Proposition 1.6.** *Assume that $a \in \mathcal{S}'(W)$ and $p \in [1,\infty]$. Then the following is true*:

(1) *the mappings $a \mapsto Aa$ and $a \mapsto a^w(x,D)$ are homeomorphisms from $s_p(W)$ to $\mathcal{I}_p$. Moreover, $\|a^w(x,D)\|_{\mathcal{I}_p} = (2\pi)^{-n/2}\|a\|_{s_p}$;*

(2) *one has that $\|a\|_{s_p} = \sup(\sum|(a,u_j)|^p)^{1/p}$, where the supremum should be taken over all sequences $(u_j) \in \mathcal{B}(W)$. It suffices to take the supremum over all $(u_j) \in \mathcal{B}^0(W)$. In particular, $\|a\|_{s_p}$ is independent of the choice of symplectic coordinates on $W$;*

(3) *if $a \in s_1(W)$, then $\mathrm{tr}(a) = (\pi/2)^{n/2}a(0)$. Moreover, $(\pi/2)^{n/2}a(0) = \|a\|_{s_1}$, if and only if $Aa$ is a positive semi-definite trace-class operator.*

In the next result we discuss extensions of the twisted convolution product.

**Proposition 1.7.** *Assume that $p,q,r \in [1,\infty]$ satisfies the Hölder condition $1/p + 1/q = 1/r$. Then the twisted convolution and the Weyl product extend uniquely to continuous bilinear mappings from $s_p(W) \times s_q(W)$ to $s_r(W)$, and for any $a \in s_p(W)$ and $b \in s_q(W)$ one has $\|a *_\sigma b\|_{s_r} \leqslant \|a\|_{s_p}\|b\|_{s_q}$. On the other hand, if $c \in s_r(W)$, then $c = a *_\sigma b$*

*and* $\|c\|_{s_r} = \|a\|_{s_p}\|b\|_{s_q}$, *for some choice of* $a \in s_p(W)$ *and* $b \in s_q(W)$. *If in addition* $c \in \mathcal{S}'_+(W)$ *and* $p = q$, *then* $c = a *_\sigma \tilde{a}$ *and* $\|c\|_{s_r} = \|a\|_{s_p}^2$ *for some* $a \in s_p(W)$.

*Moreover,* (1.2) *is still valid when* $a, b \in s_\infty(W)$.

In the following proposition we consider decomposition of elements in the $s_p$-spaces.

**Proposition 1.8.** *Assume that* $a \in s_\infty^0(W)$. *Then for some* $(u_j) \in \mathcal{B}(W)$ *and* $\lambda = (\lambda_j) \in l^\infty$ *such that* $0 \leqslant \lambda_j \to 0$ *as* $j \to \infty$, *one has that* $a = \sum \lambda_j u_j$, *with convergence in* $s_\infty(W)$. *If* $1 \leqslant p < \infty$, *then* $a \in s_p(W)$ *if and only if* $\lambda \in l^p$, *and then* $\|a\|_{s_p} = \|\lambda\|_{l^p}$ *and* $\sum \lambda_j u_j$ *converges in* $s_p(W)$.

*Moreover, if* $a \in \mathcal{S}'_+(W) \cap s_\infty^0(W)$, *then* $u_j$ *is simple when* $\lambda_j \neq 0$.

Next we discuss relations between the symplectic Fourier transform, composition by affine symplectic transformations and the twisted convolution.

**Proposition 1.9.** *The symplectic Fourier transform and composition by any affine symplectic transformation on* $W$ *are unitary mappings on* $s_\infty^0(W)$ *and on* $s_p(W)$, *for every* $1 \leqslant p \leqslant \infty$. *If* $a, b \in s_\infty(W)$, *then*

$$\mathcal{F}_\sigma(a *_\sigma b) = (\mathcal{F}_\sigma a) *_\sigma b = \check{a} *_\sigma (\mathcal{F}_\sigma b). \tag{1.4}$$

We note that if $Au = f \otimes \bar{f}$ where $f \in \mathcal{S}(V)$, and $a \in \mathcal{S}'(W)$, then $((Aa)f, f) = (Aa, Au) = (a, u)$, since $A$ is a unitary map on $L^2$. Hence Lemma 1.1, and the Propositions 1.3, 1.5, 1.7 and 1.8 give the following.

**Proposition 1.10.** *Assume that* $a \in \mathcal{S}'(W)$. *Then the following conditions are equivalent*:

(1) $(a *_\sigma \varphi, \varphi) \geqslant 0$ *for every* $\varphi \in C_0^\infty(W)$, *i.e.* $a \in \mathcal{S}'_+(W)$;
(2) $((Aa)f, f) = (2\pi)^{n/2}((\mathcal{F}_\sigma a)^w(x, D)f, f) \geqslant 0$ *for every* $f \in \mathcal{S}(V)$;
(3) $(a, u) \geqslant 0$ *for every* $u \in \mathcal{S}(W)$ *which is simple*;
(4) $(a, u) \geqslant 0$ *for every* $u \in \mathcal{S}'_+(W) \cap \mathcal{S}(W)$.

*Moreover,* $a \in s_\infty(W) \cap \mathcal{S}'_+(W)$ *if and only if* $(a *_\sigma \varphi, \varphi) \geqslant 0$ *for every* $\varphi \in L^2(W)$.

**Remark 1.11.** We note that Remark 1.4 and Proposition 1.10 implies that $\mathcal{S}'_+(W)$ is invariant under composition by linear symplectic transformations.

## 2. A generalization of Bochner–Schwartz theorem

In this section we shall give an extension of Bochner–Schwartz theorem to noncommutative convolutions of the type (0.2). In the first part we make a somewhat classical approach to the topic and present some general facts for positive elements in the algebra which are at the same time continuous functions. The considerations are similar to the usual treatments for the positive definite functions (cf. Section IX.2 in [7]).

We start by introducing the notion of $B$-positive functions, an analogue of positive definite functions.

Assume that $a \in C(\mathbf{R}^m)$. Then $a$ is said to be $B$-positive (or a $B$-positive function), if for every pairs of sequences $c_1, \dots, c_M \in \mathbf{C}$ and $x_1, \dots, x_M \in \mathbf{R}^m$, one has

$$\sum_{j,k=1}^{M} a(x_j - x_k) B(x_j, x_k) c_j \bar{c}_k \geqslant 0. \tag{2.1}$$

Here and in what follows we assume that $B \in C^\infty(\mathbf{R}^m \oplus \mathbf{R}^m; \mathbf{C})$, and that for every integer $N \geqslant 0$ we may find a polynomial $P_N$ on $\mathbf{R}^m \oplus \mathbf{R}^m$ such that

$$\left|B^{(\alpha)}(x, y)\right| + \left|B(x, y)\right|^{-1} \leqslant P_N(x, y), \quad |\alpha| \leqslant N. \tag{2.2}$$

We note that if $B(x, y) = 1$ everywhere, then we obtain the definition of positive definite functions, treated by Bochner, and if $\mathbf{R}^m$ is replaced by $W$, and that $B(X, Y) = \mathrm{e}^{2i\sigma(X,Y)}$, then we obtain the definition of $\sigma$-positive functions from the introduction. We let $C_{B,+}(\mathbf{R}^m)$ denote the set of $B$-positive functions.

It follows from (2.2) that $B(0, 0) \neq 0$, and by multiplying $a$ and $B$ with appropriate constants, we may from now on assume that $B(0, 0) = 1$.

**Proposition 2.1.** *Assume that $a \in \mathscr{D}'(\mathbf{R}^m)$. Then $a \in C_{B,+}(\mathbf{R}^m)$, if and only if $a$ is continuous and satisfies $(a *_B \varphi, \varphi) \geqslant 0$, for every $\varphi \in C_0^\infty(\mathbf{R}^m)$.*

**Proof.** If $a \in C_{B,+}$ and $\varphi \in C_0^\infty$ we obtain $(a *_B \varphi, \varphi) \geqslant 0$ if we approximate the integral expression of the left-hand side by a Riemann sum. On the other hand, if $a$ is continuous and $(a *_B \varphi, \varphi) \geqslant 0$ for every $\varphi \in C_0^\infty$, then (2.1) follows if one chooses $\varphi(x) = \sum_1^M \overline{c_j} \varepsilon^{-m} \psi((x - x_j)/\varepsilon)$ with $\psi \in C_0^\infty$, $\int \psi(x)\, dx \neq 0$ and then lets $\varepsilon$ tend to 0.

**Remark 2.2.** It follows immediately from Proposition 2.2 that $a \in C_{B,+}$, if and only if $a$ is continuous and that the operator with Schwartz kernel $a(x - y)B(x, y)$ is positive semi-definite.

**Remark 2.3.** Assume that $a \in C_{B,+}(\mathbf{R}^m)$. Then it follows from Proposition 2.1 and Remark 2.2 that

$$\begin{aligned} &a(0)B(x, x) \geqslant 0, \qquad a(x - y)B(x, y) = \overline{a(y - x)B(y, x)}, \\ &\left|a(x - y)B(x, y)\right|^2 \leqslant a(0)^2 B(x, x)B(y, y) \quad x, y \in \mathbf{R}^m. \end{aligned} \tag{2.3}$$

From this fact one obtains the following:

(1) if $0 \neq a$, then $a(0) > 0$ and $B(x, x) > 0$ for every $x \in \mathbf{R}^m$;
(2) if $(x, y) \in \Omega_B$, then $B(y, x) = c(x - y)\overline{B(x, y)}$ for some $c \in C^\infty(\omega)$ such that $\tilde{c}(x)c(x) = 1$ and $c(0) = 1$. Here $\tilde{c}(x) = \overline{c(-x)}$, $\omega = \{x - y;\ (x, y) \in \Omega_B\}$ and

$$\Omega_B = \left\{(x, y) \in \mathbf{R}^m \oplus \mathbf{R}^m;\ a(x - y) \neq 0 \text{ for some } a \in C_{B,+}\big(\mathbf{R}^m\big)\right\};$$

(3) if $B(y, x) = \overline{B(x, y)}$ for every $(x, y)$, then $\tilde{a} = a$, where $\tilde{a}(x) = \overline{a(-x)}$;

(4) if $B(x,x) \leqslant \max(|B(x,0)|^2, |B(0,-x)|^2)$ for every $x \in \mathbf{R}^m$, then $|a(x)| \leqslant a(0)$ for every $x \in \mathbf{R}^m$.

**Remark 2.4.** We note here that if $\Omega_B$ is nonempty in Remark 2.3, then we may always reduce ourselves such that $B(x, y)$ satisfies one of the following conditions:

(1) $B(x,x) = 1$ and $|B(x,0)| = 1$ for every $x \in \mathbf{R}^m$;
(2) $B(y,x) = c(x-y)\overline{B(x,y)}$ when $(x,y) \in \overline{\Omega_B}$, where $c \in C^\infty(\mathbf{R}^m)$ is even and satisfies $|c(x)| = 1$ for every $x \in \mathbf{R}^m$. If in addition $c(x) = \varphi(x)^2$, for some $\varphi \in C^\infty$, then we may reduce ourselves such that $B(y,x) = \overline{B(x,y)}$.

In fact, (1) follows if we first replace $B(x,y)$ in Proposition 2.1 by $B_1(x,y) = B(x,y)(B(x,x)B(y,y))^{-1/2}$, and then replace $a(x)$ and $B_1(x,y)$ with $a(x)B_1(x,0)$ and $B_1(x-y,0)^{-1}B_1(x,y)$ respectively. The assertion (2) follows if we replace $a(x)$ and $B(x,y)$ in Remark 2.3(2) with $a(x)|c(x)|^{-1/2}$ and $B(x,y)\cdot|c(x-y)|^{1/2}$.

According to Proposition 2.1, we say that $a \in \mathscr{D}'(\mathbf{R}^m)$ is a $B$-positive distribution if $(a *_B \varphi, \varphi) \geqslant 0$ for every $\varphi \in C_0^\infty(\mathbf{R}^m)$. The set of $B$-positive distributions is denoted by $\mathscr{S}'_{B,+}(\mathbf{R}^m)$. (A motivation for using $\mathscr{S}'_{B,+}$ instead of $\mathscr{D}'_{B,+}$ is given by Theorem 2.6 below.) Then Proposition 2.1 implies that $C_{B,+}(\mathbf{R}^m) = \mathscr{S}'_{B,+}(\mathbf{R}^m) \cap C(\mathbf{R}^m)$.

We shall now discuss our generalization of the Bochner–Schwartz theorem to weighted convolutions. It is then convenient to use $\mathscr{S}^N(\mathbf{R}^m)$, the set of all $f \in C^N(\mathbf{R}^m)$ such that

$$\|f\|_{(M)} = \|f\|_{(N,M)} \equiv \sum_{|\alpha|\leqslant M} \sum_{|\beta|\leqslant N} \sup_{x\in\mathbf{R}^m} \left|x^\alpha \partial^\beta f(x)\right|$$

is finite for every $M \geqslant 0$. Then $\mathscr{S}^N(\mathbf{R}^m)$ is a Frechét space, and it is obvious that $\mathscr{S} \subset \mathscr{S}^N \subset C^N \cap L^1 \cap L^\infty$.

In the following proposition we list some important properties for $\mathscr{S}^N$. We leave the simple proof for the reader.

**Proposition 2.5.** *Assume that $N \geqslant 0$ is an integer and that $B \in C^\infty(\mathbf{R}^m \oplus \mathbf{R}^m)$ satisfies* (2.2) *for some polynomial $P_N$. Then the following is true*:

(i) *if $T$ is a linear homeomorphism on $\mathbf{R}^m$ and $a \in \mathscr{S}^N(\mathbf{R}^m)$, then $a \circ T \in \mathscr{S}^N(\mathbf{R}^m)$*;
(ii) *the map $f \mapsto B \cdot f$ is continuous on $\mathscr{S}^N(\mathbf{R}^m \oplus \mathbf{R}^m)$*;
(iii) *the weighted convolution product $*_B$ in* (0.2) *is a continuous bilinear mapping on $\mathscr{S}^N(\mathbf{R}^m)$*;
(iv) *$C_0^\infty(\mathbf{R}^m)$ is dense in $\mathscr{S}^N(\mathbf{R}^m)$.*

Our generalization of the Bochner–Schwartz theorem is the following:

**Theorem 2.6.** *Assume that $B \in C^\infty(\mathbf{R}^m \oplus \mathbf{R}^m)$ and that for any integer $N \geqslant 0$, there is a polynomial $P_N$ on $\mathbf{R}^m \oplus \mathbf{R}^m$ such that* (2.2) *holds. Then $\mathscr{S}'_{B,+}(\mathbf{R}^m) \subset \mathscr{S}'(\mathbf{R}^m)$.*

Since any distribution has finite orders on compact sets, it follows that Theorem 2.6 is a consequence of the following result.

**Theorem 2.6′.** *Let $N_0 \geqslant 0$ be an integer and assume that $B \in C^\infty(\mathbf{R}^m \oplus \mathbf{R}^m)$ satisfies* (2.2) *for some polynomial $P_N$, where $N = 2N_0$. Assume also that $a \in \mathscr{S}'_{B,+}(\mathbf{R}^m)$ satisfies that $\chi a \in \mathscr{D}'^N(\mathbf{R}^m)$, for some $\chi \in C_0^\infty(\mathbf{R}^m)$ such that $\chi(0) \neq 0$. Then the mapping $(\varphi, \psi) \mapsto (a *_B \varphi, \psi)$ from $C_0^\infty(\mathbf{R}^m) \times C_0^\infty(\mathbf{R}^m)$ to $\mathbf{C}$ extends uniquely to a continuous mapping from $\mathscr{S}^{N_0}(\mathbf{R}^m) \times \mathscr{S}^{N_0}(\mathbf{R}^m)$ to $\mathbf{C}$. If in addition* (2.2) *is true for every $N$, for some polynomials $P_N$, then $a \in \mathscr{S}'(\mathbf{R}^m)$.*

**Proof.** Since $a \in \mathscr{S}'_{B,+}$, it follows that $(\cdot\,,\cdot)_a$ and $\|\cdot\|_a$, on $C_0^\infty$, are well-defined semi-scalar product and semi-norm respectively, where

$$(\varphi, \psi)_a \equiv (a *_B \varphi, \psi), \quad \text{and} \quad \|\varphi\|_a \equiv (\varphi, \varphi)_a^{1/2} = (a *_B \varphi, \varphi)^{1/2}, \tag{2.4}$$

when $\varphi, \psi \in C_0^\infty(\mathbf{R}^m)$. In particular one has the Schwartz inequality, $|(\varphi, \psi)_a| \leqslant \|\varphi\|_a \|\psi\|_a$. We first prove that the map $(\varphi, \psi) \mapsto (\varphi, \psi)_a$ extends to a continuous map from $\mathscr{S}^N \times \mathscr{S}^N$ to $\mathbf{C}$. By Proposition 2.5(iv) it sufficies to prove that for some constant $C > 0$ and integer $M > 0$ we have

$$\left|(\varphi, \psi)_a\right| \leqslant C\|\varphi\|_{(N_0,M)}\|\psi\|_{(N_0,M)}, \quad \varphi, \psi \in C_0^\infty\big(\mathbf{R}^m\big). \tag{2.5}$$

In order to prove (2.5) we take a neighbourhood $\Omega \subset \mathbf{R}^m$ of the origin such that $\chi \neq 0$ in a neighbourhood of $\overline{\Omega}$. By multiplying $\chi a$ with some appropriate test function if necessary, we may assume that $\chi = 1$ in $\Omega$.

Take an even and nonnegative function $\phi \in C_0^\infty(\mathbf{R}^m)$, which satisfies $\sum_{j \in J} \phi(\cdot - x_j) = 1$, for some lattice $\{x_j\}_{j \in J} \subset \mathbf{R}^m$, and such that $\operatorname{supp} \phi + \operatorname{supp} \phi \subset \Omega$. By Cauchy–Schwartz inequality we get

$$\left|(\varphi, \psi)_a\right| \leqslant \sum_{j,k \in J} \left|(\varphi_j, \psi_k)_a\right| \leqslant \sum_{j,k \in J} \|\varphi_j\|_a \|\psi_k\|_a, \quad \varphi, \psi \in C_0^\infty\big(\mathbf{R}^m\big), \tag{2.6}$$

where $\varphi_j(x) = \varphi(x)\phi(x - x_j)$, $\psi_j(x) = \psi(x)\phi(x - x_j)$. The assertion (2.5) will follow from (2.6) if we prove that for any $M_1 \geqslant 0$, there is an integer $M \geqslant 0$ and a constant $C_{M_1}$, independent of $\varphi$ and $j$ such that

$$\|\varphi_j\|_a \leqslant C_{M_1}\|\varphi\|_{(N_0,M)}\big(1 + |x_j|\big)^{-M_1}. \tag{2.7}$$

In fact, a combination of (2.6) and (2.7) gives

$$\left|(\varphi, \psi)_a\right| \leqslant \left(C_{M_1} \sum_{j \in J} \big(1 + |x_j|\big)^{-M_1}\right)^2 \|\varphi\|_{(N_0,M)}\|\psi\|_{(N_0,M)},$$

and (2.5) follows if we choose $M_1 > m$.

In order to prove (2.7), we note that $\|\varphi_j\|_a^2 = (a, \varphi_j *_{B_1} \tilde{\varphi}_j)$, where $\tilde{\varphi}_j(x) = \overline{\varphi_j(-x)}$ as before and $B_1(x, y) = \overline{B(x - y, -y)}$. Then (2.2) is fulfilled for some polynomial $P_N$, when $B$ is replaced by $B_1$. Since the support of $\varphi_j(\cdot + x_j)$ is contained in $\operatorname{supp} \phi$, it follows easily from

$$(\varphi_j *_{B_1} \tilde{\varphi}_j)(x) = \int \varphi_j(x + y + x_j)\overline{\varphi_j(y + x_j)}B_1\big(x, -(y + x_j)\big)\,dy, \tag{2.8}$$

that the support of $\varphi_j *_{B_1} \tilde{\varphi}_j$ is contained in $\Omega$, and that the function in the integral in (2.8) vanishes outside $\mathcal{M} = \{(x, y);\ x \in \Omega\ y \in \operatorname{supp} \phi\}$. Hence

$$\|\varphi_j\|_a^2 = (a, \varphi_j *_{B_1} \tilde{\varphi}_j) = (\chi a, \varphi_j *_{B_1} \tilde{\varphi}_j) \leqslant C\|\varphi_j *_{B_1} \tilde{\varphi}_j\|_{(N,0)},$$

for some constant $C$, since $\chi a$ is a distribution of order $N$.

From (2.2) and the fact that $\mathcal{M}$ is bounded we get for some constants $C$ and $M_2$, independent of $j$, that $|D^\alpha B_1(x, -(y+x_j))| \leqslant C(1+|x_j|)^{2M_2}$ when $|\alpha| \leqslant N$ and $(x, y) \in \mathcal{M}$. By partial integrations it follows also that for any $|\alpha| \leqslant N$, then $D^\alpha(\varphi_j *_{B_1} \tilde{\varphi}_j)$ is a sum of terms of the type

$$\int \big(D^\beta \varphi_j\big)(x+y+x_j)\overline{(D^\gamma \varphi_j)(y+x_j)}\big(D^\delta B_1\big)\big(x, -(y+x_j)\big)\, dy,$$

where $|\beta|, |\gamma| \leqslant N_0$ and $|\delta| \leqslant N$. This gives $\|\varphi_j *_{B_1} \tilde{\varphi}_j\|_{(N,0)} \leqslant C_1(1+|x_j|)^{2M_2}\|\varphi_j(\cdot + x_j)\|^2_{(N_0,0)}$, and we may conclude that

$$\|\varphi_j\|_a \leqslant C_2\big(1+|x_j|\big)^{M_2}\big\|\varphi_j(\cdot + x_j)\big\|_{(N_0,0)}.$$

The assertion follows therefore if we prove for every $M \geqslant 0$ that

$$\big\|\varphi_j(\cdot + x_j)\big\|_{(N_0,0)} \leqslant C_M\big(1+|x_j|\big)^{-M}\|\varphi\|_{(N_0,M)}, \tag{2.9}$$

for some constant $C_M$. By using that $(1+|x_j|) \leqslant (1+|x|)(1+|x+x_j|)$ we get for any $M \geqslant 0$ that

$$\begin{aligned}\big(1+|x_j|\big)^M\big\|\varphi_j(\cdot + x_j)\big\|_{(N_0,0)} &= \sum_{|\alpha|\leqslant N_0}\big\|\big(1+|x_j|\big)^M \partial^\alpha\big(\varphi(\cdot + x_j)\phi\big)\big\|_{L^\infty} \\ &\leqslant C\bigg(\sum_{|\alpha|\leqslant N_0}\big\|\big(1+|\cdot|\big)^M \phi^{(\alpha)}\big\|_{L^\infty}\bigg) \\ &\quad \times \bigg(\sum_{|\alpha|\leqslant N_0}\big\|\big(1+|\cdot + x_j|\big)^M \varphi^{(\alpha)}(\cdot + x_j)\big\|_{L^\infty}\bigg) \\ &\leqslant C_\phi\|\varphi\|_{(N_0,M)},\end{aligned}$$

and (2.9) follows.

It remains to prove the last part of the theorem. Assume therefore that for any $N \geqslant 0$ we may find a polynomial $P_N$ such that (2.2) holds and let $T : C_0^\infty(\mathbf{R}^m) \to \mathscr{D}'(\mathbf{R}^m)$ be the linear operator with Schwartz kernel $(x, y) \mapsto a(x-y)B(x, y)$. Since $(\varphi, \psi)_a = (T\varphi, \psi)$, it follows from (2.5) that $T$ extends uniquely to a continuous operator from $\mathscr{S}(\mathbf{R}^m)$ to $\mathscr{S}'(\mathbf{R}^m)$. Hence $a(x-y)B(x, y) \in \mathscr{S}'(\mathbf{R}^m \oplus \mathbf{R}^m)$ by the kernel theorem of Schwartz, and from (i) and (ii) in Proposition 2.5 it follows that $a \otimes 1 \in \mathscr{S}'(\mathbf{R}^m \oplus \mathbf{R}^m)$. This implies that $a \in \mathscr{S}'(\mathbf{R}^m)$, and the theorem follows.

**Remark 2.7.** In the classical Bochner–Schwartz theorem one has also that if $a \in \mathscr{S}'_{B,+}$, with $B = 1$ everywhere and that $\chi a \in \mathscr{D}'^{2N_0}$ for some integer $N_0 \geqslant 0$ and $\chi \in C_0^\infty$ such that $\chi(0) \neq 0$, then $a \in \mathscr{D}'^{2N_0}$. If this is true for general $B$ in Theorem 2.6 is not known to the author.

In the special case $B(X,Y) = \mathrm{e}^{2i\sigma(X,Y)}$ we can say more.

**Proposition 2.8.** *Let $W$ be a finite dimensional symplectic vector space, $N = 2N_0 \geqslant 0$ is even, and assume that $a \in \mathscr{S}'_+(W)$ such that $\chi a \in \mathscr{D}'^N$ for some $\chi \in C_0^\infty(W)$ which satisfies $\chi(0) \neq 0$. Then the mappings $(\varphi,\psi) \mapsto (a *_\sigma \varphi, \psi)$ and $(\varphi,\psi) \mapsto ((\mathscr{F}_\sigma a) *_\sigma \varphi, \psi)$ from $C_0^\infty(W) \times C_0^\infty(W)$ to $\mathbf{C}$ extend to continuous bilinear mappings from $\mathscr{S}^{N_0}(W) \times \mathscr{S}^{N_0}(W)$ to $\mathbf{C}$. Moreover, $\mathscr{S}'_+(W) \subset \mathscr{S}'(W)$.*

**Proof.** The result except the assertion concerning $\hat{a}$ follows immediately from Theorem 2.6 and Theorem 2.6′ with $B(X,Y) = \mathrm{e}^{2i\sigma(X,Y)}$.

In order to prove the extension for $(\varphi,\psi) \mapsto ((\mathscr{F}_\sigma a) *_\sigma \varphi, \psi)$ it follows from the proof of Theorem 2.6′ that it sufficies to prove that

$$\big|\big((\mathscr{F}_\sigma a) *_\sigma \varphi, \psi\big)\big| \leqslant C\|\varphi\|_{(N_0,M)}\|\psi\|_{(N_0,M)}, \quad \varphi,\psi \in C_0^\infty(W), \tag{2.5′}$$

for some constants $M$ and $C$. By Proposition 1.9 and Schwartz inequality we get

$$\big|\big((\mathscr{F}_\sigma a) *_\sigma \varphi, \psi\big)\big| = \big|(a *_\sigma \varphi, \mathscr{F}_\sigma \psi)\big| = \big|(\varphi, \mathscr{F}_\sigma \psi)_a\big| \leqslant \|\varphi\|_a \|\mathscr{F}_\sigma \psi\|_a,$$

and since $\|\varphi\|_a \leqslant C^{1/2}\|\varphi\|_{(N_0,M)}$, for some $M$, by (2.5), the assertion follows if we prove that $\|\mathscr{F}_\sigma \psi\|_a \leqslant C\|\psi\|_{(N_0,M)}$. From Proposition 1.9 we have that

$$\|\mathscr{F}_\sigma \psi\|_a^2 = (a *_\sigma \mathscr{F}_\sigma \psi, \mathscr{F}_\sigma \psi) = (\check{a} *_\sigma \psi, \psi) = (a *_\sigma \check{\psi}, \check{\psi}) = \|\check{\psi}\|_a^2,$$

and the result follows since $\|\check{\psi}\|_a \leqslant C\|\check{\psi}\|_{(N_0,M)} = C\|\psi\|_{(N_0,M)}$. The proof is complete.

The technique used in the proof of Theorem 2.6 works also in other situations, for example in the following result. We leave the verifications for the reader.

**Theorem 2.9.** *Assume that $T$ is a continuous and linear operator from $C_0^\infty(\mathbf{R}^m)$ to $\mathscr{D}'(\mathbf{R}^m)$ with Schwartz kernel $K \in \mathscr{D}'(\mathbf{R}^m \oplus \mathbf{R}^m)$, and let $T_\phi$ be the operator with Schwartz kernel $K_\phi(x,y) = \phi(x-y)K(x,y)$, where $\phi \in C_0^\infty(\mathbf{R}^m)$ satisfies $\phi(0) \neq 0$. Assume also that the following properties are fulfilled*:

(1) $(Tf, f) \geqslant 0$ *for every* $f \in C_0^\infty(\mathbf{R}^m)$;
(2) $T_\phi$ *extends to a continuous map from* $\mathscr{S}(\mathbf{R}^m)$ *to* $\mathscr{S}'(\mathbf{R}^m)$.

*Then $K \in \mathscr{S}'(\mathbf{R}^m \oplus \mathbf{R}^m)$, and $T$ extends uniquely to a continuous map from $\mathscr{S}(\mathbf{R}^m)$ to $\mathscr{S}'(\mathbf{R}^m)$.*

*Moreover, if $K_\phi \in \mathscr{D}'^N$, then the map $(\varphi,\psi) \mapsto (T\varphi,\psi)$ from $C_0^\infty \times C_0^\infty$ to $\mathbf{C}$ extends uniquely to a continuous mapping from $\mathscr{S}^N \times \mathscr{S}^N$ to $\mathbf{C}$.*

## 3. The $\sigma$-positive functions

In this section we make a brief discussion of $C_+(W)$, the set of $\sigma$-positive functions (cf. the introduction). We give a complete characterization of such functions, and prove that if $A$ is the mapping in (0.3) and $a \in \mathscr{S}'(W)$, then $a$ is $\sigma$-positive, if and only if $Aa$ is a positive semi-definite trace-class operator. The result is applied in different ways. It will for example be used in order to prove that $C^\infty(W) \cap C_+(W) \subset \mathscr{S}(W)$. The results apply also to the Weyl calculus, where we conclude that $a$ is $\sigma$-positive, if and only if $(\mathscr{F}_\sigma a)^w(x, D)$ is a positive semi-definite trace-class operator.

Assume that $W$ is a symplectic vector space of the finite dimension $2n$ with symplectic form $\sigma$ as before. It follows from the definitions that if $B(X, Y) = \mathrm{e}^{2i\sigma(X,Y)}$, then $C_{B,+}(W) = C_+(W)$, $B(Y, X) = \overline{B(X, Y)}$, and (1)–(2) in Remark 2.4 are fulfilled. In particular, the following two propositions are immediately consequences of Proposition 2.1 and Remark 2.3.

**Proposition 3.1.** *If $a$ is $\sigma$-positive then $a(X) = \overline{a(-X)}$ and $|a(X)| \leqslant a(0)$ for every $X \in W$.*

**Proposition 3.2.** *One has that $C_+(W) = \mathscr{S}'_+(W) \cap C(W)$.*

We recall that $\mathscr{S}'_+(W)$, and therefore $C_+(W)$ are invariant under composition with linear symplectic transformations and under multiplication by any exponential function $X \mapsto \mathrm{e}^{i\sigma(X,Y)}$, where $Y \in W$. Thus $\mathscr{F}_\sigma C_+(W)$ and $\mathscr{F}_\sigma \mathscr{S}'_+(W)$ are invariant under affine canonical transformations.

Note that it follows from Proposition 1.10 and Proposition 3.2 that if $a \in s_\infty(W)$ is bounded and continuous, then $a$ is $\sigma$-positive if and only if $Aa$ is a positive semi-definite continuous operator on $L^2$. More generally one has that $a$ is $\sigma$-positive if and only if $Aa$ is a positive semi-definite trace-class operator on $L^2(V)$, which is a consequence of the following result.

**Theorem 3.3.** *Assume that $a \in \mathscr{D}'(W)$. Then the following conditions are equivalent*:

(1) *$a \in s_1(W)$ and $a$ is $\sigma$-positive*;
(2) *$a$ is $\sigma$-positive*;
(3) *$a \in \mathscr{S}'_+(W)$, and in some neighbourhood $\Omega$ of the origin, the restriction of $a$ to $\Omega$ is a measure $d\mu$, which satisfies*

$$\liminf_{\varepsilon \to 0} \varepsilon^{-2n} \int_{|X|<\varepsilon} \left|d\mu(X)\right| < \infty; \tag{3.1}$$

(4) *$a \in \mathscr{S}'_+(W)$ and, for some $\chi \in \mathscr{S}(W)$ with nonvanishing integral one has that*

$$\liminf_{\varepsilon \to 0} (a *_\sigma \chi_\varepsilon, \chi_\varepsilon) < \infty, \tag{3.1$'$}$$

*where $\chi_\varepsilon = \varepsilon^{-2n} \chi(\cdot/\varepsilon)$.*

For the proof we shall need some lemmas.

**Lemma 3.4.** *The map $(\psi, a, \chi) \mapsto \psi *_\sigma a *_\sigma \chi$, is sequentially continuous from $C_0^\infty(W) \times \mathscr{D}'(W) \times C_0^\infty(W)$ to $C^\infty(W)$, and from $\mathscr{S}(W) \times \mathscr{S}'(W) \times \mathscr{S}(W)$ to $\mathscr{S}(W)$.*

**Proof.** We may assume that $W = T^*\mathbf{R}^n$. The first assertion is obvious. In order to prove the last assertion, we write $U_1 = A\psi$, $U_2 = Aa$ and $U_3 = A\chi$, where $\psi, \chi \in \mathscr{S}$, $a \in \mathscr{S}'$ and $A$ is given by (0.3). We let $\Phi_{(x,y)} = U_1(x,\cdot) \otimes U_3(\cdot, y)$. Then $A(\psi *_\sigma a_2 *_\sigma \chi)(x,y) = (U_1 \circ U_2 \circ U_3)(x,y) = \langle \Phi_{(x,y)}, U_2\rangle$ by (1.2), which proves the statement, for we may view $\Phi_{(x,y)}$ as a Schwartz function of $(x,y)$ with values in $\mathscr{S}(\mathbf{R}^n \times \mathbf{R}^n)$.

**Lemma 3.5.** *Assume that $a \in \mathscr{S}'_+(W)$, and let $\chi \in \mathscr{S}(W)$ be such that $\int \chi \, dX = (\pi/2)^{n/2}$, and set $a_\varepsilon = \tilde{\chi}_\varepsilon *_\sigma a *_\sigma \chi_\varepsilon$ where $\chi_\varepsilon = \varepsilon^{-2n}\chi(\cdot/\varepsilon)$. Then $a_\varepsilon \in \mathscr{S}(W) \cap C_+(W)$, and $a_\varepsilon \to a$ in $\mathscr{S}'$ as $\varepsilon \to 0$.*

**Proof.** Assume that $\varphi \in \mathscr{S}$. Then it follows from Lemma 1.1 and a simple argument of approximation that $(a_\varepsilon, \varphi) = (a, \chi_\varepsilon *_\sigma \varphi *_\sigma \tilde{\chi}_\varepsilon)$ and $(a_\varepsilon *_\sigma \varphi, \varphi) = (a *_\sigma (\chi_\varepsilon *_\sigma \varphi), \chi_\varepsilon *_\sigma \varphi) \geqslant 0$ as $a \in \mathscr{S}'_+$. Hence $a_\varepsilon \in \mathscr{S} \cap C_+$ by Lemma 3.4. Since it is easily seen that $\chi_\varepsilon *_\sigma \varphi *_\sigma \tilde{\chi}_\varepsilon$ converges in $\mathscr{S}$ to $\varphi$ as $\varepsilon \to 0$ it follows also that $a_\varepsilon \to a$ in $\mathscr{S}'$ as $\varepsilon \to 0$. The proof is complete.

**Proof of Theorem 3.3.** It is obvious that $(1) \Rightarrow (2) \Rightarrow (3)$. Assume that (3) holds. Given $\chi \in C_0^\infty$ such that $\int \chi \, dX \neq 0$ we choose $0 \leqslant \psi \in C_0^\infty$ such that $|\chi_\varepsilon *_\sigma \tilde{\chi}_\varepsilon| \leqslant \psi_\varepsilon$. Hence for $\varepsilon$ small enough, we obtain for some constants $C$ and $C'$ that

$$(a *_\sigma \chi_\varepsilon, \chi_\varepsilon) = (a, \chi_\varepsilon *_\sigma \tilde{\chi}_\varepsilon) \leqslant C'\varepsilon^{-2n} \int\limits_{|X|<C\varepsilon} \bigl|d\mu(X)\bigr|.$$

This gives (4).

It remains to prove that (4) implies (1). We assume therefore that (4) holds for some $\chi$. It is then no restriction to assume that $\int \chi \, dX = (\pi/2)^{n/2}$. Set $a_\varepsilon = \tilde{\chi}_\varepsilon *_\sigma a *_\sigma \chi_\varepsilon$. By Lemma 3.5 it follows that $a_\varepsilon \in \mathscr{S}(W) \cap C_+(W)$ and that $a_\varepsilon \to a$ in $\mathscr{S}'$ as $\varepsilon \to 0$.

We observe that (3.1′) is equivalent to $C = \liminf_{\varepsilon \to 0} a_\varepsilon(0) < \infty$, which implies that we may take a sequence $\varepsilon_j \to 0$, when $j \to \infty$ such that $\lim_{j\to\infty} a_j(0) = C$, where $a_j = a_{\varepsilon_j}$. By Proposition 1.6(3) we obtain

$$\|a_j\|_{s_1} = (\pi/2)^{n/2} a_j(0) \leqslant C' < \infty,$$

where $C'$ is independent on $j$. This gives us a bound independent of $j$ for the norm of $a_j$ in $s_1(W)$. In order to prove that $a \in s_1(W)$ we recall Lemma 1.1 and (1.3). Let $(u_k) \in \mathscr{B}^0(W)$. Then $\sum_k |(a_j, u_k)| \leqslant C'$. When $j \to \infty$ it follows from Lemma 3.5 that $\sum_k |(a, u_k)| \leqslant C'$, since $(u_k)$ is a finite sequence in $\mathscr{S}$. This implies that $a \in s_1(W)$ and $\|a\|_{s_1} \leqslant C'$ by Proposition 1.6(2). Since $s_1 \subset C_B$ by Proposition 1.5, it follows from Proposition 3.2 that $a \in C_+(W)$. Hence $a$ satisfies (1), and the proof is complete.

**Corollary 3.6.** *If $a \in s_\infty(W) \cap \mathscr{S}'_+(W)$ is locally integrable near the origin and* (3.1) *is fulfilled with $d\mu(X) = a(X)\,dX$, then $a \in s_1(W)$.*

**Corollary 3.7.** *$C_+(W)$ is the set of all functions $\tilde{v} *_\sigma v$, where $v \in L^2(W)$.*

**Corollary 3.8.** *Assume that $v \in s_\infty(W)$. Then $v \in L^2$ if and only if $\tilde{v} *_\sigma v \in L^\infty$.*

**Proof.** If $v \in L^2$ then $\|\tilde{v} *_\sigma v\|_{L^\infty} \leqslant (2/\pi)^{n/2}\|v\|_{L^2}^2 < \infty$. On the other hand, if $a = \tilde{v} *_\sigma v$ and $a \in L^\infty$, then the assumptions in Corollary 3.6 are fulfilled by $a$, which implies that $a \in s_1(W)$. Hence $U^* \circ U$ is a trace class operator if $U = Av$, and it follows that $U$ and $v$ are $L^2$ functions.

**Corollary 3.9.** *Let $W = T^*V$ and assume that $a \in \mathscr{S}'(W)$ satisfies that $\hat{a}$ is a bounded function near the origin, and that the pseudo-differential operator $a^w(x, D)$ is nonnegative. Then $a^w(x, D)$ is a trace class operator.*

**Proof.** The result follows if one combines Proposition 1.9, Proposition 1.10 and Theorem 3.3.

**Remark 3.10.** Corollary 3.9 may be generalized in the following way. Assume that $t \in \mathbf{R}$, $a \in \mathscr{S}'(T^*\mathbf{R}^n)$, and that the pseudo-differential operator

$$a_t(x, D)f(x) = (2\pi)^{-n} \iint a\big((1-t)x + ty, \xi\big) f(y) \mathrm{e}^{i\langle x-y,\xi\rangle}\, dy\, d\xi$$

is positive semi-definite. Then the following conditions are equivalent:

(1) $a_t(x, D)$ is a trace-class operator;
(2) $\hat{a}$ is bounded near the origin;
(3) $\hat{a} \in C_B(W) \cap L^2(W)$.

In fact, if $t = 1/2$, then the assertion follows from Corollary 3.9. For general $t$, we may reduce ourselves to the case $t = 1/2$ by the equivalence

$$a_t(x, D) = b^w(x, D) \quad \Leftrightarrow \quad (\mathscr{F}b)(x, \xi) = \mathrm{e}^{i(1/2-t)\langle x,\xi\rangle}(\mathscr{F}a)(x, \xi).$$

(Cf. [4] or [11].)

We may also use Corollary 3.7 in order to improve Proposition 3.1.

**Proposition 3.11.** *Assume that $0 \neq a \in C_+(W)$ and $X \neq 0$. Then $|a(X)| < a(0)$.*

**Proof.** By Corollary 3.7, we may assume that $a = \tilde{v} *_\sigma v$, where $v \in L^2$ is a unit vector. Then $a(0) = (2/\pi)^{n/2}$, and if $X \neq 0$ then $|a(X)| \leqslant (2/\pi)^{n/2}(|v(\cdot - X)|, |v|)$, by (0.1). Since $|v(\cdot - X)|$ is not proportional to $|v|$ it follows that $|a(X)| < (2/\pi)^{n/2}\|v\|_{L^2}^2 = a(0)$.

We shall next discuss the restriction of $\sigma$-positive functions to Lagrangian planes. Since the symplectic form is zero on such planes, it follows from (0.7) that the restriction of a $\sigma$-positive function to any Lagrangian plane $\Lambda$ is a positive definite function. Identifying the dual $\Lambda'$ of $\Lambda$ with $W/\Lambda$ by means of the symplectic form it follows that the restriction of $a$ to $\Lambda$ is the Fourier transform of a positive measure on $W/\Lambda$. The next result shows that this measure is absolutely continuous.

**Theorem 3.12.** *Assume that $a \in C_+(W)$ and that $\Lambda \subset W$ is a Lagrangian plane. Then there is a density $0 \leqslant a_\Lambda \in L^1(W/\Lambda)$ such that*

$$a(X) = \int_{W/\Lambda} e^{i\sigma(X,Y)} a_\Lambda(Y)\,dY, \quad X \in \Lambda.$$

**Proof.** We may assume that $W = T^*\mathbf{R}^n$ and that $\Lambda = \{(0,\xi);\ \xi \in \mathbf{R}^n\}$. Then $W/\Lambda$ might be identified with $\{(x,0);\ x \in \mathbf{R}^n\}$. From the proof of Corollary 3.8 it follows that $Aa = U_1^* \circ U_1$ for some $U_1 \in L^2$. Hence, Lemma 1.1(v) gives

$$a(0,\xi) = (2\pi)^{-n/2} \int e^{i\langle y,\xi\rangle} (U_1^* \circ U_1)(y/2, y/2)\,dy = \hat{w}(\xi),$$

where $w(y) = \int |U_1(z, -y/2)|^2\,dz$. The result follows now since $w \in L^1(\mathbf{R}^n)$.

We finish this section by presenting the following complementary result to Theorem 3.3, concerning regularity properties for $\sigma$-positive functions.

**Theorem 3.13.** *Assume that $a \in C_+(W)$ and that $a \in C^{2N}(\Omega)$ for some neighbourhood $\Omega$ of origin and some integer $N \geqslant 0$. Then the following is true*:

(1) $a \in C^{2N}(W)$ *and* $\mathscr{F}_\sigma a \in C^{2N}(W)$;
(2) $X^\alpha D^\beta a \in s_1(W)$ *and* $X^\alpha D^\beta \hat{a} \in s_1(W)$ *for any* $\alpha, \beta$ *such that* $|\alpha + \beta| \leqslant N$.

*In particular,* $C^\infty(\Omega) \cap \mathscr{S}'_+(W) \subset \mathscr{S}(W)$.

Here we note that (2) in Theorem 3.13 is independent of the choice of coordinate system for $W$. We need some preparations for the proof.

**Lemma 3.14.** *Assume that $a \in \mathscr{S}(W)$, and $Aa = \sum \lambda_j f_j \otimes g_j$, where $(f_j), (g_j) \in \mathrm{ON}(V)$. Then $f_j, g_j \in \mathscr{S}(V)$ when $\lambda_j \neq 0$.*

**Proof.** We have that $f_j = \lambda_j^{-1}(Aa)\overline{g_j} \in \mathscr{S}(V)$, since $(g_j) \in \mathrm{ON}(V)$, $a \in \mathscr{S}$ and $A$ is a homeomorphism on $\mathscr{S}$. By similar reasons we get $g_j \in \mathscr{S}$, and the proof is complete.

**Lemma 3.15.** *Assume that $Y \in W$ and let $T_Y$ be the operator $T_Y = \sigma(Y, D_X)^2 - \sigma(Y, X)^2$. Then $T_Y$ restricts to a continuous map on $\mathscr{S}'_+(W)$.*

Here and in what follows we let $\varphi(D_X)$ be multiplication by $\varphi(X)$ on the symplectic Fourier transform side.

**Proof.** If we choose some appropriate symplectic coordinates $X = (x, \xi)$, then we may assume that $W = T^*\mathbf{R}^n$ and that $Y = (y, \eta)$, where $y = (0, \dots, 0)$ and $\eta = (1, 0, \dots, 0)$. This implies that $T_Y = -\partial_{\xi_1}^2/4 - x_1^2$. It is clear that $T_Y$ is continuous on $\mathscr{S}(W)$ and on $\mathscr{S}'(W)$ for every $Y \in W$. We have to prove that $T_Y a \in \mathscr{S}'_+$ when $a \in \mathscr{S}'_+$. Since $T_Y$

is self-adjoint it follows from Proposition 1.10, Lemma 3.14 and duality that it suffices to prove that $T_Y a \in C_+ \cap \mathscr{S}$ when $a \in C_+ \cap \mathscr{S}$ is simple.

We therefore assume that $Aa = f \otimes \bar{f}$ for some $f \in \mathscr{S}$. From Lemma 1.1(v) and a straight-forward computation, we get $(T_Y a)(x,\xi) = A^{-1}(g \otimes \bar{g})$, where $g(x) = x_1 f(x)$, and the result follows since $A^{-1}(g \otimes \bar{g}) \in C_+ \cap \mathscr{S}$.

**Lemma 3.16.** *Let $(f_{1,j}), (f_{2,j})$ be two sequences in $\mathscr{S}'(V)$ such that the series $a_k = \sum_{j=1}^{\infty} u_{k,j}$, $k = 1, 2$, where $u_{k,j} = A^{-1}(f_{k,j} \otimes \overline{f_{k,j}})$, converge in $\mathscr{S}'(W)$. Then the following are true*:

(1) *the series $a_{0,c} \equiv \sum_{j=1}^{\infty} c_j A^{-1}(f_{1,j} \otimes \overline{f_{2,j}})$ converges in $\mathscr{S}'(W)$ for any bounded sequence $c = (c_j)$ in $\mathbf{C}$;*

(2) *with $T_Y$ as in Lemma 3.15 we have*

$$T_Y a_k = \sum_{j=1}^{\infty} T_Y u_{k,j} \in \mathscr{S}'_+(W), \tag{3.2}$$

*with convergence in $\mathscr{S}'$;*

(3) *if in addition $a_1, a_2 \in s_p(W)$ for some $1 \leqslant p \leqslant \infty$, then $a_{0,c} \in s_p(W)$ and*

$$\|a_{0,c}\|_{s_p} \leqslant \|c\|_{l^\infty} \big(\|a_1\|_{s_p} \|a_2\|_{s_p}\big)^{1/2}. \tag{3.3}$$

*Furthermore, $f_{j,k} \in L^2$ for every $j$ and $k$.*

**Proof.** (1) Since $A$ is a homeomorphism on $\mathscr{S}$ and $\mathscr{S}'$ it suffices to prove that $\sum_{j=1}^{\infty} c_j f_{1,j} \otimes \overline{f_{2,j}}$ converges in $\mathscr{S}'$, provided $U_1 = \sum_{j=1}^{\infty} f_{1,j} \otimes \overline{f_{1,j}}$ and $U_2 = \sum_{j=1}^{\infty} f_{2,j} \otimes \overline{f_{2,j}}$ do. Set $U_{0,N} = \sum_{j=1}^{N} c_j f_{1,j} \otimes \overline{f_{2,j}}$. By the kernel theorem of Schwartz it suffices to verify that for any $\varphi, \psi \in \mathscr{S}(V)$, the limit

$$(U_0\varphi, \psi) = \lim_{N\to\infty} (U_{0,N}, \psi \otimes \bar{\varphi})$$

exists and satisfies $|(U_0\varphi, \psi)| \leqslant \|c\|_{l^\infty} \|\varphi\| \|\psi\|$ for some seminorm $\|\cdot\|$ on $\mathscr{S}$.

Now it follows from the Cauchy–Schwartz inequality that

$$\begin{aligned}
&\left| \sum_{j=M}^{N} c_j (f_{1,j} \otimes \overline{f_{2,j}}, \psi \otimes \bar{\varphi}) \right|^2 \\
&\quad\leqslant \|c\|_{l^\infty}^2 \left( \sum_{j=M}^{N} \big|(f_{1,j}, \psi)\big| \big|(f_{2,j}, \varphi)\big| \right)^2 \\
&\quad\leqslant \|c\|_{l^\infty}^2 \left( \sum_{j=M}^{N} (f_{1,j} \otimes \overline{f_{1,j}}, \psi \otimes \bar{\psi}) \right) \left( \sum_{j=M}^{N} (f_{2,j} \otimes \overline{f_{2,j}}, \varphi \otimes \bar{\varphi}) \right).
\end{aligned}$$

From this we immediately conclude the existence of the limit and that

$$\big|(U_0\varphi, \psi)\big|^2 \leqslant \|c\|_{l^\infty}^2 \big|(U_1\varphi, \varphi)\big| \big|(U_2\psi, \psi)\big|.$$

This proves (1).

The assertion (2) follows from (1), Lemma 3.15 and duality, since $T_Y$ is continuous on $\mathscr{S}(W)$.

(3) We start to prove that $f_{1,j} \in L^2$ for every $j$ when $a_1 \in s_p$ for some $p$. Assume that $f_{1,j_0} \notin L^2$ for some $j_0$. Then $Aa_1 \geqslant f_{1,j_0} \otimes \overline{f_{1,j_0}}$ (as operators), which implies that $Aa_1 \notin \mathscr{I}_\infty$, since the right-hand side is an unbounded operator on $L^2$. Hence, $a_1 \notin s_\infty$, which gives a contradiction and proves the assertion.

Next we consider

$$\Big(\sum_k \big|\big((Aa_{0,c})h_{1,k}, h_{2,k}\big)\big|^p\Big)^{1/p} = \Big(\sum_k \Big|\sum_j c_j (f_{1,j}, h_{1,k})(f_{2,j}, h_{2,k})\Big|^p\Big)^{1/p},$$

where $(h_{1,k}) \in \mathrm{ON}_0(V)$ and $(h_{2,k}) \in \mathrm{ON}_0(V)$. By applying Cauchy–Schwartz inequality first on the inner sum, and then on the outer sum we get

$$\begin{aligned}
&\Big(\sum_k \big|\big((Aa_{0,c})h_{1,k}, h_{2,k}\big)\big|^p\Big)^{1/p} \\
&\quad\leqslant \|c\|_{l^\infty} \Big(\sum_k \Big(\sum_j \big|(f_j, h_{1,k})\big|^2\Big)^p\Big)^{1/2p} \Big(\sum_k \Big(\sum_j \big|(g_j, h_{2,k})\big|^2\Big)^p\Big)^{1/2p} \\
&\quad= \|c\|_{l^\infty} \Big(\sum_k \big|\big((Aa_1)h_{1,k}, h_{1,k}\big)\big|^p\Big)^{1/2p} \Big(\sum_k \big|\big((Aa_2)h_{2,k}, h_{2,k}\big)\big|^p\Big)^{1/2p} \\
&\quad\leqslant \|c\|_{l^\infty} \big(\|a_1\|_{s_p} \|a_2\|_{s_p}\big)^{1/2}.
\end{aligned}$$

By taking supremum over all orthonormal sequences $(h_{1,k})$ and $(h_{2,k})$ on the left-hand side we obtain (3.3), and (3) follows. The proof is complete.

**Proof of Theorem 3.13.** We may assume that $W = T^*\mathbf{R}^n$. It follows from Theorem 3.3 that the assertion holds when $N = 0$. We assume therefore that $N \geqslant 1$. Set for $j = 1, \dots, n$

$$\begin{aligned}
S_j &= (2i)^{-1}\partial_{\xi_j} - x_j, &\quad S_{n+j} &= (2i)^{-1}\partial_{x_j} - \xi_j, \\
\widetilde{S}_j &= (2i)^{-1}\partial_{\xi_j} + x_j, &\quad \widetilde{S}_{n+j} &= (2i)^{-1}\partial_{x_j} + \xi_j.
\end{aligned} \tag{3.4}$$

Then it follows easily that if $U = Au$, where $u \in \mathscr{S}'(W)$, then

$$\begin{aligned}
(A \circ S_j)u(x,y) &= x_j U(x,y), &\quad (A \circ S_{n+j})u(x,y) &= D_{x_j} U(x,y), \\
(A \circ \widetilde{S}_j)u(x,y) &= y_j U(x,y), &\quad (A \circ \widetilde{S}_{n+j})u(x,y) &= D_{y_j} U(x,y),
\end{aligned} \tag{3.5}$$

for every $j = 1, \dots, n$. We note also that $T_j \equiv S_j \widetilde{S}_j = \sigma(Y_j, D_X)^2 - \sigma(Y_j, X)^2$ for some choice of $Y_j$. By Lemma 3.16(2), it follows that $T_j$ is continuous on $\mathscr{S}'_+$ for every $j$. Since $N \geqslant 1$, Theorem 3.3 and Lemma 3.16 gives that $(T_j a) \in C_+$ for every $j$.

Now we may write $a = \sum u_k$, where $Au_k = f_k \otimes \overline{f_k}$, for some sequence $(f_k)$ in $L^2$. Then $\sum \|f_k\|_{L^2}^2 = (\pi/2)^{n/2} a(0) = \|a\|_{s_1}$, and Lemma 3.16(2) together with some straightforward computation give for every $j = 1, \dots, n$ that $T_j a = \sum_k T_j u_k \in C_+$ and $T_{n+j} a = \sum_k T_{n+j} u_k \in C_+$, where $T_j u_k = A^{-1}((x_j f_k) \otimes \overline{(x_j f_k)})$ and $T_{n+j} u_k = A^{-1}((D_j f_k) \otimes \overline{(D_j f_k)})$. This implies that

$$\sum_k \|x_j f_k\|_{L^2}^2 = (\pi/2)^{n/2}(T_j a)(0) < \infty \quad \text{and}$$
$$\sum_k \|D_j f_k\|_{L^2}^2 = (\pi/2)^{n/2}(T_{n+j} a)(0) < \infty, \quad 1 \leqslant j \leqslant n.$$

If we repeat these arguments, then we get

$$\begin{aligned} &a_\alpha^\beta \equiv \textstyle\sum_k u_{k,\alpha}^\beta \in C_+(W) \quad \text{and} \quad \big\|a_\alpha^\beta\big\|_{s_1} = \sum \big\|x^\alpha D^\beta f_k\big\|_{L^2}^2 < \infty, \\ &\text{where } Au_{k,\alpha}^\beta = \big(x^\alpha D^\beta f_k\big) \otimes \overline{\big(x^\alpha D^\beta f_k\big)}, \text{ and } |\alpha+\beta| \leqslant N. \end{aligned} \tag{3.6}$$

Now we observe that for any multi-index $\gamma$ such that $|\gamma| \leqslant 2N$, we may find polynomials $p_{\alpha,\beta,\gamma}(X)$ on $W$ such that $D^\gamma = \sum p_{\alpha,\beta,\gamma}(X) S^\alpha \widetilde{S}^\beta$, where the sum should only be taken over $|\alpha| \leqslant N$ and $|\beta| \leqslant N$. In order to prove that $a \in C^{2N}(W)$, it suffices therefore to prove that $S^\alpha \widetilde{S}^\beta a$ is continuous when $|\alpha| \leqslant N$ and $|\beta| \leqslant N$. By (3.5) it follows that if $|\alpha| \leqslant N$ and $|\beta| \leqslant N$, then $S^\alpha \widetilde{S}^\beta a$ is a linear combination of terms of the type

$$\sum_k A^{-1}\Big(\big(x^{\gamma_1} D^{\delta_1} f_k\big) \otimes \overline{\big(x^{\gamma_2} D^{\delta_2} f_k\big)}\Big),$$

where $|\gamma_1 + \delta_1| \leqslant N$ and $|\gamma_2 + \delta_2| \leqslant N$. Hence $S^\alpha \widetilde{S}^\beta a \in s_1$ by (3.6) and Lemma 3.16(3), and since $s_1 \subset C$ we conclude that $a \in C^{2N}$.

In order to prove that $X^\alpha D^\beta a \in s_1$ when $|\alpha + \beta| \leqslant N$, we note that the operators $x_j$, $\xi_j$, $D_{x_j}$ and $D_{\xi_j}$ are linear combinations of $\{S_k\}$ and $\{\widetilde{S}_k\}$ by (3.4). This implies that if $|\alpha + \beta| \leqslant N$, then $X^\alpha D^\beta a$ is a linear combination of terms of the type $S^\gamma \widetilde{S}^\delta a$, where $|\gamma| \leqslant N$ and $|\delta| \leqslant N$. Since we have already proved that $S^\gamma \widetilde{S}^\delta a \in s_1$ for such choice of $\gamma$ and $\delta$, it follows that $X^\alpha D^\beta a \in s_1$ when $|\alpha + \beta| \leqslant N$. Hence the assertions concerning $a$ follows.

If we repeat these arguments and use Lemma 1.1(ii), then the assertions concerning $\mathscr{F}_\sigma a$ follow by similar arguments. The proof is complete.

**Remark 3.17.** It follows from Theorem 3.13 and its proof that the sum $\sum_k A^{-1}(f_k \otimes \overline{f_k})$ belongs to $C^{2N}$, if and only if $\sum_k \|x^\alpha D^\beta f_k\|_{L^2}^2 < \infty$ for every $\alpha, \beta$ such that $|\alpha + \beta| \leqslant N$. In particular, if $u_k \in C_+$ for every $k \in I$ and $a = \sum_{k \in I} u_k \in C^{2N}$, then $\sum_{k \in I_0} u_k \in C^{2N} \cap C_+$ for every $I_0 \subset I$.

## 4. Positivity properties for elements in $s_p(W)$ and $\mathscr{F}_\sigma L^p(W)$

In this section we shall find necessary and sufficient conditions for elements in $\mathscr{S}'_+(W)$ to belong to $s_p(W)$ or $\mathscr{F}_\sigma L^p(W)$, when $p \in [1, \infty]$. More precisely, we prove that elements in $\mathscr{S}'_+$ belong to $s_p$ (or $\mathscr{F}_\sigma L^p$) if and only if they belong to $s_p$ (or $\mathscr{F}_\sigma L^p$) near the origin and near the infinity. We present also generalizations to weighted $\mathscr{F}_\sigma L^p$ spaces. We apply also some parts of the basic analysis, in order to refine the regularity results in Section 3 in terms of wave-front sets, and prove that if $(X, Y) \in WF(a)$ or $(X, Y) \in WF(\mathscr{F}_\sigma a)$ for some $X, Y \in W$, then $(0, Y) \in WF(a)$.

We start the section by giving a review of some Young type results for dilated multiplications and convolutions, which are needed. We omit the proof, since it might be found in [9] or in [12].

**Theorem 4.1.** *Assume that $s,t \in \mathbf{R} \setminus \{0\}$ and that $p,q,r \in [1,\infty]$ satisfy $1/p + 1/q = 1 + 1/r$ and $\pm s^2 \pm t^2 = 1$ for some choice of $\pm$ at each place. Then the following is true*:

(1) *the mappings $(a,b) \mapsto a(s\cdot)b(t\cdot)$ and $(a,b) \mapsto a(\cdot/s) * b(\cdot/t)$ on $\mathscr{S}(W)$ extends uniquely to a continuous mapping from $s_p(W) \times s_q(W)$ to $s_r(W)$. If in addition $a,b \in \mathscr{S}'_+$, then $a(s\cdot)b(t\cdot) \in \mathscr{S}'_+$;*
(2) *the convolution $(a,b) \mapsto a * b$ extends uniquely to continuous mappings from $s_p(W) \times s_q(W)$ to $L^r(W)$, and from $s_p(W) \times L^q(W)$ to $s_r(W)$.*

*Moreover, one has the estimates*

$$\begin{aligned} \|a(s\cdot)b(t\cdot)\|_{s_r} &\leqslant C^n \|a\|_{s_p}\|b\|_{s_q}, \qquad \|a*b\|_{L^r} \leqslant C^n\|a\|_{s_p}\|b\|_{s_q},\\ \|a(\cdot/s)*b(\cdot/t)\|_{s_r} &\leqslant C^n \|a\|_{s_p}\|b\|_{s_q}, \qquad \|a*c\|_{s_r} \leqslant C^n\|a\|_{s_p}\|c\|_{L^q}, \end{aligned} \tag{4.1}$$

*where the constant $C$ is independent on $a \in s_p(W)$, $b \in s_q(W)$ and $c \in L^q(W)$.*

**Corollary 4.2.** *If $\Omega \subset W$ is an arbitrary open neighbourhood of the origin, then one can find $a \in C_+(W) \cap C_0^\infty(W)$ with support in $\Omega$ such that $a \geqslant 0$ and $a(0) = 1$. On the other hand, if $a \in C_+(W)$ and $\hat{a}$ is compactly supported, then $a = 0$.*

**Proof.** In order to construct a nontrivial and nonnegative $a \in C_+(W)$ with small support, we let $\phi \in C_0^\infty(W) \setminus \{0\}$ be even and set $\psi = \phi *_\sigma \tilde{\phi}$. Then $\psi \in C_0^\infty(W) \cap C_+(W) \setminus \{0\}$ is even, and since $\psi = \tilde{\psi}$ it follows that $\psi$ is real-valued. But then an application of Theorem 4.1 shows that the function $a(X) = \psi^2(X/\sqrt{2})$ belongs to $C_+(W)$. It is nonnegative and smooth, and its support is contained in $\Omega$, if one chooses $\phi$ with a sufficiently small support.

We postpone the verification of that $\widehat{C}_+ \cap C_0 = \{0\}$ to the next section (cf. Proposition 5.1).

**Corollary 4.3.** *Assume that $a \in \mathscr{S}'(W)$ and $\chi \in \mathscr{S}(W)$. Then $a \in s_p(W)$, if and only if $\chi a \in s_p$ and $(1-\chi)a \in s_p$.*

**Remark 4.4.** In [9] and [12] one applies Theorem 4.1(2) in order to prove that $H^p_{\mu|1-2/p|} \subset s_p \subset H^p_{-\mu|1-2/p|}$, for every $\mu > 2n$. Here $H^p_s(W)$ denotes the Sobolev space of distribution with $s \in \mathbf{R}$ derivatives in $L^p(W)$.

One may also apply Theorem 4.1(2), in order to prove that $s_p \cap \mathscr{E}' = \mathscr{F}_\sigma L^p \cap \mathscr{E}'$. (See for example Corollary 2.12 in [12].) We shall need the following modification of this result.

**Proposition 4.5.** *Assume that $\psi \in \mathscr{S}(W)$. Then one may find $\chi \in \mathscr{S}(W)$ and a constant $C$, depending on $\psi$ and $\chi$ only such that*

$$\|\psi a\|_{s_p} \leqslant C\big\|\mathscr{F}_\sigma(\chi a)\big\|_{L^p}, \qquad \big\|\mathscr{F}_\sigma(\psi a)\big\|_{L^p} \leqslant C\|\chi a\|_{s_p}, \tag{4.2}$$

*for every $a \in \mathscr{S}'(W)$.*

*Moreover, one has that $\mathscr{S}(W) \cdot s_p(W) = \mathscr{S}(W) \cdot \mathscr{F}_\sigma L^p(W)$, or equivalently $\mathscr{S}(W) * s_p(W) = \mathscr{S}(W) * L^p(W)$.*

**Proof.** We may assume that $W = T^*\mathbf{R}^n$. Let $\{\varphi_j\}_{j=0}^\infty$ be a sequence of nonnegative and nonzero $C_0^\infty$-functions on $W$ such that for some constants $C_\alpha < \infty$ we have $\sum \varphi_j = 1$, $\operatorname{supp}\varphi_j \subset B_{j+1} \setminus B_{j-1}$ and $|D^\alpha \varphi_j| \leqslant C_\alpha$ for every integer $j \geqslant 0$. Here $B_r$ denotes the closed ball with center at origin and radius $r$. (We use the convention that $B_r = \emptyset$ when $r < 0$.) Set

$$\chi(X) = \sum_{j\geqslant 0} c_j^{1/2}\varphi_j(X), \quad \text{where } c_j = \sum_{|\alpha|\leqslant 2n+1} \big\|D^\alpha\psi\big\|_{L^\infty(B_{j+3}\setminus B_{j-3})}.$$

Then $\chi \in \mathscr{S}(W)$, and if $|\alpha| \leqslant 2n+1$ then $D^\alpha(\psi(X)/\chi(X))$ is continuous and rapidly decreasing to zero as $|X| \to \infty$. In particular, $\|\mathscr{F}_\sigma(\psi/\chi)\|_{L^1} < \infty$ and $\|D^\alpha(\psi/\chi)\|_{L^1} < \infty$ as $|\alpha| \leqslant 2n+1$. Hence $\psi/\chi \in H_s^1$ for some $s > 2n$, and we conclude that $\psi/\chi \in s_1 \cap L^1 \cap \mathscr{F}_\sigma L^1$ by Remark 4.4. A combination of this fact and Theorem 4.1(2) now gives

$$\begin{aligned}\|\psi a\|_{s_p} &= \big\|\mathscr{F}_\sigma\big((\psi/\chi)(\chi a)\big)\big\|_{s_p}\\ &= \pi^{-n}\big\|\big(\mathscr{F}_\sigma(\psi/\chi)\big) * \big(\mathscr{F}_\sigma(\chi a)\big)\big\|_{s_p} \leqslant C\big\|\mathscr{F}_\sigma(\chi a)\big\|_{L^p},\end{aligned}$$

for some $C < \infty$. This proves the first inequality in (4.2), and the second inequality in (4.2) is obtained by similar arguments.

In order to prove the last part, it suffices to prove that $\mathscr{S} \cdot \mathscr{F}_\sigma L^p = \mathscr{S} \cdot s_p$ by Proposition 1.9. Assume first that $a \in s_p$, and let $\chi$ be as above. Then Theorem 4.1(2) gives

$$\mathscr{F}_\sigma\big((\psi/\chi)a\big) = \pi^{-n}\mathscr{F}_\sigma(\psi/\chi) * \hat{a} \subset s_1 * s_p \subset L^p.$$

Hence $\psi a = \chi((\psi/\chi)a) \in \mathscr{S} \cdot \mathscr{F}_\sigma L^p$, which proves that $\mathscr{S} \cdot s_p \subset \mathscr{S} \cdot \mathscr{F}_\sigma L^p$. By similar arguments, an opposite inclusion is obtained. The proof is complete.

We shall combine Proposition 4.5 with the following result. Here and in what follows we let $\chi_Y(X) = \chi(X - Y)$, if nothing else is stated.

**Proposition 4.6.** *Assume that $a \in \mathscr{S}'_+(W)$ and that $\chi \in C_+(W) \cap \mathscr{S}(W)$. Then $\mathscr{F}_\sigma(\chi a)$ is a nonnegative function. If $u = (\mathscr{F}_\sigma(\chi a))^{1/2}$ and $X, Y \in W$, then*

$$\begin{aligned}&\big|\mathscr{F}_\sigma(\chi_Y a)(X)\big| \leqslant u(X+Y)u(X-Y),\\ &\big|\mathscr{F}_\sigma\big((\mathscr{F}_\sigma\check{\chi})_Y\hat{a}\big)(X)\big| \leqslant u(X+Y)u(Y-X).\end{aligned} \tag{4.3}$$

**Proof.** Since it is clear that $\bar{\chi} \in C_+$, and that $\mathscr{S}'_+$ is invariant under multiplication by exponentials, it follows from Proposition 1.10 that $\mathscr{F}_\sigma(\chi\, a)(X) = (\mathrm{e}^{2i\sigma(X,\cdot)}a, \bar{\chi})$ is nonnegative. This proves the first part of the proposition.

In order to prove the first inequality in (4.3) we take a function $\psi$ such that $\bar\chi = \tilde\psi *_\sigma \psi$, and assume first that $\psi \in \mathscr{S}$. Then $\bar\chi_Y = \tilde\psi_Y *_\sigma \phi_Y$, where $\psi_Y = \psi(\cdot + Y)$ and $\phi_Y = \mathrm{e}^{-2i\sigma(Y,\cdot)}\psi$. From the fact that $\mathrm{e}^{2i\sigma(X,\cdot)}a \in \mathscr{S}'_+$ for every $X \in W$, an application of Cauchy–Schwartz inequality gives

$$\begin{aligned} \left|\mathscr{F}_\sigma(\chi_Y a)(X)\right|^2 &= \left|\left(\mathrm{e}^{2i\sigma(X,\cdot)}a, \tilde\psi_Y *_\sigma \phi_Y\right)\right|^2 \\ &\leqslant \left(\mathrm{e}^{2i\sigma(X,\cdot)}a, \tilde\psi_Y *_\sigma \psi_Y\right)\left(\mathrm{e}^{2i\sigma(X,\cdot)}a, \tilde\phi_Y *_\sigma \phi_Y\right). \end{aligned} \tag{4.4}$$

By some simple calculations one obtains that $\tilde\psi_Y *_\sigma \psi_Y = \mathrm{e}^{2i\sigma(Y,\cdot)}\bar\chi$, and $\tilde\phi_Y *_\sigma \phi_Y = \mathrm{e}^{-2i\sigma(Y,\cdot)}\bar\chi$. This implies that

$$\left(\mathrm{e}^{2i\sigma(X,\cdot)}a, \tilde\psi_Y *_\sigma \psi_Y\right) = \left(\mathrm{e}^{2i\sigma(X-Y,\cdot)}a, \bar\chi\right) = \mathscr{F}_\sigma(\chi a)(X-Y),$$

and similarly $(\mathrm{e}^{2i\sigma(X,\cdot)}a, \tilde\phi_Y *_\sigma \phi_Y) = \mathscr{F}_\sigma(\chi a)(X+Y)$. The first inequality in (4.3) follows now in this case from the last identities and (4.4).

For general $\psi \in \mathscr{S}$ the first inequality in (4.3) follows now by a simple argument of approximation, using Lemma 3.5.

Finally, the second inequality in (4.3) follows from the first one, since

$$\mathscr{F}_\sigma\left((\mathscr{F}_\sigma\check\chi)_Y\hat a\right)(X) = \mathrm{e}^{2i\sigma(X,Y)}\mathscr{F}_\sigma(\chi_X a)(Y),$$

by Fourier's inversion formula. The proof is complete.

Proposition 4.5 and Proposition 4.6 allow us to continue the regularization discussion from Section 3 in terms of local weighted $\mathscr{F}_\sigma L^p$ spaces which we shall discuss now. Assume that $\mu$ is a nonnegative Borel measure on $W$ and that $p \in [1,\infty]$. Then we let $\mathcal{H}^p_\mu(W)$, be the set of all $a \in \mathscr{S}'(W)$ such that $\mathscr{F}_\sigma a$ is $\mu$-measurable and

$$\|a\|_{\mathcal{H}^p_\mu} \equiv \left(\int \left|\mathscr{F}_\sigma a(X)\right|^p d\mu(X)\right)^{1/p}$$

is finite. We also let $\mathcal{H}^p_{\mu,\mathrm{loc}}(W)$ (the corresponding local space) be the set of all $a \in \mathscr{D}'(W)$ such that $\chi a \in \mathcal{H}^p_\mu(W)$ for every $\chi \in C_0^\infty(W)$. In the most situations we shall deal with the subspace $\mathcal{H}^p_{\mu,\mathrm{loc},\mathscr{S}}(W)$ of $\mathcal{H}^p_{\mu,\mathrm{loc}}(W)$, consisting of all $a \in \mathscr{S}'(W)$ such that $\chi a \in \mathcal{H}^p_\mu(W)$ for every $\chi \in \mathscr{S}(W)$.

We note that $\mathcal{H}^p_\mu \subset \mathcal{H}^p_{\mu,\mathrm{loc}}$ is not true for general $\mu$. On the other hand, if $\mu$ in addition satisfies a growth condition of the type

$$d\mu(X+Y) \leqslant g(Y)\,d\mu(X), \quad X, Y \in W, \tag{4.5}$$

for some polynomial $g$ on $W$, then we have the following.

**Proposition 4.7.** *Assume that $\mu$ satisfies* (4.5) *and that $p \in [1,\infty]$, for some polynomial $g$ on $W$. Then $\mathcal{H}^p_\mu(W) \subset \mathcal{H}^p_{\mu,\mathrm{loc},\mathscr{S}}(W) \subset \mathcal{H}^p_{\mu,\mathrm{loc}}(W)$.*

In the proof of Proposition 4.7 and in some other situations we need to apply Minkowski's inequality, in a somewhat general form. We recall that for a $d\nu$-measurable function $f$ with values in the Banach space $B$ with norm $\|\cdot\|$, then Minkowski's inequality states that $\| \int f\, d\nu\| \leqslant \int \|f\|\, d\nu$. In our applications one has that $B$ is equal to $L^p(d\mu)$, for some $p \in [1, \infty]$, and then Minkowski's inequality takes the form $(\int |\int f\, d\nu|^p\, d\mu)^{1/p} \leqslant \int (\int |f|^p\, d\mu)^{1/p}\, d\nu$.

**Proof of Proposition 4.7.** It suffices to prove the first of the inclusions. Assume that $a \in \mathscr{S}'(W)$. Then Minkowski's inequality gives

$$\left(\int \left|\mathscr{F}_\sigma(\chi a)(X)\right|^p d\mu(X)\right)^{1/p} = \pi^{-n}\left(\int \left|\int \hat{\chi}(Y)\hat{a}(X-Y)\, dY\right|^p d\mu(X)\right)^{1/p}$$
$$\leqslant \pi^{-n}\int \left|\hat{\chi}(Y)\right|\left(\int \left|\hat{a}(X-Y)\right|^p d\mu(X)\right)^{1/p} dY.$$

By (4.5) it follows now that the right-hand side is less than or equal to $C\|a\|_{\mathcal{H}^p_\mu}$, where $C = \int |\hat{\chi}(Y)| g(Y)^{1/p}\, dY$ is finite. The proof is complete.

We may now prove the following result.

**Theorem 4.8.** *Assume that $p \in [1, \infty]$, and let $\mu$ be a positive Borel measure on $W$ such that $d\mu(-X) = d\mu(X)$ and that (4.5) holds for some polynomial $g$, and let $\chi, \psi \in \mathscr{S}(W)$ be chosen such that $\chi(0) \neq 0$. Then for some constant $C$, depending on $\chi$ and $\psi$ only, we have*

$$\|\psi a\|_{\mathcal{H}^p_\mu} \leqslant C\|\chi a\|_{\mathcal{H}^p_\mu} \quad \text{and} \quad \|\psi \hat{a}\|_{\mathcal{H}^p_\mu} \leqslant C\|\chi a\|_{\mathcal{H}^p_\mu} \tag{4.6}$$

*for every $a \in \mathscr{S}'_+(W)$. In particular, $a \in \mathcal{H}^p_{\mu,\mathrm{loc},\mathscr{S}}$ when $\chi a \in \mathcal{H}^p_\mu$ and $a \in \mathscr{S}'_+$. If in addition $\psi(0) \neq 0$, then for some constant $C$ we have*

$$C^{-1}\|\chi a\|_{\mathcal{H}^p_\mu} \leqslant \|\psi a\|_{\mathcal{H}^p_\mu} \leqslant C\|\chi a\|_{\mathcal{H}^p_\mu} \tag{4.7}$$

*for every $a \in \mathscr{S}'_+(W)$.*

**Proof.** Assume first that $\chi \in C_+ \cap \mathscr{S}$ is nonnegative such that $\int \chi^2\, dX = 1$. Then we have

$$\|\psi a\|_{\mathcal{H}^p_\mu} = \left(\int \left|\mathscr{F}_\sigma(\psi a)(X)\right|^p d\mu(X)\right)^{1/p}$$
$$= \left(\int \left|\int \mathscr{F}_\sigma\big(\psi \chi_Y^2 a\big)(X)\, dY\right|^p d\mu(X)\right)^{1/p}$$
$$\leqslant \pi^{-n}\left(\int \left|\iiint \left|\mathscr{F}_\sigma(\psi\chi_Y)(Z)\mathscr{F}_\sigma(\chi_Y a)(X-Z)\right| dY\, dZ\right|^p d\mu(X)\right)^{1/p}.$$

Here we consider $Y$ as fixed parameter when we apply the Fourier transformations. By applying Minkowski's inequality on the last expression we get

$$\|\psi a\|_{\mathcal{H}^p_\mu} \leqslant \pi^{-n}\iint \left|\mathscr{F}_\sigma(\psi\chi_Y)(Z)\right|\left(\int \left|\mathscr{F}_\sigma(\chi_Y a)(X-Z)\right|^p d\mu(X)\right)^{1/p} dY\, dZ.$$

If we combine the first inequality in (4.3) with (4.5) and Cauchy's inequality it follows that

$$\left( \int \left| \mathscr{F}_\sigma(\chi_Y a)(X-Z) \right|^p d\mu(X) \right)^{1/p} \leqslant \big( g(Z-Y) g(Z+Y) \big)^{1/2p} \|\chi a\|_{\mathcal{H}^p_\mu}.$$

By summing up we obtain $\|\psi a\|_{\mathcal{H}^p_\mu} \leqslant C\|\chi a\|_{\mathcal{H}^p_\mu}$, where

$$C = \pi^{-n} \iint \left| \mathscr{F}_\sigma(\psi \chi_Y)(Z) \right| \big( g(Z-Y) g(Z+Y) \big)^{1/2p} \, dY \, dZ. \tag{4.8}$$

Here $C < \infty$, since $(X, Y) \mapsto \psi(X)\chi_Y(X) = \psi(X)\chi(X-Y)$ is a tempered function, which implies that the integrand in (4.8) is rapidly decreasing to zero at infinity.

The second inequality in (4.6) follows by similar arguments if we use the second inequality instead of the first one in (4.3), and inserts $\int \hat\chi_Y^2 \, dY$ instead of $\int \chi_Y^2 \, dY$. (Note that $\hat\chi$ is real-valued since $\tilde\chi = \chi$ was chosen nonnegative, which implies that $\int \hat\chi_Y(X)^2 \, dY = \int \chi(X)^2 \, dX = 1$.)

It remains for us to prove (4.6) for arbitrary $\chi \in \mathscr{S}(W)$ such that $\chi(0) \neq 0$. By Corollary 4.2 we may take a nonnegative element $0 \neq \chi_1 \in C_+ \cap C_0^\infty$ such that $\chi \neq 0$ in a neighbourhood of $\operatorname{supp} \chi_1$. Then for some $\phi \in C_0^\infty$ we have that $\phi\chi = 1$ in $\operatorname{supp}\chi_1$. By Proposition 4.7 we get

$$\|\chi_1 a\|_{\mathcal{H}^p_\mu} = \left\| (\chi_1 \phi) \chi a \right\|_{\mathcal{H}^p_\mu} \leqslant C \|\chi a\|_{\mathcal{H}^p_\mu}$$

and (4.6) follows from this estimate and the first part of the proof. The proof is complete.

**Remark 4.9.** Assume that $\mu$ is a positive Borel measure on $W$ which satisfies (4.5) for some function $g$ such that $g$ which is bounded by some polynomial. Let $\chi, \psi \in \mathscr{S}(W)$ such that $\chi(0) \neq 0$, and set $\psi_{Y,Z}(X) = \mathrm{e}^{-2i\sigma(Y,X)}\psi(X-Z)$. Then the proof of Theorem 4.8 gives that

$$\|\psi_{Y,Z} a\|_{\mathcal{H}^p_\mu} \leqslant C \big( g(Y)^2 g(Z) g(-Z) \big)^{1/2p} \|\chi a\|_{\mathcal{H}^p_\mu}$$

for every $a \in \mathscr{S}'_+(W)$. If in addition $d\mu(-X) = d\mu(X)$, then

$$\|\psi_{Y,Z} \hat a\|_{\mathcal{H}^p_\mu} \leqslant C \big( g(Y)^2 g(Z) g(-Z) \big)^{1/2p} \|\chi a\|_{\mathcal{H}^p_\mu}$$

for every $a \in \mathscr{S}'_+(W)$.

**Corollary 4.10.** *Assume that $p \in [1, \infty]$ and that $\chi \in \mathscr{S}(W)$ such that $\chi(0) \neq 0$. Then for some constant $C$, depending on $\chi$ only, one has*

$$C^{-1} \|\chi a\|_{s_p} \leqslant \left\| \mathscr{F}_\sigma(\chi a) \right\|_{L^p} \leqslant C \|\chi a\|_{s_p}$$

*for every $a \in \mathscr{S}'_+(W)$.*

**Proof.** If $d\mu(X) = dX$ and $\psi = \chi^2$ in Theorem 4.8, then Theorem 4.1(2) and (4.8) gives

$$\left\| \mathscr{F}_\sigma(\chi a) \right\|_{L^p} \leqslant C_1 \left\| \mathscr{F}_\sigma\big(\chi^2 a\big) \right\|_{L^p} = C_2 \left\| \hat\chi * \mathscr{F}_\sigma(\chi a) \right\|_{L^p} \leqslant C_3 \|\chi a\|_{s_p}.$$

On the other hand, Proposition 4.5 and (4.7) shows that for some $\psi \in \mathscr{S}(W)$ (depending on $\chi$ only) we have

$$\|\chi a\|_{s_p} \leqslant C_1 \big\|\mathscr{F}_\sigma(\psi a)\big\|_{L^p} \leqslant C_2 \big\|\mathscr{F}_\sigma(\chi a)\big\|_{L^p},$$

and the proof follows.

**Corollary 4.11.** *Assume that $p \in [1,\infty]$ and that $\chi, \psi \in \mathscr{S}(W)$ such that $\chi(0) \neq 0$. Let $\psi_{Y,Z}(X) = \psi(X-Z)\mathrm{e}^{-2i\sigma(Y,X)}$, and let $\|\cdot\|_{(p,k)}$, $k = 1, 2$, be a collection of $s_p$-norms and $\mathscr{F}_\sigma L^p$-norms. Then for some constant $C$, depending on $\chi$ and $\psi$ only, we have*

$$\|\psi_{Y,Z} a\|_{(p,1)} \leqslant C\|\chi a\|_{(p,2)}, \qquad \|\psi_{Y,Z} * a\|_{s_p} \leqslant C\|\chi a\|_{(p,2)},$$
$$\|\psi_{Y,Z} \hat{a}\|_{(p,1)} \leqslant C\|\chi a\|_{(p,2)}, \qquad \|\psi_{Y,Z} * \hat{a}\|_{s_p} \leqslant C\|\chi a\|_{(p,2)},$$

*for every $a \in \mathscr{S}'_+(W)$ and every $p \in [1,\infty]$.*

**Proof.** The result is an immediate consequence of Proposition 1.9, Remark 4.9 and Corollary 4.10.

If we combine Theorem 3.3, Corollary 4.3 and Corollary 4.11, then we get the following.

**Theorem 4.12.** *Assume that $\chi, \psi \in \mathscr{S}(W)$ such that $\chi(0) \neq 0$. Then the following is true*:

(1) *if $a \in \mathscr{S}'_+(W)$, then $a \in s_1$ if and only if $\chi a \in s_1$;*
(2) *if $a \in \mathscr{S}'_+(W)$ and $p \in [1,\infty]$, then $a \in s_p$ if and only if $\chi a \in s_p$ and $(1-\psi)a \in s_p$.*

**Remark 4.13.** We note that if $p = 1$, then the assertion (1) is more sharp than (2) in Theorem 4.12, and the question arises whether it is possible to replace the condition $(1-\psi)a \in s_p$ in Theorem 4.12(2) by a weaker condition when $1 < p \leqslant \infty$. If such improvements exist or not is not known to the author. We note however that (2) is false when the condition $(1-\psi)a \in s_p$ is completely removed and not replaced by something else.

In fact, assume that $W = T^*\mathbf{R}^n$, and let $\chi(x,\xi) = \mathrm{e}^{-(|x|^2+|\xi|^2)}$. Then it suffices to find a sequence $a_\lambda \in s_p(W) \cap \mathscr{S}'_+(W)$ such that $\|\chi a_\lambda\|_{s_p}$ stays bounded, but $\|a_\lambda\|_{s_p} \to \infty$ as $\lambda \to 0$.

Let $a_\lambda = \lambda^{n/2p} A^{-1}(f_\lambda \otimes f_\lambda)$, where $f_\lambda(x) = \mathrm{e}^{-\lambda|x|^2}$ and $\lambda > 0$. Since $a_\lambda$ is a positive semi-definite operator of rank one we have that $a_\lambda \in C_+$ and

$$\|a_\lambda\|_{s_p} = \|a_\lambda\|_{s_1} = \lambda^{n/2p}\|f_\lambda\|_{L^2}^2 = (\pi/2)^n \lambda^{-n/2p'},$$

and it follows that $\|a_\lambda\|_{s_p} \to \infty$ as $\lambda \to 0$ and $p > 1$.

On the other hand, since $f_\lambda$ is a Gauss function, it follows that $a$, and therefore $\chi a$, are Gauss functions. In particular we may compute their $\mathscr{F}_\sigma L^p$-norms exactly, and by some straight-forward computations one achieves that $\|\mathscr{F}_\sigma(\chi a)\|_{L^p} = c_n(2\lambda+1)^n$ for some constant $c_n$, depending on $n$ only. Hence, $\|\mathscr{F}_\sigma(\chi a)\|_{s_p} \leqslant c'_n(2\lambda+1)^n$ by Corollary 4.11, which proves that $\|\mathscr{F}_\sigma(\chi a)\|_{s_p}$ stays bounded as $\lambda \to 0$, and the assertion follows.

Proposition 4.6 gives us also the opportunities to continue our discussion concerning regularity for elements in $\mathscr{S}'_+(W)$ in terms of wavefront sets (see Section 8.1 in [4]). We recall that if $a \in \mathscr{D}'(W)$ and $(X, Y) \in W \times (W \setminus \{0\})$, then $(X, Y) \notin WF(a)$ (where $WF(a)$ is the wavefront set for $a$), if and only if for some $\chi \in C_0^\infty(W)$ such that $\chi(X) \neq 0$ and some open cone $\Sigma$ containing $Y$, we may to any $N \geqslant 0$ find a constant $C_N$ such that

$$\left|\mathscr{F}(\chi a)(Z)\right| \leqslant C_N\bigl(1+|Z|\bigr)^{-N}, \quad Z \in \Sigma. \tag{4.9}$$

**Theorem 4.14.** *Assume that $a \in \mathscr{S}'_+(W)$, and that $(0, Y) \notin WF(a)$. Then for any $X \in W$ one has that $(X, Y) \notin WF(a)$ and $(X, Y) \notin WF(\mathscr{F}_\sigma a)$.*

**Proof.** We note that the statement is invariant of the choice of Fourier transform in the definition of wave-front set, and we shall use the symplectic Fourier transform $\mathscr{F}_\sigma$ instead of $\mathscr{F}$ in (4.9).

Assume that $(0, Y) \notin WF(a)$. Then for some $\chi \in C_0^\infty(W)$ such that $\chi(0) \neq 0$ we have that (4.9) is valid for every $N \geqslant 0$ and some open conic neighbourhood $\Sigma$ of $Y$. From Section 8.1 in [4] and Corollary 4.2 it follows also that we may assume also that $0 \neq \chi \in C_+ \cap C_0^\infty$. By Proposition 4.6 it follows that (4.9) holds for every $N$ when $\Sigma$ is replaced by a smaller open cone containing $Y$, and $\chi$ is replaced by $\chi_X = \chi(\cdot - X)$. This proves that $(X, Y) \notin WF(a)$, and by similar arguments it follows also that $(X, Y) \notin WF(\mathscr{F}_\sigma a)$. The proof is complete.

## 5. Some further properties

In this section we shall discuss some further properties for elements in $\mathscr{S}'_+$. We start by considering support properties for elements in $\mathscr{F}_\sigma\mathscr{S}'_+(W)$, and prove that there are no nontrivial elements which are compactly supported. We shall actually prove a more general result, that if $a \in \mathscr{F}_\sigma\mathscr{S}'_+(W)$ has support in a convex set, such that its boundary contains "symplectic corners", then $a = 0$. Thereafter we discuss dilation properties and prove that $s_1$, $s_\infty$ and $\mathscr{S}'_+$ are not invariant under dilations. In the end of the section we show that there are positive elements in the $*_\sigma$-algebra which are negative (as functions) on quite large domains.

The support result which we shall prove is the following:

**Proposition 5.1.** *Assume that $a \in \mathscr{S}'_+(W)$ such that* $\operatorname{supp}\mathscr{F}_\sigma a \subset D$*, where $D$ is given by*

$$D = \bigl\{X \in W;\ \sigma(X, Y) \leqslant C,\ \sigma(X, Z) \leqslant C\bigr\}, \quad \textit{where } \sigma(Y, Z) \neq 0. \tag{5.1}$$

*Then $a = 0$. In particular $\mathscr{F}_\sigma\mathscr{S}'_+(W) \cap \mathscr{E}'(W) = \{0\}$.*

We note that if $D$ is given by (5.1), then for some appropriate choice of symplectic coordinates $X = (x_1, \dots, \xi_n)$ and some constant $C$, we have

$$D = \bigl\{(x, \xi) \in W;\ x_1 \leqslant C,\ \xi_1 \leqslant C\bigr\}. \tag{5.1$'$}$$

**Proof.** We may assume that $W = T^*\mathbf{R}^n$, and that $D$ is given by (5.1$'$). Assume that $a \in \mathscr{S}'_+(W)$. By a simple argument of approximation using Lemma 3.5 together with the fact that $\mathscr{F}_\sigma C_+$ is invariant under translation, it follows that we may assume that $a \in C_+(W) \cap \mathscr{S}(W)$ and that $C = 0$ in (5.1′).

From Lemma 1.1 and Proposition 1.8 we have for some $(f_j) \in \mathrm{ON}(\mathbf{R}^n)$ that

$$a(x,\xi) = (2\pi)^{-n/2} \int e^{i\langle y,\xi\rangle}\big(f(y/2-x),\, f(y/2+x)\big)\,dy, \tag{5.2}$$

where $f = (\sqrt{\lambda_1}\, f_1, \sqrt{\lambda_2}\, f_2, \ldots)$ is an $L^2$ function with values in $l^2$ such that $f_j \in \mathscr{S}$ for every $j$. By (5.2) and Lemma 1.1 we have

$$\mathscr{F}_\sigma a(x,\xi) = (2\pi)^{-n/2} \int e^{i\langle y',\xi'\rangle} G(x_1, \xi_1, -y'/2 + x', y'/2 + x')\,dy',$$

where $x' = (x_2, \ldots, x_n)$, $y' = (y_2, \ldots, y_n)$ and

$$G(x_1, \xi_1, x', y') = \int e^{iy_1\xi_1}\big(f(-y_1/2 + x_1, x'),\, f(y_1/2 + x_1, y')\big)\,dy_1. \tag{5.3}$$

If we let $g = \mathscr{F}_1 f$, where $\mathscr{F}_1$ is the partial Fourier transform of $\mathscr{F}$ in (0.4) with respect to the variable $x_1$, then it follows from (5.3) and some straight-forward computations, using Fourier's inversion formula that

$$G(x_1, \xi_1, x', y') = \int e^{-ix_1\eta_1}\big(g(-\eta_1/2 + \xi_1, x'),\, g(\eta_1/2 + \xi_1, y')\big)\,d\eta_1. \tag{5.3′}$$

Since $\operatorname{supp} \hat{a} \subset D$, it follows from (5.2) and Fourier's inversion formula that $G(x_1, \xi_1, x', y') = 0$ when $x_1 \geqslant 0$ or $\xi_1 \geqslant 0$. By applying Fourier's inversion formula in (5.3) and (5.3′), we get that $(f(x), f(y))$ and $(g(\xi_1, x'), g(\eta_1, y'))$ vanish in the sets $x_1 + y_1 \geqslant 0$ and $\xi_1 + \eta_1 \geqslant 0$ respectively. Hence $\operatorname{supp}(f)$ and $\operatorname{supp}(\mathscr{F}_1 f)$ are contained in the set $\{x;\ x_1 \in \overline{\mathbf{R}_-}$ and $x' \in \mathbf{R}^{n-1}\}$, where $\overline{\mathbf{R}_-} = \{x \in \mathbf{R};\ x \leqslant 0\}$.

This means that if $F_{x',j}(x_1) = f_j(x_1, \ldots, x_n)$ is considered as a function in $x_1 \in \mathbf{R}$ with $x'$ as fix parameter, then $F_{x',j} \in \mathscr{S}(\mathbf{R})$ by Lemma 3.14, and the supports for $F_{x',j}$ and $\widehat{F}_{x',j}$ are contained in $\overline{\mathbf{R}_-}$. From the support property for $F_{x',j}$, it follows that $\widehat{F}_{x',j}$ is an analytic function in the half-space $\omega = \{z \in \mathbf{C};\ \mathrm{Im}(z) > 0\}$, which is continuous in the closure $\bar{\omega}$ of $\omega$, and zero at the positive real axis. An application of Schwartz principle of reflection implies that $\widehat{F}_{x',j}$ extends to a function which is analytic everywhere except at the negative real axis. Since this function is zero on the positive real axis, it follows that $\widehat{F}_{x',j}$ must be identical zero. Since $x'$ was arbitrary chosen, we conclude that $f_j(x_1, \ldots, x_n) = F_{x',j}(x_1) = 0$, for every $(x_1, \ldots, x_n) \in \mathbf{R}^n$. Hence $f(x) = 0$, and it follows that $a(X) = 0$. The proof is complete.

**Remark 5.3.** Assume that $a \in \mathscr{S} \cap C_+$. Then it follows from (5.2) and Fourier's inversion formula that

$$a(x,\xi) = (2\pi)^{-n/2} \int e^{-i\langle x,\eta\rangle}\big(\hat{f}(\eta/2-\xi),\, \hat{f}(\eta/2+\xi)\big)\,d\eta. \tag{5.4}$$

From Corollary 4.2, the proof of Proposition 5.1 and an inspection of the formulas (5.2) and (5.4) it follows that it is possible to construct a function $0 \not\equiv f \in L^2(\mathbf{R}^n; l^2)$ such that $(f(x), f(y)) = (\hat{f}(\xi), \hat{f}(\eta)) = 0$ when $|x - y| > \varepsilon$, $|\xi - \eta| > \varepsilon$, where $\varepsilon$ may be any small positive number. In fact, such function $f$ is obtained if one chooses $a = \tilde{v} *_\sigma v \in C_+$

in (5.2), where $0 \neq v \in C_0^\infty$ with support in a sufficiently small neighbourhood of the origin.

Next we shall prove that $s_1$ and positivity in $*_\sigma$-algebra are not invariant under dilations.

**Proposition 5.4.** *There exist simple elements $a, b \in s_1(W)$ with $b \in \mathscr{S}$ such that if $\rho \notin \{-1, 1\}$, then $a(\rho\cdot) \notin \mathscr{S}'_+ \cup s_1$ and $b(\rho\cdot) \notin \mathscr{S}'_+$.*

**Proof.** We may assume that $W = T^*\mathbf{R}^n$ and we write $Aa = f \otimes \bar{f}$, where $A$ is as in (0.3) and $f \in L^2(\mathbf{R}^n)$. We also set $a_\rho = a(\rho\cdot)$ and $b_\rho = b(\rho\cdot)$. If $\rho = 0$, then $a_\rho = (2/\pi)^{n/2}$ everywhere and it follows from Theorem 3.3 and Proposition 1.5 that $a_\rho$ is neither positive semi-definite nor in $s_1$. Assume that $\rho \notin \{0, -1, 1\}$. A simple computation shows that

$$|\rho|^n (Aa_\rho)(x, y) = f(\alpha x + \beta y)\bar{f}(\beta x + \alpha y), \tag{5.5}$$

where $\alpha = (\rho + \rho^{-1})/2$ and $\beta = (\rho^{-1} - \rho)/2$. Hence $\alpha\beta \neq 0$ and $\alpha + \beta \neq 0$. If $a_\rho \in s_1(W)$ then $Aa_\rho$ is the operator kernel of a self-adjoint trace class operator on $L^2(\mathbf{R}^n)$. Hence we may write

$$f(\alpha x + \beta y)\bar{f}(\beta x + \alpha y) = |\rho|^n (Au_\rho)(x, y) = \sum \lambda_j g_j(x)\overline{g_j}(y),$$

where the $(g_j) \in \mathrm{ON}(\mathbf{R}^n)$ and $\sum |\lambda_j| < \infty$. Replacing $y$ by $(z - \alpha x)/\beta$ we have, since $\alpha^2 = 1 + \beta^2$

$$f(z)\bar{f}\big((\alpha z - x)/\beta\big) = \sum \lambda_j g_j(x)\overline{g_j}\big((z - \alpha x)/\beta\big),$$

where the right-hand side above is a function of $(x, z)$ which is integrable over $\mathbf{R}^n \times K$ when $K$ is any cube in $\mathbf{R}^n$. An application of Fubini's theorem implies therefore that $f$ must be integrable if $a_\rho \in s_1(W)$. In order to have the condition $a_\rho \notin s_1(W)$ fulfilled it suffices for us therefore to choose $f$ in $L^2 \setminus L^1$.

We choose $f$ also such that $f$ is continuous and $f(x) = \mathrm{e}^{Q(x)}$, where $Q$ is real and continuous, and $Q(0) = 0$. We shall see that it is possible to choose $Q$ such that $a_\rho$ is not positive semi-definite for any $\rho$ as above.

A necessary condition for $Aa_\rho$ to be positive semi-definite is that

$$\big|f(\alpha x + \beta y)\bar{f}(\beta x + \alpha y)\big| \leqslant \big|f\big((\alpha + \beta)x\big)\big|\big|f\big((\alpha + \beta)y\big)\big|, \quad x, y \in \mathbf{R}^n.$$

By replacing $(x, y)$ by $(x, y)/(\alpha + \beta)$ in this inequality we see that it suffices for us to construct $Q$ such that the inequality

$$Q(\alpha x + \beta y) + Q(\beta x + \alpha y) > Q(x) + Q(y) \tag{5.6}$$

is solvable for any choice of $\alpha$ and $\beta$ with $\alpha\beta \neq 0$ and $\alpha + \beta = 1$. Then it sufficies to choose $Q$ such that $Q(x) = |x|^2$ in an open neighbourhood of the origin, and that $Q(x) = -|x|^2$ when $x$ belongs to an other open set. The proof is complete.

**Remark 5.5.** It follows from Proposition 5.4, and duality (cf. Proposition 1.5) that $s_\infty(W)$ is not invariant under dilations.

**Remark 5.6.** According to Proposition 5.4 we note here that if $a \in \mathcal{S}(W)$ and $a_\rho = a(\rho\cdot)$ is positive semi-definite for every $\rho > 0$, then $a = 0$. To see this we observe that when $a_\rho$ is positive semi-definite it follows from Lemma 3.5 that

$$(\pi/2)^{n/2}a(0) = \|a_\rho\|_{s_1} \geqslant \|a_\rho\|_{L^2} = |\rho|^{-n}\|a\|_{L^2}.$$

When $\rho \to 0$ this shows that $a = 0$.

The following result shows that the set where a $\sigma$-positive function is negative may be of infinite volume.

**Proposition 5.7.** *There exists $a \in s_1(T^*\mathbf{R}) \setminus L^1(T^*\mathbf{R})$ such that $a$ is simple, $\hat{a} = a$ and such that the set $\Omega = \{X;\ a(X) < 0\}$ contains infinitely many open disjoint parallelograms of area $\pi/4$.*

**Proof.** Let $f$ be the characteristic function of the interval $[-1, 1]$ and define $a \in L^2(T^*\mathbf{R})$ by $Aa = f \otimes f/2$. Then $a$ is simple and $\hat{a} = a$ in view of Lemma 1.1(ii). It follows from Lemma 1.1(v) and a straight-forward computation that

$$a(x, \xi) = (2\pi)^{-1/2}\xi^{-1} f(x)\sin\bigl(2(1 - |x|)\xi\bigr),$$

and it follows $a \notin L^1$, and that $\Omega$ contains

$$\Omega_n = \bigl\{(x, \xi);\ 0 < x < 1,\ (2n - 1)\pi < 2(1 - x)\xi < 2n\pi\bigr\}, \quad n = 1, 2, \ldots.$$

Each of these sets has infinite area.

We let $(s, t) = (2\xi/\pi, 1 - x)$ be new coordinates, in which $\Omega_1$ corresponds to the set $\Omega_1'$ defined by $0 < t < 1$, $1 < st < 2$. For any $\sigma > 2$, then $\Omega_1'$ contains the open parallelogram with corners at $(\sigma, 1/\sigma)$, $(\sigma, 2/\sigma)$, $(\sigma/2, 2/\sigma)$ and $(\sigma/2, 3/\sigma)$. The result follows now since there are infinitely numbers of such parallelograms, and that the area of any of the corresponding parallelograms in $\Omega_1$ is equal to $\pi/4$. The proof is complete.

**Remark 5.8.** We note that it is possible to find simple elements which are negative on quite larger sets than $\Omega$ in Proposition 5.7. More precisely, if $\varepsilon > 0$, then the following is true:

(1) there is a simple element $a$ which is negative outside

$$\bigl\{(x, \xi) \in T^*\mathbf{R};\ \varepsilon x^2 + \xi^2/\varepsilon \leqslant 1/2\bigr\};$$

(2) there is a simple element $a$ such that $\mathcal{F}_\sigma a = a$ and $a$ is negative outside

$$\bigl\{(x, \xi) \in T^*\mathbf{R};\ |x| \leqslant \varepsilon\bigl(|\xi| + 1\bigr)\bigr\}.$$

In fact, let $f(x) = (\lambda/\pi)^{1/4}\mathrm{e}^{-\lambda x^2}$, where $\lambda > 0$ and $x \in \mathbf{R}$. Then $a_0(x, \xi) = A^{-1}(f \otimes f)(x, \xi) = (2/\pi)^{1/2}\mathrm{e}^{-(\lambda x^2 + \xi^2/\lambda)}$ is simple. From Lemma 3.15 and its proof it follows that if $T = -4^{-1}\partial_\xi^2 - x^2$, then

$$a(x, \xi) = c(Ta_0)(x, \xi) = c(2/\pi)^{1/2}\lambda^{-1}\bigl(1/2 - \lambda x^2 - \xi^2/\lambda\bigr)\mathrm{e}^{-(\lambda x^2 + \xi^2/\lambda)}$$

is simple for some choice of $c > 0$. By choosing $\lambda = \varepsilon$, we obtain that $a$ satisfies (1).

In the same way one obtains (2) if we instead let $a = cT^2 a_0$, for some constant $c > 0$ and some appropriate choice of $\lambda$.

## 6. Applications to pseudo-differential calculus

In this section we shall use the results in previous sections together with Proposition 1.10 and Remark 3.10, in order to establish some positivity results in pseudo-differential calculus. For any $t \in \mathbf{R}$ and $p \in [1, \infty]$, we use the notation $s_{t,p}(\mathbf{R}^{2n})$ for the set of all $a \in \mathscr{S}'(\mathbf{R}^{2n})$ such that $a_t(x, D) \in \mathscr{I}_p$. (Note here that the general symplectic vector space $W$ is replaced by $\mathbf{R}^{2n}$, since the symplectic invariance which is valid for the Weyl quantization, i.e. the case $t = 1/2$, is not true for general $t$.)

**Proposition 6.1.** *Assume that $\Omega \subset \mathbf{R}^{2n}$ is an open neighbourhood of origin, and that $a \in \mathscr{S}'(\mathbf{R}^{2n})$ satisfies $a_t(x, D) \geqslant 0$ for some $t \in \mathbf{R}$. Then the following is true*:

(1) *if $\hat{a} \in C^{2N}(\Omega)$ for some integer $N \geqslant 0$, then $X^\alpha D_X^\beta a \in s_{t,1}$ for every $\alpha$ and $\beta$ such that $|\alpha + \beta| \leqslant N$;*
(2) *if $\chi, \psi \in \mathscr{S}(\mathbf{R}^{2n})$ such that $\chi$ has nonvanishing integral, then $\chi * a \in s_{t,p}$ and $a + \psi * a \in s_{t,p}$, if and only if $a \in s_{t,p}$. A similar result holds when the $s_{t,p}$ spaces are replaced by $L^p$ spaces;*
(3) *if $t = 1/2$ and $(X, Y) \in WF(a)$ for some $X, Y \in \mathbf{R}^{2n}$ such that $Y \neq 0$, then $(0, Y) \in WF(\mathscr{F}_\sigma a)$;*
(4) *if $t = 1/2$ and $a$ has compact support, then $a = 0$.*

**Proof.** The result follows immediately from Proposition 1.9, Proposition 1.10, Remark 3.10, Theorem 3.13, Theorem 4.12, Theorem 4.14 and Proposition 5.1.

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