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Existence of fractional neutral functional differential equations☆

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ABSTRACT

In this paper, the initial value problem is discussed for a class of fractional neutral functional differential equations and the criteria on existence are obtained.

1. Introduction

In this paper, we consider the initial value problems (IVP for short) of fractional neutral functional differential equations with bounded delay of the form

$$\begin{cases} {}^cD^\alpha(x(t) - g(t, x_t)) = f(t, x_t), & t \in (t_0, +\infty),\ t_0 \geq 0, \\ x_{t_0} = \phi, \end{cases} \tag{1}$$

where ${}^cD^\alpha$ is the standard Caputo's fractional derivative of order $0 < \alpha < 1$, $f, g : [t_0, +\infty) \times C([-r, 0], R^n) \to R^n$ are given functions satisfying some assumptions that will be specified later, $a > 0$ and $\phi \in C([-r, 0], R^n)$. If $x \in C([t_0 - r, t_0 + a], R^n)$, then for any $t \in [t_0, t_0 + a]$ define x_t by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]$.

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc.. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al. [1], Lakshmikantham et al. [2], Miller and Ross [3], Podlubny [4] and the papers in [5–19] and the references therein. In [5], Agarwal, Belmekki and Benchohra obtain existence results for semilinear functional differential inclusions involving fractional derivatives. In [7], Benchohra et al. consider the IVP for a class of fractional neutral functional differential equations with infinite delay. In [11], El-Sayed discusses a class of nonlinear functional differential equations of arbitrary orders. In [20], Lakshmikantham initiates the basic theory for fractional functional differential equations. In [17–19], Zhou et al. investigated the existence and uniqueness for fractional functional differential equations with unbounded and infinite delay.

In this paper, we discuss the initial value problem for a class of fractional neutral functional differential equations with bounded delay. We firstly deduce IVP (1) to a equivalent integral equation. Next, by using Krasnoselskii's fixed point theorem, we get that the equivalent operator has (at least) a fixed point, it means that IVP (1) has at least one solution.

2. Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Let $J \subset R$. Denote $C(J, R^n)$ be the Banach space of all continuous functions from J into R^n with the norm

$$\|x\| = \sup_{t \in J} |x(t)|,$$

where $|\cdot|$ denotes a suitable complete norm on R^n.

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Definition 2.1 (*[1,4]*). The fractional integral of order q with the lower limit t_0 for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{1-q}} \mathrm{d}s, \quad t > t_0,\ q > 0,$$

provided the right-hand side is pointwise defined on $[t_0, \infty)$, where Γ is the gamma function.

Definition 2.2 (*[1,4]*). Riemann–Liouville derivative of order q with the lower limit t_0 for a function $f : [t_0, \infty) \to R$ can be written as

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} \frac{f(s)}{(t-s)^{q+1-n}} \mathrm{d}s, \quad t > t_0, n-1 < q < n.$$

The first–and maybe the most important–property of Riemann–Liouville fractional derivative is that for $t > t_0$ and $q > 0$, we have

$$D^q(I^q f(t)) = f(t),$$

which means that Riemann–Liouville fractional differentiation operator is a left inverse to the Riemann–Liouville fractional integration operator of the same order q.

Definition 2.3 (*[1,4]*). Caputo's derivative of order q with the lower limit t_0 for a function $f : [t_0, \infty) \to R$ can be written as

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} \mathrm{d}s = I^{n-q} f^{(n)}(t), \quad t > t_0,\ n-1 < q < n.$$

Obviously, Caputo's derivative of a constant is equal to zero.

Remark 2.1. We need to mention that there exits a link between Riemann–Liouville and Caputo's fractional derivative of order q. Namely,

$$\begin{aligned} {}^cD^q f(t) &= \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} \mathrm{d}s \\ &= D^q f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t_0)}{\Gamma(k-q+1)} (t-t_0)^{k-q} \\ &= D^q \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t_0)}{k!} (t-t_0)^k \right], \quad t > t_0,\ n-1 < q < n. \end{aligned}$$

Lemma 2.1 (*Krasnoselskii's Fixed Point Theorem*). *Let X be a Banach space, let E be a bounded closed convex subset of X and let S, U be maps of E into X such that $Sx + Uy \in E$ for every pair $x, y \in E$. If S is a contraction and U is completely continuous, then the equation*

$$Sx + Ux = x$$

has a solution on E.

3. Main results

Let

$$I_0 = [t_0, t_0 + \delta],$$
$$A(\delta, \gamma) = \{x \in C([t_0 - r, t_0 + \delta], R^n) | x_{t_0} = \phi, \sup_{t_0 \le t \le t_0 + \delta} |x(t) - \phi(0)| \le \gamma\},$$

where δ, γ are positive constants.

Before stating and proving the main results, we introduce the following hypotheses.

(H_1) $f(t, \varphi)$ is measurable with respect to t on I_0,
(H_2) $f(t, \varphi)$ is continuous with respect to φ on $C([-r, 0], R^n)$,
(H_3) there exist $\alpha_1 \in (0, \alpha)$ and a real-valued function $m(t) \in L^{\frac{1}{\alpha_1}}(I_0)$ such that for any $x \in A(\delta, \gamma)$, $|f(t, x_t)| \le m(t)$, for $t \in I_0$,
(H_4) for any $x \in A(\delta, \gamma)$, $g(t, x_t) = g_1(t, x_t) + g_2(t, x_t)$,
(H_5) g_1 is continuous and for any $x', x'' \in A(\delta, \gamma)$, $t \in I_0$

$$|g_1(t, x'_t) - g_1(t, x''_t)| \le l\|x' - x''\|, \quad \text{where } l \in (0, 1),$$

(H_6) g_2 is completely continuous and for any bounded set Λ in $A(\delta, \gamma)$, the set $\{t \to g_2(t, x_t) : x \in \Lambda\}$ is equicontinuous in $C(I_0, R^n)$.

Lemma 3.1. *If there exist* $\delta \in (0, a)$ *and* $\gamma \in (0, \infty)$ *such that* (H_1)–(H_3) *are satisfied, then for* $t \in (t_0, t_0 + \delta]$, *IVP* (1) *is equivalent to the following equation*

$$\begin{cases} x(t) = \phi(0) - g(t_0, \phi) + g(t, x_t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x_s)\mathrm{d}s, & t \in I_0, \\ x_{t_0} = \phi. \end{cases} \tag{2}$$

Proof. First, it is easy to obtain that $f(t, x_t)$ is Lebesgue measurable on I_0 according to conditions (H_1) and (H_2). A direct calculation gives that $(t-s)^{\alpha-1} \in L^{\frac{1}{1-\alpha_1}}([t_0, t])$, for $t \in I_0$. In the light of the Hölder inequality and (H_3), we obtain that $(t-s)^{\alpha-1} f(s, x_s)$ is Lebesgue integrable with respect to $s \in [t_0, t]$ for all $t \in I_0$ and $x \in A(\delta, \gamma)$, and

$$\int_{t_0}^{t} |(t-s)^{\alpha-1} f(s, x_s)|\mathrm{d}s \le \|(t-s)^{\alpha-1}\|_{L^{\frac{1}{1-\alpha_1}}([t_0,t])} \|m\|_{L^{\frac{1}{\alpha_1}}(I_0)}, \tag{3}$$

where

$$\|F\|_{L^p(J)} = \left(\int_J |F(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}$$

for any L^p-integrable function $F : J \to R$.

According to Definitions 2.1 and 2.3, it is easy to see that if x is a solution of the IVP (1), then x is a solution of the Eq. (2). On the other hand, if (2) is satisfied, then for every $t \in (t_0, t_0 + \delta]$, we have

$$\begin{aligned} {}^cD^{\alpha}(x(t) - g(t, x_t)) &= {}^cD^{\alpha}\left[\phi(0) - g(t_0, \phi) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x_s)\mathrm{d}s \right] \\ &= {}^cD^{\alpha}\left[\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x_s)\mathrm{d}s \right] \\ &= {}^cD^{\alpha}(I^{\alpha} f(t, x_t)) \\ &= D^{\alpha}(I^{\alpha} f(t, x_t)) - [I^{\alpha} f(t, x_t)]_{t=t_0} \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &= f(t, x_t) - [I^{\alpha} f(t, x_t)]_{t=t_0} \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned}$$

According to (3), we know that $[I^{\alpha} f(t, x_t)]_{t=t_0} = 0$, which means that ${}^cD^{\alpha}(x(t) - g(t, x_t)) = f(t, x_t),\ t \in (t_0, t_0 + \delta]$, and this completes the proof. □

Theorem 3.1. *Assume that there exist* $\delta \in (0, a)$ *and* $\gamma \in (0, \infty)$ *such that* (H_1)–(H_6) *are satisfied. Then the IVP* (1) *has at least one solution on* $[t_0, t_0 + \eta]$ *for some positive number* η.

Proof. According to (H_4), Eq. (2) is equivalent to the following equation

$$\begin{cases} x(t) = \phi(0) - g_1(t_0, \phi) - g_2(t_0, \phi) + g_1(t, x_t) + g_2(t, x_t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1} f(s, x_s)\mathrm{d}s, & t \in I_0, \\ x_{t_0} = \phi. \end{cases}$$

Let $\widetilde{\phi} \in A(\delta, \gamma)$ be defined as $\widetilde{\phi}_{t_0} = \phi, \widetilde{\phi}(t_0 + t) = \phi(0)$ for all $t \in [0, \delta]$. If x is a solution of the IVP (1), let $x(t_0 + t) = \widetilde{\phi}(t_0 + t) + y(t),\ t \in [-r, \delta]$, then we have $x_{t_0+t} = \widetilde{\phi}_{t_0+t} + y_t,\ t \in [0, \delta]$. Thus y satisfies the equation

$$\begin{aligned} y(t) = &-g_1(t_0, \phi) - g_2(t_0, \phi) + g_1(t_0 + t, y_t + \widetilde{\phi}_{t_0+t}) + g_2(t_0 + t, y_t + \widetilde{\phi}_{t_0+t}) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(t_0 + s, y_s + \widetilde{\phi}_{t_0+s})\mathrm{d}s, \quad t \in [0, \delta]. \end{aligned} \tag{4}$$

Since g_1, g_2 are continuous and x_t is continuous in t, there exists $\delta' > 0$, when $0 < t < \delta'$,

$$|g_1(t_0 + t, y_t + \widetilde{\phi}_{t_0+t}) - g_1(t_0, \phi)| < \frac{\gamma}{3}, \tag{5}$$

and

$$|g_2(t_0 + t, y_t + \widetilde{\phi}_{t_0+t}) - g_2(t_0, \phi)| < \frac{\gamma}{3}. \tag{6}$$

Choose

$$\eta = \left\{\delta, \delta', \left(\frac{\gamma\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}{3M}\right)^{\frac{1}{(1+\beta)(1-\alpha_1)}}\right\} \tag{7}$$

where $\beta = \frac{\alpha-1}{1-\alpha_1} \in (-1, 0)$ and $M = \|m\|_{L^{\frac{1}{\alpha_1}}(I_0)}$.

Define $E(\eta, \gamma)$ as follows

$$E(\eta, \gamma) = \{y \in C([-r, \eta], R^n) \mid y(s) = 0 \text{ for } s \in [-r, 0] \text{ and } \|y\| \le \gamma\}.$$

Then $E(\eta, \gamma)$ is a closed bounded and convex subset of $C([-r, \delta], R^n)$. On $E(\eta, \gamma)$ we define the operators S and U as follows

$$Sy(t) = \begin{cases} 0, & t \in [-r, 0], \\ -g_1(t_0, \phi) + g_1(t_0 + t, y_t + \widetilde{\phi}_{t_0+t}), & t \in [0, \eta], \end{cases}$$

$$Uy(t) = \begin{cases} 0, & t \in [-r, 0], \\ -g_2(t_0, \phi) + g_2(t_0 + t, y_t + \widetilde{\phi}_{t_0+t}) & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(t_0 + s, y_s + \widetilde{\phi}_{t_0+s})\mathrm{d}s, & t \in [0, \eta]. \end{cases}$$

It is easy to see that if the operator equation

$$y = Sy + Uy \tag{8}$$

has a solution $y \in E(\eta, \gamma)$ if and only if y is a solution of Eq. (4). Thus $x(t_0 + t) = y(t) + \widetilde{\phi}(t_0 + t)$ is a solution of Eq. (1) on $[0, \eta]$. Therefore, the existence of a solution of the IVP (1) is equivalent that (8) has a fixed point in $E(\eta, \gamma)$.

Now we show that $S + U$ has a fixed point in $E(\eta, \gamma)$. The proof is divided into three steps.

Step I. $Sz + Uy \in E(\eta, \gamma)$ for every pair $z, y \in E(\eta, \gamma)$.

In fact, for every pair $z, y \in E(\eta, \gamma)$, $Sz + Uy \in C([-r, \eta], R^n)$. Also, it is obvious that $(Sz + Uy)(t) = 0$, $t \in [-r, 0]$. Moreover, for $t \in [0, \eta]$, by (5)–(7) and the condition (H_3), we have

$$\begin{aligned} |Sz(t) + Uy(t)| &\le |-g_1(t_0, \phi) + g_1(t_0 + t, z_t + \widetilde{\phi}_{t_0+t})| + |-g_2(t_0, \phi) + g_2(t_0 + t, y_t + \widetilde{\phi}_{t_0+t})| \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t |(t-s)^{\alpha-1} f(t_0 + s, y_s + \widetilde{\phi}_{t_0+s})|\mathrm{d}s \\ &\le \frac{2\gamma}{3} + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}\mathrm{d}s\right)^{1-\alpha_1}\left(\int_{t_0}^{t_0+t} (m(s))^{\frac{1}{\alpha_1}}\mathrm{d}s\right)^{\alpha_1} \\ &\le \frac{2\gamma}{3} + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}\mathrm{d}s\right)^{1-\alpha_1}\left(\int_{t_0}^{t_0+\delta} (m(s))^{\frac{1}{\alpha_1}}\mathrm{d}s\right)^{\alpha_1} \\ &\le \frac{2\gamma}{3} + \frac{M\eta^{(1+\beta)(1-\alpha_1)}}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} \\ &\le \gamma. \end{aligned}$$

Therefore,

$$\|Sz + Uy\| = \sup_{t\in[0,\eta]} |(Sz)(t) + (Uy)(t)| \le \gamma,$$

which means that $Sz + Uy \in E(\eta, \gamma)$ for any $z, y \in E(\eta, \gamma)$.

Step II. S is a contraction on $E(\eta, \gamma)$.

For any $y', y'' \in E(\eta, \gamma)$, $y'_t + \widetilde{\phi}_{t_0+t}, y''_t + \widetilde{\phi}_{t_0+t} \in A(\delta, \gamma)$. So by ($H_5$), we get that

$$\begin{aligned} |Sy'(t) - Sy''(t)| &= |g_1(t_0 + t, y'_t + \widetilde{\phi}_{t_0+t}) - g_1(t_0 + t, y''_t + \widetilde{\phi}_{t_0+t})| \\ &\le l\|y' - y''\|, \end{aligned}$$

which implies that

$$\|Sy' - Sy''\| \le l\|y' - y''\|.$$

In view of $0 < l < 1$, S is a contraction on $E(\eta, \gamma)$.

Step III. Now we show that U is a completely continuous operator.

Let

$$U_1 y(t) = \begin{cases} 0, & t \in [-r, 0], \\ -g_2(t_0, \phi) + g_2(t_0 + t, y_t + \widetilde{\phi}_{t_0+t}), & t \in [0, \eta] \end{cases}$$

and

$$U_2y(t)=\begin{cases}0, & t\in[-r,0],\\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}f(t_0+s,y_s+\widetilde{\phi}_{t_0+s})\mathrm{d}s, & t\in[0,\eta].\end{cases}$$

Clearly, $U=U_1+U_2$.

Since g_2 is completely continuous, U_1 is continuous and $\{U_1y: y\in E(\eta,\gamma)\}$ is uniformly bounded. From the condition that the set $\{t\to g_2(t,x_t): x\in\Lambda\}$ be equicontinuous for any bounded set Λ in $A(\delta,\gamma)$, we can conclude that U_1 is a completely continuous operator.

On the other hand, for any $t\in[0,\eta]$, we have

$$\begin{aligned}|U_2y(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(t_0+s,y_s+\widetilde{\phi}_{t_0+s})|\mathrm{d}s\\ &\le \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}\right)^{1-\alpha_1}\left(\int_{t_0}^{t_0+t}(m(s))^{\frac{1}{\alpha_1}}\mathrm{d}s\right)^{\alpha_1}\\ &\le \frac{M\eta^{(1+\beta)(1-\alpha_1)}}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}.\end{aligned}$$

Hence, $\{U_2y: y\in E(\eta,\gamma)\}$ is uniformly bounded.

Now, we will prove that $\{U_2y: y\in E(\eta,\gamma)\}$ is equicontinuous. For any $0\le t_1<t_2\le\eta$ and $y\in E(\eta,\gamma)$, we get that

$$\begin{aligned}|U_2y(t_2)-U_2y(t_1)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]f(t_0+s,y_s+\widetilde{\phi}_{t_0+s})\mathrm{d}s\right.\\ &\quad\left.+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(t_0+s,y_s+\widetilde{\phi}_{t_0+s})\mathrm{d}s\right|\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]|f(t_0+s,y_s+\widetilde{\phi}_{t_0+s})|\mathrm{d}s\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(t_0+s,y_s+\widetilde{\phi}_{t_0+s})|\mathrm{d}s\\ &\le \frac{M}{\Gamma(\alpha)}\left(\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}}\mathrm{d}s\right)^{1-\alpha_1}+\frac{M}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}[(t_2-s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}}\mathrm{d}s\right)^{1-\alpha_1}\\ &\le \frac{M}{\Gamma(\alpha)}\left(\int_0^{t_1}(t_1-s)^{\beta}-(t_2-s)^{\beta}\mathrm{d}s\right)^{1-\alpha_1}+\frac{M}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{\beta}\mathrm{d}s\right)^{1-\alpha_1}\\ &\le \frac{M}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}\left(t_1^{1+\beta}-t_2^{1+\beta}+(t_2-t_1)^{1+\beta}\right)^{1-\alpha_1}+\frac{M}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}(t_2-t_1)^{(1+\beta)(1-\alpha_1)}\\ &\le \frac{2M}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}(t_2-t_1)^{(1+\beta)(1-\alpha_1)},\end{aligned}$$

which means that $\{U_2y: y\in E(\eta,\gamma)\}$ is equicontinuous. Moreover, it is clear that U_2 is continuous. So U_2 is a completely continuous operator. Then $U=U_1+U_2$ is a completely continuous operator.

Therefore, Krasnoselskii's fixed point theorem shows that $S+U$ has a fixed point on $E(\eta,\gamma)$, and hence the IVP (1) has a solution $x(t)=\phi(0)+y(t-t_0)$ for all $t\in[t_0,t_0+\eta]$. This completes the proof. □

In the case where $g_1\equiv 0$, we get the following result.

Theorem 3.2. *Assume that there exist $\delta\in(0,a)$ and $\gamma\in(0,\infty)$ such that* (H_1)–(H_3) *hold and*

$(\mathrm{H}_5)'$ *g is continuous and for any $x',x''\in A(\delta,\gamma)$, $t\in I_0$*

$$|g(t,x'_t)-g(t,x''_t)|\le l\|x'-x''\|,\quad \text{where } l\in(0,1).$$

Then the IVP (1) has at least one solution on $[t_0,t_0+\eta]$ for some positive number η.

In the case where $g_2\equiv 0$, we have the following result.

Theorem 3.3. *Assume that there exist $\delta\in(0,a)$ and $\gamma\in(0,\infty)$ such that* (H_1)–(H_3) *hold and*

$(\mathrm{H}_6)'$ *g is completely continuous and for any bounded set Λ in $A(\delta,\gamma)$, the set $\{t\to g(t,x_t): x\in\Lambda\}$ is equicontinuous on $C(I_0,R^n)$.*

Then the IVP (1) has at least one solution on $[t_0,t_0+\eta]$ for some positive number η.

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