In this paper we consider a mathematical model describing static elastic contact problems with the Hooke constitutive law and subdifferential boundary conditions. We treat boundary hemivariational inequalities which are weak formulations of contact problems. We establish existence and uniqueness of solutions to hemivariational inequalities. Using the notion of $H$-convergence of elasticity tensors we investigate the limit behavior of the sequence of solutions to hemivariational inequalities.

Abstract

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Keywords: Elasticity; $H$-convergence; Hemivariational inequality; Homogenization; Static process; Frictional contact; Clarke subdifferential; Weak solution

1. Introduction

In this paper we deal with the boundary hemivariational inequalities. The study is motivated by a mathematical model describing static elastic contact problems with the Hooke constitutive law and subdifferential boundary conditions. The mechanical problem concerns a linear elastic body which may come in contact with a foundation. The dependence of the normal stress on the normal displacement is assumed to have nonmonotone character of the subdifferential form. We model the friction assuming that the tangential shear on the contact surface is given as a nonmonotone and possibly multivalued function of the tangential displacement. Due to the nonmonotone character of these multivalued boundary conditions, a convex analysis approach to the problem cannot be employed. It leads to hemivariational inequality models involving the Clarke subdifferential of a locally Lipschitz functional. There is a large class of mechanical problems with nonconvex energy functions which are generally nonsmooth. For example, considering the contact between an elastic structure and a granular medium (or a composite material) we arrive to multivalued boundary conditions of the subdifferential type, cf. [13,14].

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The notion of hemivariational inequality was introduced by Panagiotopoulos in the 1980s (cf. Panagiotopoulos [14]) as generalizations of variational inequalities. For motivation and mathematical results on hemivariational inequalities we refer to Panagiotopoulos [14], Nanciewicz and Panagiotopoulos [13] and the references therein. On the other hand in many problems of physics one has to solve boundary value problems in periodic media considering equations with highly oscillating coefficients. Quite often the size of the period is small compared to the size of a simple of the medium and an averaging process is needed to reduce the complexity of the problem. In the mathematical theory of homogenization, the problem is embedded into a sequence of similar problems and an asymptotic analysis is performed as the lengthscale goes to zero. The most general theory of homogenization is that of $H$-convergence which was introduced by De Giorgi and Spagnolo under the name of $G$-convergence and further generalized by Murat and Tartar who described the so-called energy method. The variational theory of homogenization is based on the $\Gamma$-convergence due to De Giorgi, cf. [3] and an extensive literature therein.

In this paper we treat a hemivariational inequality which is weak formulation of a model contact problem. First, we establish the existence of solutions to this hemivariational inequality. This result is a consequence of surjectivity result for multivalued operators. Next, we deliver sufficient conditions under which the solution to a hemivariational inequality is unique. These results are quite general and they allow to deduce existence and uniqueness of solutions to a class of elasticity models with nonmonotone and possible multivalued boundary conditions. Then, using the notion of $H$-convergence of elasticity tensors we investigate the limit behavior of the sequence of solutions to hemivariational inequalities. The limit is of the same form and corresponds to the homogenized tensor. We use the $H$-convergence adopted to the elasticity setting by Francfort and Murat [6] and Tartar [15]. We prove that the $H$-convergence defined by the convergence of solutions to homogeneous Dirichlet problems implies not only the convergence of local solutions (cf. Proposition 1.4.6 of [1] and Section 12.2 of [16]) but also the convergence of solutions to boundary value problems with multivalued and nonmonotone boundary conditions.

To our knowledge the homogenization of hemivariational inequalities has not been considered in the literature till now. Finally, we remark that an extension of our results to dynamic problems with nonmonotone boundary conditions seems to be an open problem. We hope to report on our efforts in this direction in a forthcoming paper.

We also mention that a general method for the study of contact problems involving subdifferential boundary conditions was presented in [12]. Within the framework of hemivariational inequalities, this method represents a new approach which unifies several methods used in the study of frictional contact problems for viscoelastic materials and allows to obtain new existence and uniqueness results. The reader is referred to [8] for the results on $H$-convergence of elliptic equations with nonhomogeneous Dirichlet and Neumann boundary conditions, to [9] for the corresponding results on evolution problems and to [10] for the boundary homogenization approach applied to inverse problems.

The paper is structured as follows. In Section 2 we give a preliminary material and describe the mechanical model of frictional contact between an elastic body and a foundation. The process is static and the contact is modeled with subdifferential boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation of the problem which is in the form of a hemivariational inequality for the displacement field. Then in Section 4 we state our main existence and uniqueness result, Theorem 5, which deals with the unique weak solvability of frictions (cf. Proposition 1.4.6 of [1] and Section 12.2 of [16]) but also the convergence of solutions to homogeneous Dirichlet problems implies not only the convergence of local solu-

### 2. Preliminaries and mechanical model

In this section we present the notation, some preliminary material which will be used in the next sections and we describe a mechanical model which motivates our study. For further details, we refer to [2,4,5,7,14].

Let $d$ be a positive integer and denote by $S_d$ the linear space of second order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space $\mathbb{R}^{d \times d}$ of symmetric matrices of order $d$. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $S_d$ are given by

$$u \cdot v = u_i v_i, \quad \|v\|_d = (v \cdot v)^{1/2} \quad \text{for all } u, v \in \mathbb{R}^d,$$

$$\sigma : \tau = \sigma_{ij} \tau_{ij}, \quad \|\tau\|_{S_d} = (\tau : \tau)^{1/2} \quad \text{for all } \sigma, \tau \in S_d.$$

In this paper the indices $i, j, k, l$ run from 1 to $d$ and summation convention over repeated indices is used. Let $M_d = \{A = (a_{ijkl}) \mid a_{ijkl} = a_{klij} = a_{iklj} = a_{ijlk}\}$ be the space of symmetric fourth order tensors acting on symmetric
matrices. Given $\alpha$ and $\beta$ two positive constants such that $\alpha \beta \leq 1$, we define the subspace of $\mathcal{M}_d$ which consists of coercive tensors with coercive inverses

$$
\mathcal{M}_{\alpha, \beta} = \{ A \in \mathcal{M}_d \mid \alpha \| \tau \|_{S_d}^2 \leq A \tau : \tau, \quad \beta \| \tau \|_{S_d}^2 \leq A^{-1} \tau : \tau \text{ for all } \tau \in S_d \}.
$$

It is easy to remark that if $A \in \mathcal{M}_{\alpha, \beta}$, then $\alpha \| \tau \|_{S_d}^2 \leq A \tau : \tau \leq \frac{1}{\alpha} \| \tau \|_{S_d}^2$ for all $\tau \in S_d$.

Given a Banach space $X$, we denote its norm by $\| \cdot \|_X$. The dual space is denoted by $X^*$ and $\langle \cdot, \cdot \rangle_{X^* \times X}$ is the duality pairing between $X$ and $X^*$. For a set $U \subset X$, we write $\| U \|_X = \sup \{ \| x \|_X : x \in U \}$. The symbol $w-X$ denotes the space $X$ endowed with its weak topology and the notation $\mathcal{L}(X, Y)$ stands for the space of linear bounded operators from a Banach space $X$ to a Banach space $Y$.

**Definition 1.** (See [2,4].) Let $p : X \rightarrow \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of $p$ at $x \in X$ in the direction $v \in X$, denoted by $p^0(x; v)$, is defined by

$$
p^0(x; v) = \limsup_{y \to x, \lambda \downarrow 0} \frac{p(y + \lambda v) - p(y)}{\lambda}.
$$

The generalized gradient of $p$ at $x$, denoted by $\partial p (x)$, is the subset of $X^*$ given by

$$
\partial p (x) = \{ \xi \in X^* \mid p^0(x; v) \geq \langle \xi, v \rangle_{X^* \times X} \text{ for all } v \in X \}.
$$

A locally Lipschitz function $p$ is called regular (in the sense of Clarke) at $x \in X$ if for all $v \in X$ the one-sided generalized directional derivative $p'(x; v)$ exists and satisfies $p^0(x; v) = p'(x; v)$ for all $v \in X$.

We state the properties of the generalized directional derivative and the generalized gradient which are needed in the sequel (cf. Theorem 2.3.10 of [2]).

**Proposition 2.** Let $X$ and $Y$ be Banach spaces, $L \in \mathcal{L}(Y, X)$ and let $p : X \rightarrow \mathbb{R}$ be a locally Lipschitz function. Then

(i) $(p \circ L)^0(x; z) \leq p^0(Lx; Lz)$ for $x, z \in Y$,

(ii) $\partial (p \circ L)(x) \subseteq L^* \partial p (Lx)$ for $x \in Y$,

where $L^* \in \mathcal{L}(X^*, Y^*)$ denotes the adjoint operator to $L$. If in addition either $p$ or $-p$ is regular, then (i) and (ii) are replaced by the corresponding equalities.

Let us conclude this section with the formulation of the contact problem of linear elasticity.

We consider a linear elastic body which occupies the domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$ in applications) with a smooth boundary $\partial \Omega = \Gamma$ and a unit outward normal $n$. The body is acted upon by body forces of density $f_0$. We consider a partition of $\Gamma$ into three open disjoint parts $\Gamma_D$, $\Gamma_N$ and $\Gamma_C$ such that $\text{meas}(\Gamma_D) > 0$. The body is clamped on $\Gamma_D$ and thus, the displacement field vanishes there. Surface tractions of density $f_N$ act on $\Gamma_N$. In the reference configuration the body may come in contact over $\Gamma_C$ with an obstacle, the so-called foundation. The process is assumed to be static, i.e. the inertial terms in the momentum balance equations are neglected (cf. [7]).

We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, i.e. $u_N = u \cdot n$, $u_T = u - u_N n$, $\sigma_N = (\sigma n) \cdot n$ and $\sigma_T = \sigma n - \sigma_N n$.

The contact problem under consideration is as follows.

**Problem (P).** Find a displacement field $u : \Omega \rightarrow \mathbb{R}^d$ and a stress field $\sigma : \Omega \rightarrow S_d$ such that

\begin{align*}
\text{div } \sigma (u) + f_0 &= 0 \quad \text{in } \Omega, \quad (1) \\
\sigma (u) &= A e(u) \quad \text{in } \Omega, \quad (2) \\
u &= 0 \quad \text{on } \Gamma_D, \quad (3) \\
n \sigma n &= f_N \quad \text{on } \Gamma_N, \quad (4) \\
-\sigma_N &= \partial_j N (x, u_N) \quad \text{on } \Gamma_C, \quad (5) \\
-\sigma_T &= \partial_j T (x, u_T) \quad \text{on } \Gamma_C. \quad (6)
\end{align*}
Equation (1) is the steady state equation for the stress, in which div denotes the divergence operator for tensor valued functions, i.e. \( \text{div} \sigma = (\sigma_{ij,j}) \). The index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable.

Equation (2) characterizes the elastic body and represents the elastic linear constitutive law (the Hooke law) between the linearized strain tensor \( e(u) \) and the stress tensor \( \sigma(u) \) (the so-called strain-stress law), where \( A \in \mathcal{M}_d \) is the elasticity tensor. Recall that

\[
e(u) = (e_{ij}(u)), \quad e_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i}).
\]

The conditions (3) and (4) are the displacement and traction boundary conditions. Our main interest is in the boundary conditions (5) and (6) which describe the contact and the frictional conditions on the potential contact surface \( \Gamma_C \). Here \( j_N \) and \( j_T \) are given functions and the notation \( \partial j_N \) and \( \partial j_T \) stands for the Clarke subdifferentials of \( j_N(x, \cdot) \) and \( j_T(x, \cdot) \).

### 3. Variational formulation

To present the variational formulation of Problem (P) we need some notations and preliminaries. We start by introducing the spaces

\[
\mathcal{H} = \left\{ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega) \right\} = L^2(\Omega; S_d),
\]

\[
\mathcal{H}_1 = \left\{ u \in H \mid e(u) \in \mathcal{H} \right\} = H^1(\Omega; \mathbb{R}^d), \quad \mathcal{H}_1 = \{ \tau \in \mathcal{H} \mid \text{div} \tau \in H \},
\]

where \( e \) and div denote the deformation and the divergence operators, defined above. The spaces \( H, \mathcal{H}, \mathcal{H}_1 \) and \( \mathcal{H}_1 \) are Hilbert spaces equipped with the inner products

\[
\langle u, v \rangle_H = \int_{\Omega} u \cdot v \, dx, \quad \langle \sigma, \tau \rangle_{\mathcal{H}} = \int_{\Omega} \sigma : \tau \, dx,
\]

\[
\langle u, v \rangle_{\mathcal{H}_1} = \langle u, v \rangle_H + \langle e(u), e(v) \rangle_{\mathcal{H}}, \quad \langle \sigma, \tau \rangle_{\mathcal{H}_1} = \langle \sigma, \tau \rangle_{\mathcal{H}} + \langle \text{div} \sigma, \text{div} \tau \rangle_H.
\]

Given \( v \in H^1/2(\Gamma; \mathbb{R}^d) \) we denote by \( v_N \) and \( v_T \) the normal and the tangential components of \( v \) on the boundary \( \Gamma \), i.e. \( v_N = v \cdot n \) and \( v_T = v - v_N n \). Similarly, for a tensor field \( \sigma \in C^1(\Gamma; S_d) \), we define its normal and tangential components by \( \sigma_N = (\sigma n) \cdot n \) and \( \sigma_T = \sigma n - \sigma N n \). Recall the following Green formula

\[
\langle \sigma, e(v) \rangle_{\mathcal{H}} + \langle \text{div} \sigma, v \rangle_H = \int_{\Gamma} \sigma n \cdot v d\Gamma \quad \text{for} \; \sigma \in \mathcal{H}_1, \; v \in \mathcal{H}_1.
\]

For the displacement field we introduce the space

\[
V = \left\{ v \in H^1(\Omega; \mathbb{R}^d) \mid v = 0 \; \text{on} \; \Gamma_D \right\},
\]

which is a closed subspace of \( \mathcal{H}_1 \). On \( V \) we consider the inner product and the corresponding norm given by

\[
\langle u, v \rangle_V = \langle e(u), e(v) \rangle_{\mathcal{H}}, \quad \| v \|_V = \| e(v) \|_{\mathcal{H}} \quad \text{for all} \; u, v \in V.
\]

Since \( \text{meas}(\Gamma_D) > 0 \), it follows from the Korn inequality that \( (V, \| \cdot \|_V) \) is a Hilbert space. Let \( Z = H^3(\Omega; \mathbb{R}^d) \) where \( \delta \in (1/2, 1) \) and let \( \gamma : Z \rightarrow L^2(\Gamma; \mathbb{R}^d) \) denote the trace operator and \( \gamma^* : L^2(\Gamma; \mathbb{R}^d) \rightarrow Z^* \) stands for its adjoint. Moreover, we denote by \( \langle \cdot, \cdot \rangle_{V^* \times V} \) the duality pairing of \( V \) and \( V^* \). It is well known that \( V \subset Z \subset H \subset Z^* \subset V^* \) continuously and compactly.

In what follows we need the following hypotheses.

\( (H_0) \) \( f_0 \in H, f_N \in L^2(\Gamma_N; \mathbb{R}^d) \).

\( H(j_N) \) \( j_N : \Gamma_C \times \mathbb{R} \rightarrow \mathbb{R} \) is such that

(i) \( j_N(\cdot, r) \) is measurable for all \( r \in \mathbb{R} \) and \( j_N(\cdot, 0) \in L^1(\Gamma_C) \);

(ii) \( j_N(x, \cdot) \) is locally Lipschitz for a.e. \( x \in \Gamma_C \).
The function $\vartheta(t) = \int_0^t \vartheta(s) ds$ and we find that

\[ \int f_N(x, r; -r) \leq d_N(1 + |r|) \text{ for all } r \in \mathbb{R}, \text{ a.e. } x \in \Gamma_C \text{ with } d_N \geq 0. \]

\[ H(j_T) : \Gamma_C \times \mathbb{R}^d \rightarrow \mathbb{R} \text{ is such that} \]

(iii) $|\partial j_N(x, r)| \leq c_N(1 + |r|)$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_C$ with $c_N > 0$;

(iv) $f_N^0(x, r; -r) \leq d_N(1 + |r|)$ for all $r \in \mathbb{R}$, a.e. $x \in \Gamma_C$ with $d_N \geq 0$.

Next, define $f$ and $\sigma$ are regular functions which solve (1)–(6). Let $v \in V$. From the equilibrium equation (1) and the Green formula (7), we find that

\[ \{ \sigma(u), e(v) \}_{\mathcal{H}} = \int \int \sigma_n \cdot v d\Gamma. \]  

We take into account the boundary conditions (3) and (4) to see that

\[ \int \sigma_n \cdot v d\Gamma = \int f_N \cdot v d\Gamma + \int (\sigma_N v_N + \sigma_T \cdot v_T) d\Gamma. \]  

On the other hand, from the definition of the Clarke subdifferential and the inclusions (5) and (6), we get

\[ -\sigma_N v_N \leq j_N^0(x, u_N; v_N), \quad -\sigma_T \cdot v_T \leq j_T^0(x, u_T; v_T) \text{ on } \Gamma_C \]

and

\[ \int (\sigma_N v_N + \sigma_T \cdot v_T) d\Gamma \geq - \int (j_N^0(x, u_N; v_N) + j_T^0(x, u_T; v_T)) d\Gamma. \]

Next, define $f \in V^*$ by

\[ \langle f, v \rangle_{V^* \times V} = \langle f_0, v \rangle_{\mathcal{H}} + \langle f_N, v \rangle_{L^2(\Gamma_C; \mathbb{R}^d)} \quad \text{for all } v \in V. \]

Hence using (8)–(11), we have

\[ \{ \sigma(u), e(v) \}_{\mathcal{H}} + \int (j_N^0(x, u_N; v_N) + j_T^0(x, u_T; v_T)) d\Gamma \geq \langle f, v \rangle_{V^* \times V}. \]

We substitute (2) in (12) and derive the following variational formulation of Problem (P), in terms of displacement field.

**Problem (HVI).** Find a displacement field $u \in V$ such that

\[ \{ A e(u), e(v) \}_{\mathcal{H}} + \int (j_N^0(x, u_N; v_N) + j_T^0(x, u_T; v_T)) d\Gamma \geq \langle f, v \rangle_{V^* \times V} \quad \text{for all } v \in V. \]

The above problem is called a boundary hemivariational inequality. In the next section we prove existence and uniqueness of solutions for Problem (HVI).

Finally, we comment on the boundary conditions (5) and (6).

**Example 3.** The condition (5) is a general expression of the normal compliance contact condition which models the relationship between the normal stress and the normal displacement. Consider the condition (5) with the function $j_N : \mathbb{R} \rightarrow \mathbb{R}$ defined by $j_N(t) = \int_0^t \vartheta(s) ds$ for $t \in \mathbb{R}$ (for simplicity we have omitted the $x$-dependence), where the function $\vartheta \in L^\infty_{\text{loc}}(\mathbb{R})$ is such that $|\vartheta(s)| \leq k(1 + |s|)$ for $s \in \mathbb{R}$ with $k > 0$. It is well known (see [2,12]) that
\[ \vartheta(s) = \hat{\vartheta}(s) \text{ for } s \in \mathbb{R}, \text{ where the multivalued function } \hat{\vartheta} : \mathbb{R} \rightarrow 2^\mathbb{R} \text{ is given by } \hat{\vartheta}(s) = [\vartheta_1(s), \vartheta_2(s)] \text{ ([·]) denotes an interval in } \mathbb{R} \text{ and } \]
\[ \vartheta_1(s) = \lim_{\varepsilon \rightarrow 0^+} \text{ess inf } \vartheta(\tau), \quad \vartheta_2(s) = \lim_{\varepsilon \rightarrow 0^+} \text{ess sup } \vartheta(\tau). \]

In this case \( H(j_N) \) holds and (5) takes the form \(-\sigma_N \in \hat{\vartheta}(u_N)\) on \( \Gamma_C \). If additionally \( \vartheta \) is a continuous function (as in [7]), then (5) reduces to \(-\sigma_N = \vartheta(u_N)\) on \( \Gamma_C \).

We provide two concrete examples. In the first one, let \( j_N : \mathbb{R} \rightarrow \mathbb{R} \) be given by
\[ \vartheta(s) = \begin{cases} ks & \text{if } s \in (-\infty, -1) \cup (1, +\infty), \\ 2ks & \text{if } s \in (-1, 1), \end{cases} \]
where \( k > 0 \). Then \( |\vartheta(s)| \leq k(1 + |s|) \) for \( s \in \mathbb{R} \) and the nonconvex function \( j_N : \mathbb{R} \rightarrow \mathbb{R} \) can be expressed as a minimum of two convex functions, i.e. \( j_N(s) = \min\{j_1(s), j_2(s)\} \), where \( j_1(s) = ks^2 \) and \( j_2(s) = \frac{k}{2}(s^2 + 1) \) for \( s \in \mathbb{R} \). Then
\[ \partial j_N(s) = \begin{cases} ks & \text{if } s \in (-\infty, -1) \cup (1, +\infty), \\ 2ks & \text{if } s \in (-1, 1), \\ [-2k, -k] & \text{if } s = -1, \\ [k, 2k] & \text{if } s = 1. \end{cases} \]

This model example can be modified to obtain nonmonotone zig-zag relations which describe the adhesive contact problems and contact laws for a granular material and a reinforced concrete (see Sections 2.4 and 7.2 of [14] and Section 4.6 of [13]).

In the second example, we consider the nonmonotone Winkler law. Let \( \vartheta \in L^\infty_\text{loc}(\mathbb{R}) \) be defined by
\[ \vartheta(s) = \begin{cases} 0 & \text{if } s \in (-\infty, 0) \cup (e_0, +\infty), \\ k_0s & \text{if } s \in (0, e_0), \end{cases} \]
where \( e_0 \) is a small positive constant and \( k_0 > 0 \) is the Winkler coefficient. Then \( |\vartheta(s)| \leq k_0e_0 \) for \( s \in \mathbb{R} \) and \( j_N(s) = \min\{g_1(s), g_2(s)\} \),
\[ g_1(s) = \frac{k_0}{2} s^2 \quad \text{if } s < 0, \quad \text{and } g_2(s) = \frac{k_0}{2} e_0^2 \quad \text{if } s \geq 0, \]
for \( s \in \mathbb{R} \). Assuming that the tangential forces are known \( \sigma_T = C_T \), \( C_T = C_T(x) \) is given on \( \Gamma_C \), the condition (5) can be interpreted as follows. In the noncontact region \( u_N < 0 \) and we have \( \sigma_N = 0 \). For \( u_N \in [0, e_0) \) the contact is idealized by the Winkler law \(-\sigma_N = k_0u_N \). If \( u_N = e_0 \), then we deal with destruction of the support and we have \(-\sigma_N \in [0, k_0e_0] \). When \( u_N > e_0 \), then \( \sigma_N = 0 \) and it holds in a region where the support has been destructed. The support can maintain the maximal value of reactions given by \( k_0e_0 \).

**Example 4.** The multivalued condition (6) appears in several mechanical problems in elasticity. In the simplest case, if \( j_T = 0 \), we are lead to frictionless contact. We point out the reaction-displacement diagrams in Chapter 4.6 of [13] which imitate nonmonotone variants of the friction law of Coulomb. Analogous situations arise in geomechanics and rock interface analysis as well as in friction laws between reinforcement and concrete in concrete structures. The sawtooth laws generated by nonconvex superpotentials \( j_T \) (see [14, Chapter 2.4]) describe the partial cracking and crushing of the adhesive bonding material.

Consider the function \( j_T : \mathbb{R}^d \rightarrow \mathbb{R} \) given by \( j_T(\xi) = \min\{\varphi_1(\xi), \varphi_2(\xi)\}, \xi \in \mathbb{R}^d \), where \( \varphi_i : \mathbb{R}^d \rightarrow \mathbb{R}, i = 1, 2 \), are convex and quadratic. Using Theorem 2.5.1 of [2], we know that \( j_T(\xi) \subset \text{co}(\nabla \varphi_1(\xi), \nabla \varphi_2(\xi)) \) and hence the subdifferential has at most a linear growth. Next, we have \( j_T^0(\xi; \eta) = \max\{\xi^* \cdot \eta | \xi^* \in j_T(\xi)\} \), cf. Proposition 2.1.2 of [2]. Therefore
\[ j_T^0(\xi; -\xi) = \max\{\xi^* \cdot (-\xi) | \xi^* = \lambda \nabla \varphi_1(\xi) + (1 - \lambda) \nabla \varphi_2(\xi), \lambda \in (0, 1)\} \leq 0, \]
because \( \nabla \varphi_i(\xi) \cdot \xi \geq 0 \) for \( i = 1, 2 \). Hence the function \( j_T \) satisfies hypothesis \( H(j_T) \).

The concrete examples of two- and three-dimensional nonconvex superpotential laws of the form (6) which are defined as minima and maxima of convex quadratic functions (e.g., the friction law with a locking effect, the adhesive contact law, etc.) are detailed in Section 4.6.1 of [13].
4. Existence and uniqueness results

The goal of this section is to demonstrate the following unique weak solvability of Problem (HVI).

**Theorem 5.** Let $A$ be a fourth order tensor in $L^\infty(\Omega; \mathcal{M}_{a,b})$. Under the hypotheses $(H_0)$, $H(j_N)$ and $H(j_T)$, Problem (HVI) admits a solution. If in addition we suppose that

$$H(j_N)(v) \text{ either } j_N(x, \cdot) \text{ or } -j_N(x, \cdot) \text{ is regular and } (\eta_1 - \eta_2)(r_1 - r_2) \geq -m_N |r_1 - r_2|^2 \text{ for all } \eta_i \in \partial j_N(x, r_i), r_i \in \mathbb{R}, i = 1, 2, \text{ a.e. } x \in \Gamma_C \text{ with } m_N > 0;$$

$$H(j_T)(v) \text{ either } j_T(x, \cdot) \text{ or } -j_T(x, \cdot) \text{ is regular and } (\eta_1 - \eta_2) \cdot (\xi_1 - \xi_2) \geq -m_T \|\xi_1 - \xi_2\|_{\mathbb{R}^d}^2 \text{ for all } \eta_i \in \partial j_T(x, \xi_i), \xi_i \in \mathbb{R}^d, i = 1, 2, \text{ a.e. } x \in \Gamma_C \text{ with } m_T > 0;$$

$$(H1) \alpha > \max\{m_N, m_T\} c_0^2 \|\gamma\|^2,$$

where $\|\gamma\|$ denotes the norm of the trace operator $\gamma$ in the space $\mathcal{L}(Z, L^2(\Gamma_C; \mathbb{R}^d))$, $Z = H^\delta(\Omega; \mathbb{R}^d)$ with a fixed $\delta \in (1/2, 1)$ and $c_0$ is the embedding constant of $V$ into $Z$, then Problem (HVI) has a unique solution.

The proof will be given in several steps. First, we need the properties of the integral functional associated with superpotentials $j_N$ and $j_T$. Consider the functional $J : L^2(\Gamma_C; \mathbb{R}^d) \to \mathbb{R}$ defined by

$$J(w) = \int_{\Gamma_C} (j_N(x, w_N(x)) + j_T(x, w_T(x))) \, d\Gamma \quad \text{for } w \in L^2(\Gamma_C; \mathbb{R}^d). \quad (13)$$

**Lemma 6.** Under hypotheses $H(j_N)(i)$–(iv) and $H(j_T)(i)$–(iv), the functional defined by (13) satisfies

(i) $J$ is locally Lipschitz on $L^2(\Gamma_C; \mathbb{R}^d)$;

(ii) $\|\partial J(w)\|_{L^2(\Gamma_C; \mathbb{R}^d)} \leq c(1 + \|w\|_{L^2(\Gamma_C; \mathbb{R}^d)})$ for all $w \in L^2(\Gamma_C; \mathbb{R}^d)$ with $c > 0$;

(iii) the following inequality holds

$$J^0(w; z) \leq \int_{\Gamma_C} (j_N(x, w_N; z_N) + j_T^0(x, w_T; z_T)) \, d\Gamma \quad \text{for } w, z \in L^2(\Gamma_C; \mathbb{R}^d); \quad (14)$$

(iv) $J^0(w; -w) \leq d(1 + \|w\|_{L^2(\Gamma_C; \mathbb{R}^d)})$ for all $w \in L^2(\Gamma_C; \mathbb{R}^d)$ with $d > 0$;

(v) if additionally either $j_N(x, \cdot)$ or $-j_N(x, \cdot)$ and either $j_T(x, \cdot)$ or $-j_T(x, \cdot)$ are regular in the sense of Clarke, then $J$ is regular and in (14) we have equality.

For the proof we refer to Lemma 5 of [11].

**Proof of Theorem 5.** Assume first that $(H_0)$, $H(j_N)$ and $H(j_T)$ hold. Define the operator $A : V \to V^*$ by

$$\langle Au, v \rangle_{V^* \times V} = \int_\Omega A e(u) : e(v) \, dx \quad \text{for } u, v \in V. \quad (15)$$

It is easy to observe that if $A \in L^\infty(\Omega; \mathcal{M}_{a,b})$, then the operator $A$ is linear continuous and coercive, i.e. $\langle Au, v \rangle_{V^* \times V} \geq \alpha \|v\|^2_V$ for all $v \in V$. Consider the auxiliary problem: find $u \in V$ such that

$$Au + \gamma^* \partial J(\gamma u) \ni f, \quad (16)$$

where $J$ and $A$ are given by (13) and (15), respectively.

**Claim 1.** Every solution to problem (16) is a solution to Problem (HVI).

Indeed, let $u \in V$ satisfies (16). Then $Au + z = f$ with $z = \gamma^* \xi$ and $\xi \in \partial J(\gamma u)$. For every $v \in V$, we have $\langle Au, v \rangle_{V^* \times V} + \langle z, v \rangle_{Z^* \times Z} = \langle f, v \rangle_{V^* \times V}$ and by Lemma 6 it follows that
\[ \langle z, v \rangle_{Z^* \times Z} = \{ y^*z, v \}_{Z^* \times Z} = \langle z, y^* \rangle_{L^2(\mathcal{G}; \mathbb{R}^d)} \leq \|y^*\|_{L^2(\mathcal{G}; \mathbb{R}^d)} \leq \langle 0, y^* \rangle \leq \int_{\mathcal{G}} \left( f_N(x, u_N; v_N) + \int_{\mathcal{G}} f_T(x, u_T; v_T) \right) d\Gamma. \]

Hence, we have
\[ \langle Au, v \rangle_{V^* \times V} + \int_{\mathcal{G}} \left( f_N(x, u_N; v_N) + \int_{\mathcal{G}} f_T(x, u_T; v_T) \right) d\Gamma \geq \langle f, v \rangle_{V^* \times V} \quad \text{for all } v \in V, \]

which means that \( u \) is a solution to (HVI).

**Claim 2.** For every \( f \in V^* \), the operator inclusion (16) admits a solution.

Define the multivalued operator \( F : V \rightarrow 2^{V^*} \) by \( Fv = \{ w \in Z^* ; \ w \in y^* \partial J(y^*v) \} \) for \( v \in V \). We will establish the following properties of the operator \( F \).

(a) \( \| Fv \|_{Z^*} \leq \tilde{c}(1 + \| v \|_V) \) for all \( v \in V \) with \( \tilde{c} > 0 \);
(b) for all \( v \in V \), \( Fv \) is a nonempty convex and weakly compact subset of \( Z^* \);
(c) \( \langle Fv, v \rangle_{V^* \times V} \geq -c_1 \| v \|_V - c_2 \) for all \( v \in V \) with \( c_1, c_2 > 0 \);
(d) for all \( v_n, v \in V \) with \( v_n \rightarrow v \) in \( Z \) and \( w_n, w \in Z^* \) with \( w_n \rightarrow w \) weakly in \( Z^* \), if \( w_n \in Fv_n \), then \( w \in Fv \).

For the proof of (a), let \( v \in V \) and \( w \in Fv \). Hence \( w = y^*z \) with \( z \in \partial J(y^*v) \). Using \( H(J)(ii) \), we have
\[ \| z \|_{L^2(\mathcal{G}; \mathbb{R}^d)} \leq c(1 + \| y^*v \|_{L^2(\mathcal{G}; \mathbb{R}^d)}) \leq c(1 + \| y^* \|_{L^2(\mathcal{G}; \mathbb{R}^d)}) \leq c(1 + c_0 \| y^* \|_{L^2(\mathcal{G}; \mathbb{R}^d)}) \]
where the constant \( c_0 > 0 \) satisfies \( \| \cdot \|_{Z^*} \leq c_0 \| \cdot \|_V \). Hence \( \| w \|_{Z^*} = \| y^* \|_{L^2(\mathcal{G}; \mathbb{R}^d)}^* \leq c^* \| y^* \|^* (1 + c_0 \| y^* \|_{L^2(\mathcal{G}; \mathbb{R}^d)}) \)
and (a) follows.

For the proof of (b), we recall (cf. Proposition 2.1.2 of [2]) that the values of \( \partial J \) are nonempty, weakly compact and convex subsets of \( L^2(\mathcal{G}; \mathbb{R}^d) \). Thus \( Fv \) is a nonempty and convex subset in \( Z^* \). To show that \( Fv \) is weakly compact in \( Z^* \), we show that it is closed in \( Z^* \). Indeed, let \( v \in V \), \( \{ w_n \} \subset Fv \) and \( w_n \rightarrow w \) in \( Z^* \). Since \( w_n \in y^* \partial J(y^*v) \) and the latter is a closed subset of \( Z^* \), we obtain \( w \in y^* \partial J(y^*v) \) and thus \( w \in Fv \). Therefore, the set \( Fv \) is closed in \( Z^* \) and convex, so it is also weakly closed in \( Z^* \). Because \( Fv \) is a bounded set in a reflexive Banach space \( Z \), we get that \( Fv \) is weakly compact in \( Z^* \).

Proof of (c). Consider \( v \in V \) and \( w \in Fv \). Therefore \( w = y^*z \) and \( z \in \partial J(y^*v) \). Using \( H(J)(iv) \), we have
\[ -\langle z, y^* v \rangle_{L^2(\mathcal{G}; \mathbb{R}^d)} \leq \langle 0, y^* v \rangle \leq d(1 + \| y^* v \|_{L^2(\mathcal{G}; \mathbb{R}^d)}) \leq d(1 + c_0 \| y^* \|_{L^2(\mathcal{G}; \mathbb{R}^d)}) \leq c_1 \| v \|_V + c_2 \]
where \( c_1, c_2 > 0 \). Hence, it follows \( \langle w, v \rangle_{V^* \times V} = \langle w, v \rangle_{Z^* \times Z} = \langle y^*z, v \rangle_{Z^* \times Z} = \langle z, y^*v \rangle_{L^2(\mathcal{G}; \mathbb{R}^d)} \geq -c_1 \| v \|_V - c_2 \)
which implies property (c).

For the proof of (d), let \( w_n \in Fv_n, v_n, v \in V, v_n \rightarrow v \) in \( Z \), \( w_n, w \in Z^* \) and \( w_n \rightarrow w \) weakly in \( Z^* \). Then \( w_n = y^*z_n \) and \( z_n \in \partial J(y^*v_n) \). The continuity of the trace implies \( y^*v_n \rightarrow y^*v \) in \( L^2(\mathcal{G}; \mathbb{R}^d) \) and \( H(J)(ii) \) implies that at least for a subsequence, we have \( z_n \rightarrow z \) weakly in \( L^2(\mathcal{G}; \mathbb{R}^d) \) with some \( z \in L^2(\mathcal{G}; \mathbb{R}^d) \). By the equality \( w_n = y^*z_n \) we easily get \( w = y^*z \). Since the graph of \( \partial J \) is closed in \( L^2(\mathcal{G}; \mathbb{R}^d) \times w \), \( w \in L^2(\mathcal{G}; \mathbb{R}^d) \), \( w = L^2(\mathcal{G}; \mathbb{R}^d) \) denotes the space \( L^2(\mathcal{G}; \mathbb{R}^d) \) furnished with the weak topology (cf. Proposition 5.6.10 in [4]), from \( z_n \in \partial J(y^*v_n) \) we obtain \( z \in \partial J(y^*v) \) and subsequently \( w \in y^* \partial J(y^*v) \), i.e. \( w \in Fv \).

Next, we will show that the operator \( F : V \rightarrow 2^{V^*} \) given by \( Fv = Av + Fv \) for \( v \in V \) is coercive and pseudomonotone. Using the property (c) of the operator \( F \) and the coercivity of \( A \), we get
\[ \langle Fv, v \rangle_{V^* \times V} = \langle Av, v \rangle_{V^* \times V} + \langle Fv, v \rangle_{Z^* \times Z} \geq \alpha \| v \|^2 - c_1 \| v \|_V - c_2 \]
for all \( v \in V \).

In order to demonstrate the pseudomonotonicity of \( F \), we apply Proposition 6.3.66 of [5]. It says that a generalized pseudomonotone operator with nonempty, bounded, closed and convex values is pseudomonotone. From the property (b) of the operator \( F \), it follows that \( F \) has nonempty, convex and closed values. From (a), it is clear that \( F \) is a bounded map.
We will show that $F$ is a generalized pseudomonotone operator. Let $v_n \to v$ weakly in $V$, $v_n^* \to v^*$ weakly in $V^*$, $v_n^* \in Fv_n$ and $\limsup (v_n^*, v_n - v)_{V^* \times V} \leq 0$. We will show that $v^* \in Fv$ and $(v_n^*, v_n)_{V^* \times V} \to (v^*, v)_{V^* \times V}$. We have $v_n^* = A v_n + w_n$ where $w_n \in F v_n$. Since the embedding $V \subset Z$ is compact, it follows that

$$v_n \to v \quad \text{in } Z.$$  \hfill (17)

By the boundedness of $F$, by passing to a subsequence if necessary, we have

$$w_n \to w \quad \text{weakly in } Z^* \text{ with some } w \in Z^*.$$  \hfill (18)

From the property (d) we infer that $w \in Fv$. Furthermore, from the equality $\langle v_n^*, v_n - v \rangle_{V^* \times V} = (A v_n, v_n - v)_{V^* \times V} + \langle w_n, v_n - v \rangle_{Z^* \times Z}$, we obtain $\limsup (A v_n, v_n - v)_{V^* \times V} = \limsup (\langle v_n^*, v_n - v \rangle_{V^* \times V} \leq 0$. Exploiting the pseudomonotonicity of $A$ (recall that $A$ is continuous and monotone, cf. Chapter 6 of [5]), we deduce

$$A v_n \to A v \quad \text{weakly in } V^*$$  \hfill (19)

and

$$\lim (A v_n, v_n - v)_{V^* \times V} = 0.$$  \hfill (20)

Therefore, by passing to the limit in the equation $v_n^* = A v_n + w_n$, we have $v^* = A v + w$ which together with $w \in Fv$ implies $v^* \in A v + Fv = Fv$. Next, from the convergences (17)--(20) we get

$$\lim (v_n^*, v_n - v)_{V^* \times V} = \lim (A v_n, v_n - v)_{V^* \times V} + \lim (w_n, v_n)_{Z^* \times Z} = (A v, v)_{V^* \times V} + (w, v)_{Z^* \times Z}$$

which completes the proof that $F$ is pseudomonotone. Applying a surjectivity result (cf., e.g., Theorem 1.3.70 of [5]), since $F$ is coercive and pseudomonotone, it is surjective which means that the problem (16) admits a solution.

Combining Claims 1 and 2, we deduce that Problem (HVI) possesses a solution.

Now, in addition to $(H_0)$, $(H(j_N))$ and $H(j_T)$, we assume $H(j_N)(v)$, $H(j_T)(v)$ and $(H_1)$.

**Claim 3.** The functional $J$ defined by (13) satisfies the condition

$$\langle z_1 - z_2, w_1 - w_2 \rangle_{L^2(I_C; \mathbb{R}^d)} \geq -\max\{m_N, m_T\} \|w_1 - w_2\|_{L^2(I_C; \mathbb{R}^d)}^2$$

for all $z_i \in \partial J(w_i), w_i \in L^2(I_C; \mathbb{R}^d), i = 1, 2$. \hfill (21)

In fact, let $j : I_C \times \mathbb{R}^d \to \mathbb{R}$ be defined by $j(x, \xi) = j_N(x, \xi_N) + j_T(x, \xi_T)$ for $(x, \xi) \in I_C \times \mathbb{R}^d$. From Lemma 5 of [11], it follows that

$$\partial j(x, \xi) \subset \partial j_N(x, \xi_N) n + \partial j_T(x, \xi_T) = \partial j_T(x, \xi_T)$$

for $(x, \xi) \in I_C \times \mathbb{R}^d$.

Let $z_i \in \partial J(w_i), w_i, z_i \in L^2(I_C; \mathbb{R}^d), i = 1, 2$. By the formula (cf. Theorem 2.7.5 of [2])

$$\partial J(v) \subseteq \int_{I_C} \partial j(x, v(x)) \, d\Gamma \quad \text{for } v \in L^2(I_C; \mathbb{R}^d),$$

we obtain $z_i(x) = a_i(x) n + (b_i(x))_T$, where $a_i(x) \in \partial j_N(x, w_i(x))$ and $b_i(x) \in \partial j_T(x, w_i(x))$, $i = 1, 2$. Using $H(j_N)(v)$, $H(j_T)(v)$ and the equality $\eta \cdot \xi_T = \eta_T \cdot \xi$ for $\eta, \xi \in \mathbb{R}^d$, we have

$$\langle z_1(x) - z_2(x), (w_1(x) - w_2(x)) \rangle$$

$$= (a_1(x) - a_2(x) n \cdot (w_1(x) - w_2(x)) + (b_1(x) - b_2(x))_T \cdot (w_1(x) - w_2(x))$$

$$= (a_1(x) - a_2(x)) (w_1(x) - w_2(x)) + (b_1(x) - b_2(x))_T \cdot (w_1(x) - w_2(x))$$

$$\geq -m_N \|w_1(x) - w_2(x)\|^2 - m_T \|w_1(x) - w_2(x)\|^2_{\mathbb{R}^d}$$

for a.e. $x \in I_C$. Therefore
\[\langle z_1 - z_2, w_1 - w_2 \rangle_{L^2(G_C; \mathbb{R}^d)} = \int_{G_C} (z_1(x) - z_2(x)) \cdot (w_1(x) - w_2(x)) \, d\Gamma \]
\[\geq - \max\{m_N, m_T\} \int_{G_C} \|w_1(x) - w_2(x)\|_{\mathbb{R}^d}^2 \, d\Gamma \]
\[= - \max\{m_N, m_T\} \|w_1 - w_2\|_{L^2(G_C; \mathbb{R}^d)}^2 \]
which implies (21).

**Claim 4.** The solution of the inclusion (16) is unique.

Let \( f \in V^* \) and let \( u_1, u_2 \in V \) be solutions to (16). Therefore, there exist \( z_i \in \partial J(\gamma u_i), z_i \in L^2(G_C; \mathbb{R}^d), i = 1, 2, \) such that
\[Au_i + \gamma^* z_i = f \quad \text{for } i = 1, 2.\]
Subtracting the above two equations, multiplying the result by \( u_1 - u_2 \) and using the coercivity of \( A \), we have
\[\alpha \|u_1 - u_2\|_V^2 + (\gamma^* z_1 - \gamma^* z_2, u_1 - u_2)_{Z^* \times Z} = 0.\]
By the condition (21), we infer
\[\langle \gamma^* z_1 - \gamma^* z_2, u_1 - u_2 \rangle_{Z^* \times Z} = \langle z_1 - z_2, \gamma u_1 - \gamma u_2 \rangle_{L^2(G_C; \mathbb{R}^d)} \geq - \max\{m_N, m_T\} \|\gamma u_1 - \gamma u_2\|_{L^2(G_C; \mathbb{R}^d)}^2 \geq - \max\{m_N, m_T\} c_N^2 \|\gamma\|^2 \|u_1 - u_2\|_V^2.\]
Hence \( \alpha \|u_1 - u_2\|_V^2 \geq - \max\{m_N, m_T\} c_N^2 \|\gamma\|^2 \|u_1 - u_2\|_V^2 \leq 0 \), which in view of (H1), implies \( u_1 = u_2 \) and subsequently \( z_1 = z_2 \) which completes the proof of Claim 4.

Finally, in order to show the uniqueness for Problem (HVI), we prove, under the regularity hypotheses that Problem (HVI) and (16) are equivalent. By virtue of Claim 1, it is enough to demonstrate that every solution to Problem (HVI) is also a solution to (16).

Let \( u \in V \) be a solution to Problem (HVI). Applying Lemma 6(v) and Proposition 2(i), we have
\[\langle Au, v \rangle_{V^* \times V} + (J \circ \gamma)^0(u; v) = \langle A\hat{e}(u), \hat{e}(v) \rangle_H + J^0(\gamma u; \gamma v) \geq (f, v)_{V^* \times V}\]
for all \( v \in V \). From Proposition 2(ii) and the definition of the subdifferential, we get \( f - Au \in \partial(J \circ \gamma)(u) = \gamma^* \partial J(\gamma u) \) which means that \( u \) is a solution to (16).

The uniqueness of a solution to (HVI) follows now from Claim 4. The proof of the theorem is complete. \( \square \)

We remark that if \( j_N(x, \cdot) \) and \( j_T(x, \cdot) \) are convex functions, then \( H(j_N)(v) \) and \( H(j_T)(v) \) hold with \( m_N = m_T = 0 \) and (H1) is trivially satisfied. We conclude this section with a simple example of a superpotential which satisfies \( H(j_N) \). An analogous example can be given for the function \( j_T \).

**Example 7.** Let \( \varphi : G_C \times \mathbb{R} \rightarrow \mathbb{R} \) be defined by \( \varphi(x, r) = g(x) h(r) \) with
\[h(r) = \begin{cases} 0 & \text{if } r < 0, \\ -\frac{1}{2} r^2 + r & \text{if } 0 \leq r < 1, \\ \frac{1}{2} & \text{if } r \geq 1 \end{cases}\]
and \( g \in L^\infty(G_C), g \geq 0 \) a.e. on \( G_C \). Then the subdifferential of \( \varphi(x, \cdot) \) is of the form \( \partial \varphi(x, r) = g(x) \partial h(r) \) with
\[\partial h(r) = \begin{cases} 0 & \text{if } r < 0, \\ [0, 1] & \text{if } r = 0, \\ -r + 1 & \text{if } 0 < r < 1, \\ 0 & \text{if } r \geq 1 \end{cases}\]
It is clear that \( H(j_N)(i)-(ii) \) hold. Since \( |\partial \varphi(x, r)| \leq g(x)(1 + |r|) \) for a.e. \( x \in G_C \) and \( r \in \mathbb{R} \), we see that \( H(j_N)(iii) \) holds with \( c_N = \|g\|_{L^\infty(G_C)} \). Using the formula \( \varphi^0(x, r; -r) = \max\{\eta(-r) \mid \eta \in \partial \varphi(x, r)\} \), it is easy to check that
$H(j_N)(iv)$ holds with $d_N = 0$. Next, we verify that $\eta_1 \leq \eta_2 - (r_1 - r_2)$ for all $r_1 < r_2$ and $\eta_i \in \partial \varphi(x, r_i), \ i = 1, 2$, which implies the condition in $H(j_N)(v)$ with $m_N = 1$. Finally, $\varphi$ can be represented as the difference of convex functions (it is of d.c. type), i.e. $\varphi(x, r) = \varphi_1(x, r) - \varphi_2(x, r)$ with $\varphi_1(x, r) = g(x), h_1(r)$, where

$$h_1(r) = \begin{cases} \frac{1}{7}r^2 - r + 1 & \text{if } r < 0, \\ \frac{1}{2}r^2 - r + \frac{3}{2} & \text{if } r \geq 1, \end{cases}$$

and $\varphi_2(x, r) = g(x)(\frac{1}{7}r^2 - r + 1)$. Since $\varphi_1(x, \cdot)$ and $\varphi_2(x, \cdot)$ are convex functions, $\partial \varphi_1(x, \cdot)$ and $\partial \varphi_2(x, \cdot)$ have a sublinear growth with $\partial \varphi_2(x, \cdot)$ being a singleton, we deduce that either $\varphi(x, \cdot)$ or $-\varphi(x, \cdot)$ is regular with $\partial \varphi(x, r) = \partial \varphi_1(x, r) - \partial \varphi_2(x, r)$.

5. $H$-convergence

The goal of this section is to study the behavior of a sequence of solutions to boundary hemivariational inequalities (HVI) under $H$-convergence of elasticity tensors.

Denoting by $\{\varepsilon\}$ a sequence of positive reals converging to zero, we consider a sequence of tensors $A_\varepsilon \in L^\infty(\Omega; M_{\alpha, \beta})$. We recall the definition of $H$-convergence of the fourth order tensors, cf. [1,6,16].

**Definition 8.** A sequence $A_\varepsilon \in L^\infty(\Omega; M_{\alpha, \beta})$ is said to converge in the sense of homogenization (or to $H$-converge) to a tensor $A \in L^\infty(\Omega; M_{\alpha, \beta})$ if for any $f \in H^{-1}(\Omega; \mathbb{R}^d)$ the sequence $u_\varepsilon$ of solutions of

$$\begin{cases} -\text{div}(A_\varepsilon e(u_\varepsilon)) = f & \text{in } \Omega, \\ u_\varepsilon \in H^1_0(\Omega; \mathbb{R}^d) \end{cases}$$

(22)

satisfies

$$u_\varepsilon \rightharpoonup u \ \text{weakly in } H^1_0(\Omega; \mathbb{R}^d),$$

$$A_\varepsilon e(u_\varepsilon) \rightharpoonup Ae(u) \ \text{weakly in } \mathcal{H},$$

where $u$ is the unique solution of the homogenized equation

$$\begin{cases} -\text{div}(A e(u)) = f & \text{in } \Omega, \\ u \in H^1_0(\Omega; \mathbb{R}^d). \end{cases}$$

**Remark 9.** Since $A_\varepsilon$ is coercive and $\| \cdot \|_V$ is a norm on $H^1_0(\Omega; \mathbb{R}^d)$ equivalent to the usual one, it follows from the Lax–Milgram lemma that for every $\varepsilon > 0$, the problem (22) has a unique weak solution $u_\varepsilon \in H^1_0(\Omega; \mathbb{R}^d)$ and

$$\|u_\varepsilon\|_{H^1_0(\Omega; \mathbb{R}^d)} \leq C \|f\|_{H^{-1}(\Omega; \mathbb{R}^d)}$$

with $C > 0$ independent of $\varepsilon$.

Moreover, since the tensors in $L^\infty(\Omega; M_{\alpha, \beta})$ are symmetric by definition, $H$-convergence is equivalent to $G$-convergence in the elasticity setting. The basic properties of $H$-convergence include the uniqueness of $H$-limit, the local character, convergence of arbitrary solutions, convergence of energy, ordering properties, corrector results and can be found in [1,6,16] and the references therein.

The definition of $H$-convergence makes sense because of the following compactness result, cf. [1,16].

**Theorem 10.** For any sequence $A_\varepsilon \in L^\infty(\Omega; M_{\alpha, \beta})$ there exist a subsequence $\{\varepsilon\}'$ of $\{\varepsilon\}$ and a homogenized tensor $A \in L^\infty(\Omega; M_{\alpha, \beta})$ such that $A_{\varepsilon}'$ $H$-converges to $A$.

The following lemma which will be used later helps to overcome a difficulty cause by passing to the limit in a product of weakly convergent sequences.
Lemma 11 (Compensated compactness). Let \( \{u_\varepsilon\} \) be a sequence converging to \( u \) weakly in \( H^1(\Omega; \mathbb{R}^d) \) and let \( \{g_\varepsilon\} \) be a sequence converging to \( g \) weakly in \( L^2(\Omega; S_d) \) such that \( \text{div} \, g_\varepsilon \to \text{div} \, g \) in \( H^{-1}(\Omega; \mathbb{R}^d) \). Then \( g_\varepsilon : e(u_\varepsilon) \to g : e(u) \) in the sense of distributions on \( \Omega \), i.e.
\[
\int_\Omega g_\varepsilon \cdot e(u_\varepsilon) \, \varphi \, dx \to \int_\Omega g : e(u) \, \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).
\]

Proof. It is enough to observe that for every \( \varepsilon, \varphi \in C_0^\infty(\Omega) \), we have
\[
\langle -\text{div} \, g_\varepsilon, u_\varepsilon \varphi \rangle_{H^{-1}(\Omega; \mathbb{R}^d) \times H^1_0(\Omega; \mathbb{R}^d)} = \int_\Omega g_\varepsilon \cdot e(u_\varepsilon) \, \varphi \, dx = \int_\Omega g_\varepsilon \cdot e(u) \, \varphi \, dx + \int_\Omega g_\varepsilon : (e(u_\varepsilon \nabla \varphi)^	op) \, dx
\]
and we can pass to the limit in the above equality because of \( u_\varepsilon \varphi \to u \varphi \) weakly in \( H^1_0(\Omega; \mathbb{R}^d) \) and \( e(u_\varepsilon \nabla \varphi)^	op \to u(\nabla \varphi)^	op \) in \( L^2(\Omega; S_d) \). \( \square \)

Given \( A_\varepsilon \in L^\infty(\Omega; \mathbb{M}_{\alpha,\beta}) \) and \( f \in V^* \), we denote by \( S_\varepsilon(f) \) the set of solutions to the problem
\[
\langle A_\varepsilon e(u_\varepsilon), e(v) \rangle_{H^1} + \int_{\Gamma_C} (j_N^0(x,u_\varepsilon;v_N) + j_T^0(x,u_\varepsilon;v_T)) \, d\Gamma \geq \langle f, v \rangle_{V^* \times V} \quad \text{for all } v \in V.
\]

Theorem 12. Let the hypotheses (H0), \( H(j_N) \) and \( H(j_T) \) hold and let \( A_\varepsilon \) be a sequence in \( L^\infty(\Omega; \mathbb{M}_{\alpha,\beta}) \) that \( H \)-converges to \( A \). If \( f_\varepsilon, f \in V^* \), \( f_\varepsilon \to f \) in \( V^* \), \( u_\varepsilon \in S_\varepsilon(f_\varepsilon) \) and \( u_\varepsilon \to u \) weakly in \( H^1(\Omega; \mathbb{R}^d) \), then \( u \in S(f) \), where \( S(f) \) denotes the solution set to the following problem
\[
\langle A e(u), e(v) \rangle_{H^1} + \int_{\Gamma_C} (j_N^0(x,u;v_N) + j_T^0(x,u;v_T)) \, d\Gamma \geq \langle f, v \rangle_{V^* \times V} \quad \text{for all } v \in V.
\]

Proof. Let \( f_\varepsilon \to f \) in \( V^* \). From Theorem 5, we know that both \( S_\varepsilon(f_\varepsilon) \) and \( S(f) \) are nonempty. Let \( u_\varepsilon \in V \) be a sequence satisfying
\[
\langle A_\varepsilon e(u_\varepsilon), e(v) \rangle_{H^1} + \int_{\Gamma_C} (j_N^0(x,u_\varepsilon;v_N) + j_T^0(x,u_\varepsilon;v_T)) \, d\Gamma \geq \langle f_\varepsilon, v \rangle_{V^* \times V}
\]
for every \( v \in V \) and \( u_\varepsilon \to u \) weakly in \( H^1(\Omega; \mathbb{R}^d) \). It is clear that \( u \in V \). Since \( j_N^0(u_\varepsilon;0) = j_T^0(u_\varepsilon;0) = 0 \), from (25), we deduce
\[
\int_\Omega A_\varepsilon e(u_\varepsilon) \cdot e(v) \, dx = \langle f_\varepsilon, v \rangle_{V^* \times V} \quad \text{for all } v \in C_0^\infty(\Omega; \mathbb{R}^d),
\]
i.e. \( -\text{div}(A_\varepsilon e(u_\varepsilon)) = f_\varepsilon \) in \( \Omega \). The convergence \( u_\varepsilon \to u \) weakly in \( H^1(\Omega; \mathbb{R}^d) \) implies \( e(u_\varepsilon) \to e(u) \) weakly in \( L^2(\Omega; S_d) \) and thus the sequence \( \{A_\varepsilon e(u_\varepsilon)\} \) is bounded in \( L^2(\Omega; S_d) \). Hence, by passing to a subsequence, if necessary, we may assume that
\[
A_\varepsilon e(u_\varepsilon) \to \eta \quad \text{weakly in } L^2(\Omega; S_d)
\]
with \( \eta \in L^2(\Omega; S_d) \). We will prove that \( \eta = A e(u) \) in \( \Omega \). To this end we consider an auxiliary sequence of problems. Let \( z \in C_0^\infty(\Omega; \mathbb{R}^d) \) and \( g = -\text{div}(A e(z)) \). Consider the sequence \( \{z_\varepsilon\} \subset V \) of (unique) solutions to
\[
\begin{cases}
-\text{div}(A_\varepsilon e(z_\varepsilon)) = g & \text{in } \Omega, \\
z_\varepsilon \in H^1_0(\Omega; \mathbb{R}^d).
\end{cases}
\]
From the definition of \( H \)-convergence, we know that
\[
z_\varepsilon \to z \quad \text{weakly in } H^1_0(\Omega; \mathbb{R}^d),
\]
\[
A_\varepsilon e(z_\varepsilon) \to A e(z) \quad \text{weakly in } L^2(\Omega; S_d).
\]
Now, let \( \varphi \in C_0^\infty(\Omega) \). We have the following equality

\[
\int_\Omega A_\varepsilon e(u_\varepsilon) : e(z_\varepsilon) \varphi \, dx = \int_\Omega A_\varepsilon e(z_\varepsilon) : e(u_\varepsilon) \varphi \, dx.
\]  

(28)

Having in mind the hypotheses and convergences (26) and (27), we apply Lemma 11 twice to the following pairs of sequences \( A_\varepsilon e(u_\varepsilon), z_\varepsilon \) and \( A_\varepsilon e(z_\varepsilon), u_\varepsilon \). By passing to the limit as \( \varepsilon \to 0 \), from (28), we have

\[
\int_\Omega \eta : e(z) \varphi \, dx = \int_\Omega A e(z) : e(u) \varphi \, dx.
\]

Using again the equality \( \int_\Omega A e(z) : e(u) \varphi \, dx = \int_\Omega A e(u) : e(z) \varphi \, dx \), we obtain

\[
\int_\Omega (\eta - A e(u)) : e(u) \varphi \, dx = 0.
\]

Since \( z \in C_0^\infty(\Omega; \mathbb{R}^d) \) and \( \varphi \in C_0^\infty(\Omega) \) are arbitrary, we deduce that \( \eta = A e(u) \) in \( \Omega \) which by (26) gives

\[
A_\varepsilon e(u_\varepsilon) \to A e(u) \quad \text{weakly in } L^2(\Omega; S_d).
\]  

(29)

In virtue of the continuity of the trace operator \( \gamma : H^1(\Omega; \mathbb{R}^d) \to H^{1/2}(\Gamma_C; \mathbb{R}^d) \) and the compactness of the embedding \( H^{1/2}(\Gamma_C; \mathbb{R}^d) \subset L^2(\Gamma_C; \mathbb{R}^d) \), we get \( \gamma u_\varepsilon \to \gamma u \) in \( L^2(\Gamma_C; \mathbb{R}^d) \). Hence

\[
\gamma u_\varepsilon N \to \gamma u_N \quad \text{in } L^2(\Gamma_C), \quad \gamma u_\varepsilon T \to \gamma u_T \quad \text{in } L^2(\Gamma_C; \mathbb{R}^d),
\]

so passing to the next subsequence, we may assume that

\[
u_\varepsilon N(x) \to u_N(x) \quad \text{in } \mathbb{R}, \quad u_\varepsilon T(x) \to u_T(x) \quad \text{in } \mathbb{R}^d
\]  

(30)

for a.e. \( x \in \Gamma_C \) and

\[
|u_\varepsilon N(x)| \leq h_1(x), \quad \|u_\varepsilon T(x)\|_{\mathbb{R}^d} \leq h_2(x), \quad \text{a.e. } x \in \Gamma_C
\]  

(31)

with \( h_1, h_2 \in L^2(\Gamma_C) \). Exploiting the upper semicontinuity of \( j^0_N(\cdot; v_N) \) and \( j^0_T(\cdot; v_T) \) (cf. Proposition 5.6.6 of [4]) and (30), we have

\[
\limsup_{\varepsilon \to 0} (j^0_N(x, u_\varepsilon N(x); v_N(x)) + j^0_T(x, u_\varepsilon T(x); v_T(x))) \leq j^0_N(x, u_N(x); v_N(x)) + j^0_T(x, u_T(x); v_T(x))
\]

for a.e. \( x \in \Gamma_C \).

We apply the Fatou lemma and get

\[
\limsup_{\varepsilon \to 0} \int_{\Gamma_C} (j^0_N(x, u_\varepsilon N(x); v_N(x)) + j^0_T(x, u_\varepsilon T(x); v_T(x))) \, d\Gamma \\
\leq \int_{\Gamma_C} \limsup_{\varepsilon \to 0} (j^0_N(x, u_\varepsilon N(x); v_N(x)) + j^0_T(x, u_\varepsilon T(x); v_T(x))) \, d\Gamma \\
\leq \int_{\Gamma_C} (j^0_N(x, u_N(x); v_N(x)) + j^0_T(x, u_T(x); v_T(x))) \, d\Gamma.
\]  

(32)

The use of the Fatou lemma is justified by the following two estimates (cf. \( H(j_N), H(j_T) \) and (31))

\[
j^0_N(x, u_\varepsilon N(x); v_N(x)) = \max \{ \xi \cdot v_N(x) \mid \xi \in \partial j_N(x, u_\varepsilon N(x)) \} \leq c_N |v_N(x)| (1 + |u_\varepsilon N(x)|) \\
\leq c_N |v_N(x)| (1 + h_1(x)) = h(x),
\]

\[
j^0_T(x, u_\varepsilon T(x); v_T(x)) \leq c_T \|v_T(x)\|_{\mathbb{R}^d} (1 + h_2(x)) = \tilde{h}(x)
\]

for a.e. \( x \in \Gamma_C \), for all \( v \in V \), where \( h, \tilde{h} \in L^1(\Gamma_C) \). From (29), (32) and (25) we obtain
\[ \langle A e(u), e(v) \rangle_H + \int_{H_c} (j^0_N(x, u_N(x); v_N(x)) + j^0_T(x, u_T(x); v_T(x))) \, d\Gamma \\
\geq \limsup_{\epsilon \to 0} \int_{H_c} (j^0_N(x, u_{\epsilon N}(x); v_N(x)) + j^0_T(x, u_{\epsilon T}(x); v_T(x))) \, d\Gamma + \lim_{\epsilon \to 0} \langle A_{\epsilon} e(u_{\epsilon}), e(v) \rangle_H \geq \langle f, v \rangle_{V^* \times V} \]

for all \( v \in V \),

which implies that \( u \in S(f) \) and completes the proof. □

**Remark 13.** Although homogenization is not restricted to periodic problems, the main application is for the asymptotic analysis of periodic structures. Let \( A \in L^\infty(\Omega; \mathcal{M}_{a,\beta}) \) be a periodic tensor and \( A_\epsilon(x) = \frac{1}{\epsilon} A(x) \). Then \( A_\epsilon \) \( H \)-converges to \( A_0 \), as \( \epsilon \to 0 \), where \( A_0 \) is independent of the spatial variable and can be found by solving the minimization problem

\[ A_0 \tau : \tau = \inf \{ A(\tau + e(u)) : (\tau + e(u)) \mid u \in H^1(\Omega; \mathbb{R}^d) \} \quad \text{for} \quad \tau \in S_d, \]

where \( Q \) is the basic cube of periods, for details cf. [1,16].

The following is one of the most important results of the paper.

**Corollary 14.** Let \((H_0), (H_1), H(j_N)(\cdot)\rangle\langle \cdot, v \rangle \text{ and } H(j_T)(\cdot)\rangle\langle \cdot, v \rangle \) hold, let \( A_\epsilon \) be a sequence in \( L^\infty(\Omega; \mathcal{M}_{a,\beta}) \) that \( H \)-converges to \( A \) and \( f_\epsilon \to f \) in \( V^* \). Then the sequence of unique solutions \( u_\epsilon \) of the hemivariational inequality (23) corresponding to \( f_\epsilon \) converges weakly in \( H^1(\Omega; \mathbb{R}^d) \) to a unique solution to (24) corresponding to \( f \).

The physical idea of homogenization is the description of the macroscopic properties of media with highly heterogeneities of lengthscale \( \epsilon \) described by tensors \( A_\epsilon \) (for instance, composites with mixed periodically distributed different phases, fiber materials, stratified or porous media). From the mechanical point of view, the asymptotic analysis when \( \epsilon \to 0 \) determines the large scale properties of the material without determining its fine scale structure. The limit homogenized tensor \( A \) defines an effective properties of the medium. Thus the solution \( u_\epsilon \) of the hemivariational inequality when \( \epsilon \) is small is replaced by the solution \( u \) of the hemivariational inequality of the same type corresponding to the limit tensor. For further comments on the physical and mechanical aspects of averaging, we refer to [1,3,6,15] and the references therein.

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**References**