On Stability of Symplectic Maps

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Received April 12, 1982

The aim of this short note is to give a simple geometrical proof of a result due to Cushman and Kelley [4] giving a characterization of strongly stable symplectic maps. The original proof relied on the use of normal forms.

The following is a slight reformulation of the main result of [4].

**THEOREM.** An infinitesimally stable symplectic matrix $A$ is strongly stable iff its centralizer $C(A)$ (in the set $\text{sp}(n)$ defined below) consists of stable matrices.

We recall first some definitions (see [3–7, 9, 10, 12]). Any $2n \times 2n$ matrix of the form

$$A = JH, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad H^T = H$$

is called infinitesimally symplectic; let $\text{sp}(n)$ be the set of such matrices. Any matrix $A$ is called stable iff $\|e^{At}\|$ is bounded for all (positive and negative) $t$. Matrix $A = JH \in \text{sp}(n)$ is called strongly stable if any matrix $B = JK$ with $K = K^T$ sufficiently close to $H$ is stable.

Centralizer $C(A)$ of a matrix $A$ is, by definition, the set of all matrices in $\text{sp}(n)$ commuting with $A$.

Before proceeding with the proof, we will need one perturbation result [4, 1].

**LEMMA.** Any matrix $B \in \text{sp}(n)$ sufficiently close to a stable matrix $A \in \text{sp}(n)$ can be expressed as

$$B = S^{-1}(A + C)S \quad \text{with} \quad C \in C(A), \quad S = e^T, \quad T \in \text{sp}(n).$$

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' Definitions follow the statement of this theorem.
Proof of the Theorem (followed by the proof of the lemma). 1. Assume that \( C(A) \) consists of stable matrices. Any \( B \in \text{sp}(n) \) close to \( A \) can be written as
\[
B = S^{-1}(A + C) S, \quad C \in C(A),
\]
according to the above lemma. Therefore, \( B \) is stable, being similar to stable matrix \( A + C \in C(A) \); strong stability of \( A \) is proven.

2. Conversely, assume that \( A \) is strongly stable; choose any \( B \in C(A) \) and show its stability. For \( \epsilon \) small enough we have, for some \( c > 0 \),
\[
\|e^{-\epsilon t}\|, \quad \|e^{(A + \epsilon B)t}\| < c \text{ for all } t,
\]
since \( A \) is strongly stable. Using the fact that \( A \) and \( B \) commute (\( B \in C(A) \)), we have
\[
\|e^{\epsilon Bt}\| = \|e^{-A\epsilon} e^{(A + \epsilon B)t}\| < c^2,
\]
which proves stability of \( B \). Q.E.D.

Proof of the Lemma. Introduce a map
\[
M : \text{ran } \text{ad}_A \oplus \ker \text{ad}_A \rightarrow \text{sp}(n),
\]
given by
\[
M(T, C) = e^{-\tau}(A + C) e^	au;
\]
here \( \text{ad}_A X = [A, X] \). Wishing to apply the implicit function theorem to \( M \) near \( T = C = 0 \), we calculate its derivative:
\[
DM(0, 0)(T, C) = -[A, T] + C \in \text{ran } \text{ad}_A \oplus \ker \text{ad}_A = \text{sp}(n).
\]
The last equality follows from the fact that \( \text{ad}_A \) is semisimple (i.e., diagonalizable), which in turn is the consequence of stability (and thus diagonalizability) of \( A \).

This shows that \( DM(0, 0) \) maps \( \text{ran } \text{ad}_A \oplus \ker \text{ad}_A \) onto itself; by the implicit function theorem for any \( B \in \text{sp}(n) \) there exists \( T \in \text{ran } \text{ad}_A \), \( C \in \ker \text{ad}_A = C(A) \) with
\[
B = M(C, T) = e^{-\tau}(A + C) e^\tau = S^{-1}(A + C) S. \quad \text{Q.E.D.}
\]

References