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Almost periodic structures and the semiconjugacy problem

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ABSTRACT

The description of almost periodic or quasiperiodic structures has a long tradition in mathematical physics, in particular since the discovery of quasicrystals in the early 80's. Frequently, the modelling of such structures leads to different types of dynamical systems which include, depending on the concept of quasiperiodicity being considered, skew products over quasiperiodic or almost periodic base flows, mathematical quasicrystals or maps of the real line with almost periodic displacement. An important problem in this context is to know whether the considered system is semiconjugate to a rigid translation. We solve this question in a general setting that includes all the above-mentioned examples and also allows the treatment of scalar differential equations that are almost periodic both in space and time. To that end, we study a certain class of flows that preserve a one-dimensional foliation and show that a semiconjugacy to a minimal translation flow exists if and only if a boundedness condition, concerning the distance of orbits of the flow to those of the translation, holds.

1. Introduction

A classical topic in the theory of dynamical systems is the study of cohomological equations. For minimal base transformations, the well-known Gottschalk–Hedlund Theorem states that the existence of continuous solutions is equivalent to the boundedness of the associated cocycle.

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Theorem 1.1 *(Gottschalk–Hedlund). (See [10].) Suppose Ω is a compact metric space, $\Phi : \mathbb{T} \times \Omega \to \Omega$, $(t, x) \mapsto \Phi_t(x)$ is a minimal flow with time $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$, $\rho \in \mathbb{R}$ and $\alpha : \mathbb{T} \times \Omega \to \mathbb{R}$, $(t, x) \mapsto \alpha_t(x)$ is a continuous cocycle over Φ, that is,*

$$\alpha_{s+t}(x) = \alpha_s\big(\Phi_t(x)\big) + \alpha_t(x). \tag{1.1}$$

Then the cohomological equation

$$\alpha_t = \gamma - \gamma \circ \Phi_t + t\rho \quad \forall t \in \mathbb{T} \tag{1.2}$$

has a continuous solution $\gamma : \Omega \to \mathbb{R}$ if and only if there exists $x \in \Omega$ such that $\sup_{t\in\mathbb{T}} |\alpha_t(x) - t\rho| < \infty$. Moreover, in this case γ can be defined by

$$\gamma(x) = \limsup_{t\to\infty} \alpha_t(x) - t\rho. \tag{1.3}$$

If Φ is not minimal, then (1.3) yields at least a bounded solution of (1.1).

A closely related question is that of the existence of a semiconjugacy, also called a factor map, from a given map to a rigid translation. For example, suppose f is an orientation-preserving homeomorphism of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with lift $F : \mathbb{R} \to \mathbb{R}$, and $\rho = \lim_{n\to\infty}(F^n(x) - x)/n$ is the rotation number of f. We say a continuous onto map $h : \mathbb{T}^1 \to \mathbb{T}^1$ is a *semiconjugacy* from f to the rotation R_ρ with angle ρ if there holds $h \circ f = R_\rho \circ h$. Writing $h(x) = x + \gamma(x) \bmod 1$ with continuous $\gamma : \mathbb{T}^1 \to \mathbb{R}$, it is easy to see that h is a semiconjugacy if γ is a solution of the cohomological equation (1.2) with $\Phi_n(x) = F^n(x)$ and $\alpha_n(x) = F^n(x) - x$. In comparison to the Gottschalk–Hedlund Theorem, it is noteworthy that this equation always has a continuous solution provided that ρ is irrational, so that R_ρ is minimal, independent of any minimality assumption on f. This is the main content of the celebrated Poincaré Classification Theorem.

Theorem 1.2 *(Poincaré). Suppose f is an orientation-preserving homeomorphism of the circle with lift $F : \mathbb{R} \to \mathbb{R}$. Then the limit $\rho(F) = \lim_{n\to\infty}(F^n(x) - x)/n$ exists and is independent of $x \in \mathbb{R}$. Furthermore, there holds:*

(i) *$\rho(F)$ is rational if and only if f has a periodic orbit.*
(ii) *$\rho(F)$ is irrational if and only if f is semiconjugate to the irrational rotation $R_{\rho(F)}$.*

The core part of this result is the existence of a semiconjugacy to an irrational rotation in (ii). In general, we call the task of finding equivalent conditions for the existence of a semiconjugacy to a given minimal translation the *semiconjugacy problem*. (Here 'translation' will be understood in a broad sense, made precise below.) Recently, the authors have independently made some progress on this question. In [9], a Poincaré-like classification is given for skew products on the two-torus over irrational rotations (so-called quasiperiodically forced circle homeomorphisms). For maps on mathematical quasicrystals, a similar classification was obtained in [1]. While in both situations strong use is made of the order-preserving properties of the considered systems, corresponding to the existence of a one-dimensional foliation that is preserved by the dynamics, the methods employed are quite different.

In order to obtain a unified proof of these results, we introduce a general setting below that also allows to treat a number of other system classes, including minimally forced circle homeomorphisms, increasing maps on the real line with almost periodic displacement (see [12]) and scalar differential equations with almost periodicity both in space and time. All these different types of systems have been studied rather independently in the literature so far. The proposed framework should allow their study from a more global point of view and to identify similarities and differences between them. To that end, we provide some structure results for our general class of flows, which identify important subcases to which some of the above examples can be associated.

Suppose Ω is a compact metric space, $\varphi : \mathbb{R} \times \Omega \to \Omega$, $(t,x) \mapsto \varphi_t(x) = x \odot t$ is a continuous flow on Ω and $\omega : \mathbb{T} \times \Omega \to \Omega$, $(t,x) \mapsto \omega_t(x)$ is a flow with discrete or continuous time (that is, $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$) which commutes with φ, that is,

$$\omega_s(x \odot t) = \omega_s(x) \odot t. \tag{1.4}$$

We say $\rho \in \mathbb{R}$ is (φ,ω)-*irrational* if the *translation flow*

$$T^\rho : (t,x) \mapsto T^\rho_t(x) = \omega_t(x) \odot t\rho \tag{1.5}$$

is minimal. Otherwise we say that ρ is (φ,ω)-*rational*. Now, assume that $\tau : \mathbb{T} \times \Omega \to \mathbb{R}$ is a continuous function that satisfies

$$\tau_{s+t}(x) = \tau_s\big(\omega_t(x) \odot \tau_t(x)\big) + \tau_t(x), \tag{1.6}$$

so that

$$\Phi = \Phi^{\varphi,\omega,\tau} : \mathbb{T} \times \Omega \to \Omega, \quad (t,x) \mapsto \Phi_t(x) = \omega_t(x) \odot \tau_t(x) \tag{1.7}$$

defines a flow with time $\mathbb{T}$ on Ω. We call ω *transversal action* and τ *translation function* of Φ.

An interpretation of the structure of (1.7) is that there is a one-dimensional foliation on Ω given by the orbits of φ, and the flow Φ preserves this foliation in the sense that *leaves* $\varphi_{\mathbb{R}}(x) = \{\varphi_t(x) \mid t \in \mathbb{R}\}$ are send to leaves $\varphi_{\mathbb{R}}(\omega_t(x))$ by Φ_t. We will further require Φ preserves the order on these leaves, which amounts to say that

$$s + \tau_t(x \odot s) \geqslant \tau_t(x) \quad \forall x \in \Omega,\ t \in \mathbb{R},\ s \geqslant 0. \tag{1.8}$$

(See Remark 1.5 below.) If the limit

$$\rho(\Phi) = \lim_{t\to\infty} \tau_t(x)/t \tag{1.9}$$

exists and is independent of $x \in \Omega$, we call it the *translation number* of Φ. Further, we say Φ has *bounded mean motion* if there exists a constant $C > 0$ such that

$$\big|\tau_t(x) - t\rho(\Phi)\big| \leqslant C \quad \forall x \in \Omega,\ t \in \mathbb{T}. \tag{1.10}$$

Theorem 1.3. *Suppose $\rho(\Phi)$ is (φ,ω)-irrational. Then the cohomological equation*

$$\tau_t(x) = \gamma(x) - \gamma \circ \Phi_t(x) + t\rho(\Phi) \tag{1.11}$$

has a continuous solution $\gamma : \Omega \to \mathbb{R}$ if and only if Φ has bounded mean motion. In this case, the map

$$h(x) = x \odot \gamma(x) \tag{1.12}$$

provides a semiconjugacy from Φ to the translation flow $T^{\rho(\Phi)}$, that is, $h \circ \Phi_t = T^{\rho(\Phi)}_t \circ h$ for all $t \in \mathbb{T}$. Moreover, h can be chosen to be order-preserving on the leaves $\varphi_{\mathbb{R}}(x)$, $x \in \Omega$, that is, (1.8) *holds with $\tau = \gamma$.*

A further immediate consequence we obtain from the proof in Section 2 is the following.

Corollary 1.4. *Under the assertions of Theorem* 1.3 *the flow Φ has a unique minimal set.*

The application of these results to the different examples mentioned above will be discussed in detail in Section 4. We close the introduction with a remark on (1.8) and the order-preservation on the leaves.

Remark 1.5. We call $\varphi_{\mathbb{R}}(x)$ the *leaf* of x and say the leaf is periodic if there exists $t_0 > 0$ with $\varphi_{t_0}(x) = x$. The natural order on $\mathbb{R}$, respectively $\mathbb{R}/t_0\mathbb{Z}$, induces a canonical order on $\varphi_{\mathbb{R}}(x)$. If the leaf is aperiodic, that is, $\varphi_t(x) \neq x$ for all $t \neq 0$, then this order is linear and given by $x \leqslant y$: $\Leftrightarrow y = x \odot t$ for some $t \geqslant 0$. When the leaf is periodic with period t_0, then the order is circular and is declared by $x \leqslant y \leqslant z$ being equivalent to the existence of $0 \leqslant s \leqslant t \leqslant t_0$ with $y = x \odot s$ and $z = x \odot t$ (y lies between x and z in positive direction). In both cases, (1.8) implies that the flow Φ preserves the order on the leaves, that is, $x \leqslant y \leqslant z$ implies $\Phi_t(x) \leqslant \Phi_t(y) \leqslant \Phi_t(z)$ for all $t \in \mathbb{T}$.

Given $y \in \varphi_{\mathbb{R}}(x)$, we write $[x,z] = \{y \in \varphi_{\mathbb{R}}(x) \mid x \leqslant y \leqslant z\}$ to denote intervals in the leaves with respect to the canonical ordering.

2. Proof of Theorem 1.3

Due to (1.6) the function τ is a cocycle over the flow Φ, that is,

$$\tau_{s+t}(x) = \tau_s\bigl(\Phi_t(x)\bigr) + \tau_t(x). \tag{2.1}$$

By (1.10) and the Gottschalk–Hedlund Theorem

$$\gamma(x) = \limsup_{t\to\infty}\bigl(\tau_t(x) - t\rho\bigr) \tag{2.2}$$

defines a bounded solution $\gamma : \Omega \to \mathbb{R}$ of the cohomological equation (1.11), such that the restriction of γ to every minimal set is continuous. As a consequence, the map $h(x) = x \odot \gamma(x)$ satisfies

$$h \circ \Phi_t = T_t^{\rho(\Phi)} \circ h \quad \forall t \in \mathbb{T}. \tag{2.3}$$

Moreover, (2.2) and (1.8) imply that

$$\gamma(x \odot s) = \limsup_{t\to\infty}\bigl(\tau_t(x\odot s) - t\rho\bigr) \geqslant \limsup_{t\to\infty}\bigl(\tau_t(x) - t\rho\bigr) - s \geqslant \gamma(x) - s, \quad s \geqslant 0, \tag{2.4}$$

which means that h also preserves the order on leaves (see Remark 1.5). It remains to show that γ is continuous on all of Ω.

In order to do so, fix any minimal set M and let $\rho = \rho(\Phi)$. Since $\gamma_{|M}$ is continuous, $h_{|M}$ is continuous as well. Due to (2.3) the set $h(M)$ is T^ρ-invariant, so that by (φ,ω)-irrationality of ρ we have $h(M) = \Omega$. Given $x \in \Omega$, we let $\mathcal{R}_M(x) = \{t \in \mathbb{R} \mid x \odot t \in M\}$. Since γ is bounded and h maps $x \odot \mathbb{R} \cap M$ onto $x \odot \mathbb{R}$, the set $\mathcal{R}_M(x)$ is relatively dense and the size of its maximal gap is at most $2\sup_{x\in\Omega}|\gamma(x)|$. Given $x \in M$, define

$$t^+(x) = \inf\{t \geqslant 0 \mid x \odot t \in M\}, \qquad t^-(x) = \sup\{t \leqslant 0 \mid x \odot t \in M\}, \tag{2.5}$$

$$x^+ = x \odot t^+(x) \quad \text{and} \quad x^- = x \odot t^-(x). \tag{2.6}$$

Then t^+ and t^- are bounded and due to the compactness of M the function t^+ is lower semi-continuous and t^- is upper semi-continuous. Furthermore, since h preserves the order on the leaves and maps $x \odot \mathbb{R} \cap M$ onto $x \odot \mathbb{R}$, it must be constant on the interval $[x^-, x^+]$, that is,

$$h\bigl(x^-\bigr) = h\bigl(x^+\bigr) = h(x) = h\bigl(x'\bigr) \quad \forall x' \in \bigl[x^-, x^+\bigr]. \tag{2.7}$$

In particular, this means that

$$\gamma(x) = \gamma\big(x^{\pm}\big) + t^{\pm}(x). \tag{2.8}$$

Fix $x \in \Omega$ and suppose x'_n is a sequence in Ω converging to x. Let x_n be a subsequence of x'_n such that $\gamma(x_n)$ converges. Let $t_n^+ = t^+(x_n)$ and $t_n^- = t^-(x_n)$. Taking a subsequence again if necessary, we may assume that $t_n^+ \to s^+$, $t_n^- \to s^-$, $x_n^+ \to y^+$ and $x_n^- \to y^-$ as $n \to \infty$. We have $y^+ = x \odot s^+$, $y^- = x \odot s^-$ and, since $x_n^{\pm} \in M$, the points y^+ and y^- also belong to M. By definition of $t^{\pm}$, we therefore have $s^+ \geqslant t^+$ and $s^- \leqslant t^-$. Further

$$\lim_{n\to\infty} \gamma(x_n) \overset{(2.8)}{=} \lim_{n\to\infty} \gamma\big(x_n^+\big) + t_n^+ = \gamma\big(y^+\big) + s^+ = \gamma\big(x \odot s^+\big) + s^+ \geqslant \gamma(x), \tag{2.9}$$

where the last inequality is due to $s^+ \geqslant t^+$, (2.4) and (2.8). In the same way, we get $\lim_{n\to\infty} \gamma(x_n) = \gamma(y^-) + s^- \leqslant \gamma(x)$, which implies $\lim_{n\to\infty} \gamma(x_n) = \gamma(x)$. This argument shows that any convergent subsequence of $\gamma(x'_n)$ has limit $\gamma(x)$, and since γ is bounded, we get $\lim_{n\to\infty} \gamma(x'_n) = \gamma(x)$. Since x and x'_n were arbitrary, this shows the continuity of γ.

Proof of Corollary 1.4. Suppose for a contradiction that M_1 and M_2 are two different minimal sets of Φ. Then $M_1 \cap M_2 = \emptyset$ by minimality, so that $M_2 \subseteq M_1^c$. Since M_1 is compact, the set $\varphi_{\mathbb{R}}(x) \cap M_1^c$ consists of an at most countable union of open segments in $\varphi_{\mathbb{R}}(x)$, each of which is mapped to a single point by h (see (2.7)). However, this means that $h(\varphi_{\mathbb{R}}(x) \cap M_2)$ is at most countable, contradicting the fact that $h(\varphi_{\mathbb{R}}(x) \cap M_2) = \varphi_{\mathbb{R}}(x)$ since $h(M_2) = \Omega$. $\square$

3. Two structure results

In this section, we study the structure of the joint action of φ and ω and the implications for Φ. In particular, we concentrate on some situations where the general setting introduced above reduces to a simpler one. As a standing assumption, we assume the minimality of the joint $\mathbb{R} \times \mathbb{T}$-action of φ and ω. Note that otherwise no (φ, ω)-irrational numbers can exist.

Given an ω-minimal set W, a skew product circle flow on $W \times \mathbb{T}^1$ over ω is a flow $F : \mathbb{T} \times W \times \mathbb{T}^1 \to W \times \mathbb{T}^1$ of the form (see also Section 4.1)

$$F_t(x, \theta) = \big(\omega_t(x), f_t^x(\theta)\big).$$

Our aim is to show that if φ has a periodic leaf, then Φ 'almost' reduces to a skew product circle flow.

Proposition 3.1. *Suppose that the joint action of φ and ω is minimal, and that φ has a periodic leaf. Then there exist an ω-minimal set W and a skew product flow Ψ on $W \times \mathbb{T}^1$ over ω which is a finite-to-one extension of the flow Φ defined in* (1.7)*.*

Further, there exists a continuous lift $\hat{\Psi} : \mathbb{T} \times W \times \mathbb{R} \to W \times \mathbb{R}$ of Ψ whose displacement function $\tilde{\tau}_t(x, s) = \pi_2(\hat{\Psi}_t(x, s)) - s$ is a scalar multiple of $\tau \circ g \circ p$, where $g : W \times \mathbb{T}^1 \to \Omega$ is the finite-to-one factor map and $p : W \times \mathbb{R} \to W \times \mathbb{T}^1$ is the canonical projection.

Note that the second part of the statement implies that for all questions related to the translation number, one may equivalently consider Ψ instead of Φ.

In order to prove the above statement, we start with some general considerations. First, given any closed set W, we define its invariant group H_W as

$$H_W := \{\rho \in \mathbb{R} \mid W \odot \rho = W\}.$$

From the continuity of φ, it is clear that H_W is a closed subgroup of $\mathbb{R}$. This means that H_W is either trivial, $\mathbb{R}$, or a lattice in $\mathbb{R}$.

Lemma 3.2. *Suppose that the joint action of φ and ω is minimal. Then there exists a closed subgroup $H = H_{\varphi,\omega}$ of $\mathbb{R}$ such that $H_W = H$ for all (non-empty) ω-minimal sets W.*

Proof. Fix a (non-empty) ω-minimal set W and set $H = H_W$. We deal with the three possible cases for H separately.

First, suppose that $H = \mathbb{R}$. Then W is invariant for the joint action. Since this action is minimal, it follows that $W = X$ and therefore there is only one non-empty ω-minimal set, from which the conclusion is direct.

Second, suppose H is a lattice, that is, $H = s_0\mathbb{Z}$ for some $s_0 > 0$. On one hand, due to the continuity of φ we have that

$$Y = \bigcup_{t\in[0,s_0)} W \odot t$$

is closed. Since it is also invariant for the joint action, it follows that $Y = X$. On the other hand, it is easy to see that $W \odot t$ is an ω-minimal set for every $t \in \mathbb{R}$. This implies that $t - s \in s_0\mathbb{Z}$ whenever $W \odot t \cap W \odot s$ is not empty. It follows that $\{W \odot t\colon\ t \in [0, s_0)\}$ is a partition of X into ω-minimal sets. In particular, $H_W = s_0\mathbb{Z}$ for all ω-minimal sets W.

Finally, if $H_W = \{0\}$, then for every ω-minimal set W' we have $H_{W'} = \{0\}$. Indeed, if we suppose otherwise, then by the above H_W is either $\mathbb{R}$ or a lattice, which is absurd. $\square$

We now focus on the case where H is a lattice, that is, $H = s_0\mathbb{Z}$. In this case, it turns out that φ is conjugate to a suspension flow, which we define as follows. Let $\sigma : W \to W$, $x \mapsto x \odot s_0$. Note that σ is continuous and commutes with $\omega_{|W}$. Let $\sim_\sigma$ be the smallest equivalence relation on $W \times \mathbb{R}$ that satisfies $(x, s + ks_0) \sim_\sigma (\sigma^k(x), s)$ $\forall (x, s) \in W \times \mathbb{R}$, $k \in \mathbb{Z}$. We denote by $[x, s]_\sigma$ the equivalence class of $(x, s) \in W \times \mathbb{R}$ with respect to $\sim_\sigma$ and let $W_\sigma = W \times \mathbb{R}/\sim_\sigma$ be the *suspension space*. Then we define the *suspension flow* $\varphi^{W,\sigma}$ on W_σ by $\varphi^{W,\sigma}_t[x, s]_\sigma = [x, s + t]_\sigma$.

Now, it is easy to check that the map $h : W_\sigma \to \Omega$, $h([x, s]_\sigma) = x \odot s$ is a homeomorphism. Further, we have $h \circ \varphi^{W,\sigma}_t([x, s]_\sigma) = x \odot (s + t) = h([x, s]_\sigma) \odot t = \varphi_t \circ h([x, s]_\sigma)$, which means that h conjugates $\varphi^{W,\sigma}$ and φ. We have thus obtained the following statement.

Corollary 3.3. *If H is a lattice then, for every ω-minimal set W, the flow φ is conjugate to the suspension flow $\varphi^{W,\sigma}$ defined above.*

As a consequence, we can also view the flow Φ as a flow on the suspension space W_σ. Namely, let $\tilde{\Phi} : W_\sigma \times \mathbb{R} \to W_\sigma$, $\tilde{\Phi}_t([x, s]_\sigma) = [\omega(x), s + \tau_t(x \odot s)]_\sigma$. Then it is easy to check that h defined above conjugates $\tilde{\Phi}$ and Φ.

Corollary 3.4. *If H is a lattice, then the flow Φ from* (1.7) *is conjugate to the flow $\tilde{\Phi}$ on the suspension space W_σ.*

We now turn to the situation where φ has a periodic leaf. In this case, all leaves are periodic.

Lemma 3.5. *Suppose that the joint action of φ and ω is minimal and φ has a periodic leaf. Then all the leaves of φ are periodic with the same period.*

Proof. Suppose t is the period of a leaf $\varphi_{\mathbb{R}}(x)$, such that $x \odot t = x$. Since ω commutes with φ, it is clear that $\omega_s(x) \odot t = \omega_s(x)$. Hence, ω sends periodic leaves to periodic leaves. Fix any $y \in X$. Due to the minimality of the joint action of φ and ω there exist sequences $(t_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ in $\mathbb{T}$ such that $y_k = \omega_{t_k}(x) \odot s_k$ converges to y. Since all y_k are t-periodic, it follows that $y \odot t = \lim_{k\to\infty} y_k \odot t = \lim_{k\to\infty} y_k = y$. From this, we easily obtain that all leaves have the same period. $\square$

Proof of Proposition 3.1. Fix an ω-minimal set W and let s_0, σ and W_σ be as above. By Lemma 3.5, all the orbits of φ are periodic with period t_0. In particular we have $W \odot t_0 = W$, which means that H_W is the lattice $s_0\mathbb{Z}$ and $t_0 = ks_0$ for some $k \in \mathbb{Z}$. Consequently, σ^k is the identity map on W.

Now we define a flow $\hat{\Psi}$ on $W \times \mathbb{R}$ by $\hat{\Psi}_t(x,s) = (\omega_t(x), s + t_0^{-1}\tau_t(x \odot t_0 s))$. Further, we let $\hat{g} : W \times \mathbb{R} \to \Omega$ be given by $g(x,s) = x \odot t_0 s$. Then

$$\begin{aligned}\hat{g} \circ \hat{\Psi}_t(x,s) &= \omega_t(x) \odot \big(t_0 s + \tau_t(x \odot t_0 s)\big)\\ &= \omega_t(x \odot t_0 s) \odot \tau_t(x \odot t_0 s) = \Phi_t(x \odot t_0 s) = \Phi_t \circ \hat{g}(x,s).\end{aligned}$$

Hence, $\hat{g}$ is a semiconjugacy from $\hat{\Psi}$ to Φ. Furthermore, $\hat{\Psi}$ projects to a skew product circle flow Ψ on $W \times \mathbb{T}^1$ and $\hat{g}$ factors to a map $g : W \times \mathbb{T}^1 \to \Omega$. It is easy to check that this map g is a finite-to-one factor map (or semiconjugacy) from Ψ to Φ. The fact that the displacement function of the lift $\hat{\Psi}$ has the required form follows immediately from the above definition. □

Example 3.6. We exhibit a simple example where the factor map g in Proposition 3.1 is 2-to-1. Consider $\Omega = \mathbb{T}^2$, ρ an irrational number, $\omega(x,y) = (x+\rho, y)$ and $\varphi_t(x,y) = (x + t/2, y + t)$. Then Φ has the form

$$\Phi(x,y) = \big(x + \rho + \tau(x,y)/2,\, y + \tau(x,y)\big).$$

Clearly, the smallest period of φ is 2, but if we choose τ such that $\tau(x,0) = 0$ for all $x \in \mathbb{T}^1$, then $W = \mathbb{T}^1 \times \{0\}$ is ω-minimal due to the irrationality of ρ. On the other hand, it is easy to see that $H_W = \mathbb{Z}$, which means that $t_0 = 2$ and $s_0 = 1$ and hence from the proof of Proposition 3.1 we have that g is 2-to-1. Of course, it can be checked directly that the skew product

$$\Psi : (x,y) \mapsto \big(x + \rho,\, y + \tau(x,y)/2\big)$$

is a 2-to-1 extension of Φ with the semiconjugacy being:

$$\pi : (x,y) \mapsto (x + y, 2y).$$

A second situation in which the general setting simplifies is when there exists an invariant leaf.

Proposition 3.7. *Suppose that the joint action of φ and ω is minimal and that there is a φ-orbit that is invariant by ω. Then there exists $\rho \in \mathbb{R}$ such that $\omega_t(x) = x \odot t\rho$ for all $x \in X$ and $t \in \mathbb{T}$.*

Proof. Suppose $\varphi_{\mathbb{R}}(x)$ is ω-invariant. Given $t \in \mathbb{T}$ there exists $\rho(t) \in \mathbb{R}$ such that $\omega_t(x) = x \odot t\rho(t)$. Due the minimality of the joint action, for every $y \in X$ we can find sequences $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ and $(s_k)_{k\in\mathbb{N}} \subset \mathbb{Z}$ such that the sequence $(y_k)_{k\in\mathbb{N}}$ defined by $y_k := \omega_{t_k}(x) \odot s_k$ converges to y. It follows from the continuity of ω that $\omega_t(y_k)$ converges to $\omega_t(y)$. Since the actions of ω and φ commute, we deduce that

$$\begin{aligned}\omega_t(y) &= \lim_{k\to\infty} \omega_t(y_k)\\ &= \lim_{k\to\infty} \omega_{t_k}\big(\omega_t(x)\big) \odot s_k = \lim_{k\to\infty} \big(\omega_{t_k}(x) \odot t\rho(t)\big) \odot s_k\\ &= \lim_{k\to\infty} \big(\omega_{t_k}(x) \odot s_k\big) \odot t\rho(t) = y \odot t\rho(t)\end{aligned}$$

for all $y \in \Omega$ and $t \in \mathbb{T}$, such that for all $t \in \mathbb{T}$ there exists a unique $\rho(t)$ with $\omega_t(y) = y \odot t\rho(t)$ $\forall y \in \Omega$. A simple compatibility check yields that $\rho(t)$ is the same for all rational (or integer) $t \in \mathbb{T}$. If $\mathbb{T} = \mathbb{Z}$ this completes the proof, otherwise the statement follows from the continuity of ω. □

4. Applications

Before we turn to more sophisticated applications, let us indicate how the two classical examples mentioned in the introduction fit into this framework. For the Gottschalk–Hedlund Theorem, we choose $\mathbb{T}=\mathbb{Z}$ and $\varphi_t=\mathrm{Id}_\Omega$ $\forall t\in\mathbb{T}$. Thus $\Phi=\omega$, and in particular Φ is independent of τ. This implies that every $\rho\in\mathbb{R}$ is (φ,ω)-irrational provided that ω is minimal. Hence, in this particular case, we see that Theorem 1.3 is equivalent to Gottschalk–Hedlund Theorem.

In order to obtain part (ii) of the Poincaré Classification Theorem we let $\Omega=\mathbb{T}^1$, $\varphi_t(x)=x+t \bmod 1$ and choose ω to be trivial, that is, $\omega_t=\mathrm{Id}_\Omega$ $\forall t\in\mathbb{T}=\mathbb{Z}$. Further, given $t\in\mathbb{Z}$ we choose $\tau_t(x)=F^t(x)-x$, where $F:\mathbb{R}\to\mathbb{R}$ is a lift of the orientation-preserving circle homeomorphism f. Then Φ and T^ρ with $\rho=\rho(\Phi)=\rho(F)$ are given by their time-one maps $\Phi_1=f$ and $T_1^\rho=R_\rho$. Further, $|F^t(x)-x-t\rho|\leqslant 1$ $\forall x\in\mathbb{R}$, $t\in\mathbb{Z}$ (see, for example, [10]). Hence, when ρ is irrational, such that T^ρ is minimal, then Theorem 1.3 provides the required semiconjugacy from f to R_ρ.

We now want to apply Theorem 1.3 to some important classes of examples. The skew products in Section 4.1 correspond to the case of periodic leafs described in Proposition 3.1. The examples from Section 4.2 all have in common that the transversal flow ω is trivial, such that all leaves are preserved one-by-one as in Proposition 3.7. Only in Section 4.3 we then make full use of the generality of (1.7), by considering systems that have both non-trivial transversal action and non-periodic leafs.

4.1. Skew products circle flows over a minimal base

Suppose Ω_0 is a compact metric space and $\tilde{\omega}:\mathbb{T}\times\Omega_0\to\Omega_0$ a minimal flow with time $\mathbb{T}=\mathbb{Z}$ or $\mathbb{R}$. Let $\Omega=\Omega_0\times\mathbb{T}^1$ and write $x=(x_0,x')\in\Omega$. We consider skew product flows

$$\Phi:\mathbb{T}\times\Omega\to\Omega,\quad \Phi_t\big(x_0,x'\big)=\big(\tilde{\omega}_t(x_0),\Phi_t^{x_0}\big(x'\big)\big) \tag{4.1}$$

over a *base flow* ω. We say $\hat{\Phi}$ is a lift of Φ if it is a continuous map

$$\hat{\Phi}:\mathbb{T}\times\Omega_0\times\mathbb{R}\to\Omega_0\times\mathbb{R},\quad \big(t,x_0,x'\big)\mapsto\big(\tilde{\omega}_t(x_0),\hat{\Phi}_t^{x_0}\big(x'\big)\big), \tag{4.2}$$

that satisfies $\hat{\Phi}_t^{x_0}(x') \bmod 1=\Phi_t^{x_0}(x' \bmod 1)$.

Corollary 4.1. *Suppose Φ is a skew product flow of the form* (4.1) *with lift $\hat{\Phi}$ and the product flow $T^\rho:(t,x_0,x')\mapsto(\tilde{\omega}_t(x_0),x'+t\rho \bmod 1)$ is minimal. Then Φ is semiconjugate to T^ρ if and only if*

$$\sup_{t\in\mathbb{T},\ x\in\Omega}\big|\hat{\Phi}_t^{x_0}\big(x'\big)-x'-t\rho\big|\leqslant\infty. \tag{4.3}$$

Proof. It suffices to note that Φ is of the form (1.7) with $\varphi_t(x_0,x')=(x_0,x'+t \bmod 1)$, $\omega_t(x_0,x')=(\tilde{\omega}_t(x_0),x')$ and $\tau_t(x_0,x')=\hat{\Phi}_t^{x_0}(x')-x'$. $\square$

We note that so far the result was only known in the case where ω_0 is an irrational rotation of the circle [9]. The proof given in [9] can be extended to irrational rotations of higher-dimensional tori, but does not carry over to the general situation described here.

Corollary 1.4 further implies that under the hypotheses of Corollary 4.1 there is a unique minimal sets. However, in this case a result by Huang and Yi yields the uniqueness of the minimal set also when mean motion is unbounded, that is, when (4.3) fails [7, Theorem 7]. This leads to the following

Corollary 4.2. *Suppose Φ is a skew product flow of the form* (4.1) *and the product flow $T^\rho:(t,x_0,x')\mapsto(\tilde{\omega}_t(x_0),x'+t\rho \bmod 1)$ is minimal almost periodic. Then Φ has a unique minimal set.*

Finally, we note that the uniqueness of the rotation number of skew product flows depends on the unique ergodicity of the base flow.

Theorem 4.3. *(See [6].) Suppose Φ is a skew product circle flow of the form* (4.1) *and the base flow ω is uniquely ergodic. Then the limit*

$$\rho(\hat{\Phi}) = \lim_{t\to\infty} \Phi_t^{x_0}\big(x'\big)/t$$

exists and does not depend on $x_0 \in \Omega_0$ or $x' \in \mathbb{R}$.

4.2. Aperiodic order on the real line

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous non-decreasing map. The translation number of f at x is defined by

$$\rho(f,t) := \lim_{n\to\infty} \frac{f^n(t)-t}{n}$$

provided the limit exists. As pointed out in the introduction, in the case that f is the lift of an orientation-preserving homeomorphism of the circle, the translation number of f at x exists for every $x\in\mathbb{R}$ and does not depend on x. Notice that in this case the displacement function, which is defined as $\phi := f - id$ is 1-periodic. In this section, we are interested in the case when ϕ is no longer periodic but has some sort of "quasiperiodicity".

4.2.1. Almost periodic displacements

We first deal with the case where ϕ is almost periodic. Recall that a function ϕ is almost periodic (in the sense of Bohr), if the family $\{\phi(\cdot+\tau)\colon\ \tau\in\mathbb{R}\}$ has compact closure in $C(\mathbb{R})$ endowed with the topology of uniform convergence. We write X_ϕ for the closure and $\xi_t = \phi(\cdot+t)$ for all $t\in\mathbb{R}$. There is a natural multiplication defined on X_ϕ. Namely if $\xi = \lim_n \xi_{t_n}$ and $\xi' = \lim_n \xi_{s_n}$ belong to X_ϕ, then

$$\xi\cdot\xi' := \lim_n \xi_{t_n+s_n}$$

is well-defined and $(X_\phi,\cdot)$ is a compact Abelian group (see, for instance, [3] for details) with identity ξ_0. Note that $\xi\cdot\xi_t(.) = \xi(.+t)$, and this defines a natural flow $\phi:(t,\xi)\mapsto \xi\cdot\xi_t =: \xi\odot t$ on X_ϕ.

The map $f:\mathbb{R}\to\mathbb{R}$ given by $f(x) = x+\phi(x)$ can now be studied as follows. We let $\tau: X_\phi\to\mathbb{R}$, $\xi\mapsto\xi(0)$ and define $F: X_\phi\to X_\phi$ by

$$F(\xi) = \xi\odot\tau(\xi). \tag{4.4}$$

Then $F(\xi_t) = \xi_t\odot\xi_t(0) = \xi_t\odot\phi(t) = \xi_{f(t)}$. Consequently $h:\mathbb{R}\to X_\phi$, $t\mapsto\xi_t$ satisfies $h\circ f(t) = \xi_{f(t)} = F(\xi_t) = F\circ h(t)$, such that f can be identified with $F_{|\{\xi_t|t\in\mathbb{R}\}}$. In this setting, uniqueness of the rotation number was shown by Kwapisz.

Theorem 4.4. *(See [12].) If f is orientation-preserving and ϕ is bounded away from zero, then F has a unique translation number*

$$\rho(F) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\tau\circ F^i(\xi) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\tau\circ f^i(t),$$

meaning that the limits exist and are independent of $\xi\in X_\phi$ and $t\in\mathbb{R}$.

Note that the system obtained in (4.4) can be written in the general form (1.7) with time $\mathbb{T}=\mathbb{Z}$ and trivial transversal action $\omega = \mathrm{Id}$. It follows that Theorem 1.1 can be applied to give a solution to the semiconjugacy problem for maps of this type, which was explicitly raised in [12]. However, before stating the result, we first want to understand which numbers are (φ,ω)-rational.

It is well known that every continuous almost periodic function q has a mean value

$$M(q) = \lim_{N\to+\infty} \frac{1}{N} \int_{-N}^{N} q(t)\,dt.$$

The *frequency module* of ϕ is defined to be the set

$$\mathcal{M} = \left\{ \sum_{v} j_v \lambda_v,\ j_v \in \mathbb{Z} \right\},$$

where the $\lambda_v \in \mathbb{R}$ are the (countably many) values of λ for which $M(\phi e^{-i\lambda x}) \neq 0$. We say $t \in \mathbb{R}$ is *rationally independent of* $\mathcal{M}$ if $kt \in \mathcal{M}$ with $k \in \mathbb{Z}$ implies $k = 0$. Further, we call $\lambda \in \mathbb{R}$ a *continuous eigenvalue* of the flow $(X_\phi, \mathbb{R})$ if there exists a continuous map $g : X_\phi \to \mathbb{R}$ which satisfies $g(\xi \odot t) = e^{i\lambda t} g(\xi)$ for all $\xi \in X_\phi$ and $t \in \mathbb{R}$.

Proposition 4.5. *(See [13].) The frequency module of f coincides with the group of continuous eigenvalues of $(X_\phi, \mathbb{R})$.*

Lemma 4.6. *(See [5, 4.24.1].) Let $(X, \{T_t\}_{t\in\mathbb{R}})$ be a minimal compact flow. Then for every $t \in \mathbb{R} \setminus \{0\}$, the system (X, T_t) is not minimal if and only if there is a continuous eigenvalue λ of $(X, \{T_t\}_{t\in\mathbb{R}})$ such that $t\lambda$ is an integer.*

This allows the characterisation of the (φ, ω)-irrational times.

Proposition 4.7. *Suppose the flow Φ in* (1.7) *is of the form given by* (4.4)*. Then ρ is (φ, ω)-irrational if and only if ρ^{-1} is rationally independent of $\mathcal{M}$.*

Proof. By definition, ρ is (φ, ω)-irrational iff the multiplication by ξ_ρ is minimal. By Lemma 4.6, this happens iff for every continuous eigenvalue λ the quantity $\lambda\rho$ is not an integer. By Proposition 4.5, this is equivalent to the fact that $k\rho^{-1}$ is not a frequency in $\mathcal{M}$ for all $k \in \mathbb{Z} \setminus \{0\}$, which means that ρ^{-1} is rationally independent of $\mathcal{M}$. $\square$

Corollary 4.8. *Suppose that the hypotheses of Theorem* 4.4 *are satisfied. Further, assume that $\rho(F)^{-1}$ is rationally independent of $\mathcal{M}$ and that*

$$\sup_{n,x} \left| F^n(x) - x - n\rho(x) \right| < \infty. \tag{4.5}$$

Then F is semiconjugate to the multiplication by $\xi_{\rho(F)}$.

4.2.2. Pattern-equivariant displacements with respect to quasicrystals

We now show how Theorem 1.1 can be applied to the case where the displacement ϕ is pattern-equivariant with respect to a mathematical quasicrystal of the real line. We keep the exposition brief, for more details see [1]. Recall that a Delone set in $\mathbb{R}$ is an increasing sequence $X = (\theta_n)_{n\in\mathbb{Z}}$ of points in $\mathbb{R}$ such that $0 < \inf_n |\theta_n - \theta_{n-1}|$ and $\sup_n |\theta_n - \theta_{n-1}| < +\infty$. If the set $\{\theta_n - \theta_{n-1}\}_{n\in\mathbb{Z}}$ is finite, we say that X has *finite local complexity*. Given $x \in X$ and $R > 0$, the R-patch of X centred at x is defined as $P(x, R) = X \cap [x - R, x + R]$. Two patches $P(x, R)$ and $P(y, R)$ are equivalent if $P(x, R) - x = P(y, R) - y$. We say that X is *repetitive* if for every patch $P = P(x, R)$, there exists $M > 0$ such that every interval of length M contains the centre of a patch equivalent to P. Given a patch $P(x, R)$, $t \in \mathbb{R}$ and $N > 0$, let

$$n(P, N, t) := \#\left\{ y \in [t - N, t + N] \;\middle|\; y \in X \text{ and } P(y, R) \sim P(x, R) \right\}$$

$$LL$$
$$LSLS$$
$$LSLLSL$$
$$LSLLSLSLLS$$
$$LSLLSLSLLSLLSLSL$$

Fig. 1. The Fibonacci tiling.

denote the number of patches of X that are equivalent to P and whose centre is included in the interval $[t-N, t+N]$. We say that X has *uniform patch frequencies* if $n(P, N, t)/2N$ converges uniformly in t to a limit $\nu(P)$, which is independent of t, when $N \to +\infty$. Finally, we say that X is a *quasicrystal* if X is non-periodic, repetitive and has uniform patch frequencies. A function $\phi:\mathbb{R} \to \mathbb{R}$ is *strongly X-equivariant* if there exists $R > 0$ such that $X - x \cap [-R, R] = X - y \cap [-R, R]$ implies that

$$\phi(x) = \phi(y).$$

A continuous function $\phi:\mathbb{R} \to \mathbb{R}$ is *X-equivariant* if it is the uniform limit of a sequence of strongly X-equivariant continuous functions. A continuous function ϕ is *pattern-equivariant* if there exists a quasicrystal X such that ϕ is X-equivariant. The most well-known example for a quasicrystal of the real line is given by the Fibonacci tiling, which can be constructed by iterating the substitution

$$L \to LS,$$
$$S \to L$$

starting with the sequence $L.L$ as depicted in Fig. 1 and then replacing the infinite sequence thus obtained by a tiling in which each L is replaced by an interval of length τ, where τ is the golden mean number, and each S by a unit-interval.

Suppose that the displacement ϕ is X-equivariant for some quasicrystal X. Then, similar to almost periodic functions, there exists a compact metric space Ω, called the hull of X, which is the completion of the set

$$\{X - t \mid t \in \mathbb{R}\}$$

endowed with an appropriate metric (we omit the precise definition of this metric here, see [1] for details). The elements of Ω are all Delone sets, and there is a continuous action Γ on Ω given by translations $\Gamma_t Y = Y - t$ for all $t \in \mathbb{R}$ and $Y \in \Omega$. The repetitivity of X is equivalent to the minimality of Γ and the uniform pattern frequencies property is equivalent to the unique ergodicity of Γ (see [1] and references therein for details). Moreover, there is a unique continuous function $\tau:\Omega \to \mathbb{R}$ such that $\tau(X - t) = \phi(t)$ for all $t \in \mathbb{R}$, and thus a continuous map $F:\Omega \to \Omega$ defined by

$$F(Y) = Y - \tau(Y), \quad Y \in \Omega. \tag{4.6}$$

The map F is related to f by the relation $F(X - t) = X - f(t)$ for all $t \in \mathbb{R}$.

Theorem 4.9. *(See [1].) Suppose that $F : \Omega \to \Omega$ has the form of* (4.6)*. If Φ does not have zeros, then F has a unique translation number $\rho(F)$, defined as*

$$\rho(F) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \tau\big(F^k(X)\big).$$

The following result (which already appears in [1]) is again a direct consequence of Theorem 1.1.

Corollary 4.10. *Suppose that $\Gamma_{\rho(F)}$ is minimal, and that* $\sup_n |\sum_{k=0}^{n-1} \tau(F^k(X)) - n\rho(F)| < \infty$*. Then F is semiconjugate to $\Gamma_{\rho(F)}$.*

4.2.3. Special flows

Let us end this section by noticing that both of the above hull constructions, for functions with almost periodic and pattern-equivariant displacements, lead to examples of special flows. Recall the definition of a special flow. Let $\tilde{X}$ be a compact metric space, $\sigma : X \to X$ a minimal transformation and $h : \tilde{X} \to \mathbb{R}^+$ a continuous function. Let $X = \tilde{X} \times \mathbb{R}/\sim$, where $\sim$ is the smallest equivalence relation such that $(\tilde{x}, h(\tilde{x})) \sim (\tilde{\sigma}(x), 0)$ for all $\tilde{x} \in \tilde{X}$. It is well known that the natural flow over $\tilde{X} \times \mathbb{R}/\sim$ descends to a well-defined flow over X, which satisfies $\varphi_t[(\tilde{x}, s)] = [(\tilde{x}, s+t)]$ for all $\tilde{x} \in X$ and $t, s \in \mathbb{R}$, where $[(x, s)]$ denotes the equivalence class of (x, t). The dynamical system (X, φ) is known as the special flow over σ with roof function h.

In the case of almost periodic functions, the map σ can be chosen as the first return map to $X_{\phi,s}$ for $s > 0$, where $X_{\phi,s}$ is the closure of $\{\xi_{ns} \mid n \in \mathbb{Z}\}$ in the uniform topology. In the case of quasicrystals, σ is given by the first return map to the canonical transversal $\Omega_0 = \{\Omega \in X \mid 0 \in \Omega\}$.

4.3. Scalar differential equations almost periodic in time and space

The study of differential equations with almost periodic right side was already initiated by Favard [4] in 1928 and has a long tradition since. [14] gives a good overview and more recent references. We first recall some basic facts. Given $f : \mathbb{R} \times \mathbb{R}^d$, $(t, x) \mapsto f(t, x)$ we define $f_t : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ by $f_t(s, x) = f(s + t, x)$. Then, similar to the above, f is called *uniformly almost periodic* in t if $\{f_t \mid t \in \mathbb{R}\}$ has compact closure in $\mathcal{C}^0(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ in the topology of uniform convergence. In this case

$$\mathcal{H}(f) := \overline{\{f_t \mid t \in \mathbb{R}\}}$$

is called the *hull* of f and the flow

$$\omega : \mathbb{R} \times \mathcal{H}(f) \to \mathcal{H}(f), \quad (t, g) \mapsto (t, g_t)$$

is uniformly almost periodic [3] and minimal. An almost periodic scalar differential equation is of the form

$$x' = f(t, x) \tag{4.7}$$

with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ uniformly almost periodic in t. It gives rise to a skew product flow of the form

$$\hat{\Phi} : \mathbb{R} \times \mathcal{H}(f) \times \mathbb{R} \to \mathcal{H}(f) \times \mathbb{R}, \quad (t, g, x) \mapsto \big(g_t, \xi(t, g, x)\big),$$

where $t \mapsto \xi(t, g, x)$ is the solution of the scalar differential equation $x' = g(t, x)$ with initial values $t_0 = 0$ and $x_0 = x$. (Here, we make the usual assumptions of Lipschitz-continuity in x in order to guarantee the existence and uniqueness of solutions.) It suffices to make these assumptions on f, since they carry over to the hull [14, Theorem 3.1 in Part 1].

When no further structure concerning the dependence of f on x is assumed, the study of (4.7) is mostly restricted to the description of bounded solutions and their orbit closures [8]. Much more can be said when f is periodic in x and therefore factors to a function on $\mathbb{R} \times \mathbb{T}^1$. In this case $\hat{\Phi}$ projects to a skew product circle flow Φ over the minimal base $(\mathcal{H}(f), \omega)$, and an extensive theory exists to describe the dynamical behaviour [7]. As mentioned above, in this situation the rotation number is unique by Theorem 4.3 and Corollary 4.1 provides an equivalent condition for the existence of a semiconjugacy to the translation flow $(t, g, x) \mapsto (g_t, x + t\rho \bmod 1)$.

However, the main interest of the general setting introduced in Section 1 in this context comes from the fact that it can also be applied when f is only almost periodic in x. Given $f : \mathbb{R}^2 \to \mathbb{R}$ we define $f_{t,x} : \mathbb{R}^2 \to \mathbb{R}$ by $f_{t,x}(s, y) = f(s+t, y+x)$ and say that f is *uniformly almost periodic in t and x* if $\{f_{t,x} \mid (t, x) \in \mathbb{R}^2\}$ has compact closure in $\mathcal{C}^0(\mathbb{R}^2, \mathbb{R})$. Further we let

$$\Omega(f) := \overline{\left\{f_{t,x} \mid (t, x) \in \mathbb{R}^2\right\}}.$$

Then (4.7) induces a flow

$$\Phi : \mathbb{R} \times \Omega(f) \to \Omega(f), \quad (t, g) \mapsto g_{t,\xi(t,g)}, \tag{4.8}$$

where $t \mapsto \xi(t, g)$ is the solution of $x' = g(t, x)$ with initial values $t_0 = x_0 = 0$. Again, Theorem 1.3 can be applied (with $\omega_t(g) = g_{t,0}$, $\varphi_t(g) = g_{0,t}$ and $\tau_t(g) = \xi(t, g)$) to determine whether Φ is semiconjugate to the translation flow $(t, g) \mapsto g_{t,t\rho}$ on $\Omega(f)$.

Corollary 4.11. *Suppose that the translation flow $T^\rho : (t, g) \mapsto g_{t,t\rho}$ on Ω is minimal. Then the flow Φ given by* (4.8) *is semiconjugate to T^ρ if and only if the solution $t \mapsto \xi(t, f)$ of $x' = f(t, x)$ satisfies*

$$\sup_{t\in\mathbb{R}} \left|\xi(t, f)\right| < \infty. \tag{4.9}$$

Note that by continuity (4.9) implies $\sup_{t\in\mathbb{R},\, g\in\Omega(f)} |\xi(t, g)| < \infty$.

In order to give a somewhat more explicit example, suppose $F : \mathbb{R}^4 \to \mathbb{R}$ is periodic with least period 1 in all coordinates, such that it projects to a function on the four-torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$. Fix a totally irrational vector $(\alpha, \beta) \in \mathbb{R}^4$, $\alpha, \beta \in \mathbb{R}^2$, and let $f(t, x) = F(t \cdot \alpha, x \cdot \beta)$. Then the hull $\Omega(f)$ is homeomorphic to $\mathbb{T}^4$, and the skew product flow Φ is of the form

$$\Phi : \mathbb{R} \times \mathbb{T}^4 \to \mathbb{T}^4, \quad (t, z) \mapsto \big(u + t\alpha, v + \tau_t(z)\beta\big),$$

where $z = (u, v)$ and $\tau : \mathbb{R} \times \mathbb{T}^4 \to \mathbb{R}$ satisfies (1.6). Assuming that $\alpha = (1, \alpha_1)$, the Poincaré section of this flow in $\{0\} \times \mathbb{T}^3$ gives rise to a skew product map of the form

$$\Psi : \mathbb{T}^3 \to \mathbb{T}^3, \quad \zeta \mapsto \big(\theta + \alpha_1, \xi + \tau_1(0, \xi)\beta\big), \tag{4.10}$$

where $\zeta = (\theta, \xi)$ with $\theta \in \mathbb{T}^1$ and $\zeta \in \mathbb{T}^2$. One may say that this is the simplest type of dynamical systems that exhibits quasiperiodicity in both variables. Note that (the discrete-time flow generated by) Ψ can be written in the general form of (1.7) with $\omega_t(\zeta) = \zeta + t \cdot (\alpha_1, 0, 0)$, $\varphi_t(\zeta) = \zeta + t \cdot (0, \beta)$ and $\tau_t(\zeta) = \tau_t(0, \zeta)$, $t \in \mathbb{Z}$.

We believe that maps of this type are a good starting point for studying further questions concerning the type of flows introduced in Section 2, reflecting the full generality of the setting while at the same time having a rather simple structure. Hence, we end by pointing out the following still open problems.

Questions 4.12.

(a) Does Ψ always have a unique rotation number whenever 0 is not contained in the rotation interval? We observe that this question already appeared in [11].
(b) Suppose Ψ is semiconjugate to a minimal rotation on $\mathbb{T}^3$. Does this imply that Ψ is uniquely ergodic?
(c) Suppose $\rho(\Phi)$ is (φ, ω)-irrational and let

$$\Phi_t^\varepsilon(z) = \bigl(u + t\alpha, v + \bigl(\tau_t(z) + \varepsilon\bigr)\beta\bigr).$$

Then, is the map $\varepsilon \mapsto \rho(\Phi^\varepsilon)$ strictly increasing in $\varepsilon = 0$? (*Absence of mode-locking.*)
For skew product circle flows, the answer is positive [2].

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References

[1] J. Aliste-Prieto, Translation numbers for a class of maps on the dynamical systems arising from quasicrystals in the real line, Ergodic Theory Dynam. Systems 30 (02) (2010) 565–594.
[2] K. Bjerklöv, T. Jäger, Rotation numbers for quasiperiodically forced circle maps – mode-locking vs strict monotonicity, J. Amer. Math. Soc. 22 (2) (2009) 353–362.
[3] R. Ellis, Lectures on Topological Dynamics, W.A. Benjamin, 1969.
[4] J. Favard, Sur les équations différentielles linéaires a coefficients presque-périodiques, Acta Math. 51 (1) (1928) 31–81.
[5] E. Glasner, Ergodic Theory via Joinings, Math. Surveys Monogr., vol. 101, American Mathematical Society, Providence, RI, 2003.
[6] M. Herman, Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2, Comment. Math. Helv. 58 (1983) 453–502.
[7] W. Huang, Y. Yi, Almost periodically forced circle flows, preprint, 2007.
[8] R. Johnson, Bounded solutions of scalar, almost periodic linear equations, Illinois J. Math. 25 (4) (1981) 632–643.
[9] T. Jäger, J. Stark, Towards a classification for quasiperiodically forced circle homeomorphisms, J. Lond. Math. Soc. 73 (3) (2006) 727–744.
[10] A. Katok, B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1997.
[11] J. Kwapisz, Homotopy and dynamics for homeomorphisms of solenoids and Knaster continua, Fund. Math. 168 (2001) 251–278.
[12] J. Kwapisz, Poincaré rotation number for maps of the real line with almost periodic displacement, Nonlinearity 13 (5) (2000) 1841.
[13] A. Miller, J. Rosenblatt, Characterizations of regular almost periodicity in compact minimal abelian flows, Trans. Amer. Math. Soc. 356 (12) (2004) 4909–4930.
[14] W. Shen, Y. Yi, Almost automorphic and almost periodic dynamics in skew product semiflows, Mem. Amer. Math. Soc. 136 (647) (1998).