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Mixed volume techniques for embeddings of Laman graphs ☆

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A B S T R A C T

Determining the number of embeddings of Laman graph frameworks is an open problem which corresponds to understanding the solutions of the resulting systems of equations. In this paper we investigate the bounds which can be obtained from the viewpoint of Bernstein's Theorem. The focus of the paper is to provide methods to study the mixed volume of suitable systems of polynomial equations obtained from the edge length constraints. While in most cases the resulting bounds are weaker than the best known bounds on the number of embeddings, for some classes of graphs the bounds are tight.

1. Introduction

Let $G = (V, E)$ be a graph on n vertices with $2n - 3$ edges. If each subset of k vertices spans at most $2k - 3$ edges, we say that G has the *Laman property* and call it a *Laman graph* (see [20]). A *framework* is a tuple (G, L) where $G = (V, E)$ is a graph and $L = \{l_{i,j}\colon [v_i, v_j] \in E\}$ is a set of $|E|$ positive numbers interpreted as edge lengths. For generic edge lengths, Laman graph frameworks are minimally rigid (see [7]), i.e. they are rigid and they become flexible if any edge is removed.

A *Henneberg sequence* for a graph G is a sequence $(G_i)_{3 \leqslant i \leqslant r}$ of Laman graphs such that G_3 is a triangle, $G_r = G$ and each G_i is obtained by G_{i-1} via one of the following two types of steps: A *Henneberg I step* adds one new vertex v_{i+1} and two new edges, connecting v_{i+1} to two arbitrary vertices of G_i. A *Henneberg II step* adds one new vertex v_{i+1} and three new edges, connecting v_{i+1} to three vertices of G_i such that at least two of these vertices are connected via an edge e of G_i and this certain edge e is removed (see Fig. 1).

Any Laman graph G can be constructed via a Henneberg sequence and any graph constructed via a Henneberg sequence has the Laman property (see [25,27]). We call G a *Henneberg I graph* if it is constructable using only Henneberg I steps. Otherwise we call it *Henneberg II*.

Given a Laman graph framework we want to know how many embeddings, i.e. maps $\alpha : V \to \mathbb{R}^2$, exist such that the Euclidean distance between two points in the image is exactly $l_{i,j}$ for all $[v_i, v_j] \in E$. Since every rotation or translation of an embedding gives another one, we ask how many embeddings exist *modulo rigid motions*. Due to the minimal rigidity property, questions about embeddings of Laman graphs arise naturally in rigidity and linkage problems (see [16,28]). Graphs with less edges will have zero or infinitely many embeddings modulo rigid motions, and graphs with more edges do not have any embeddings for a generic choice of edge lengths.

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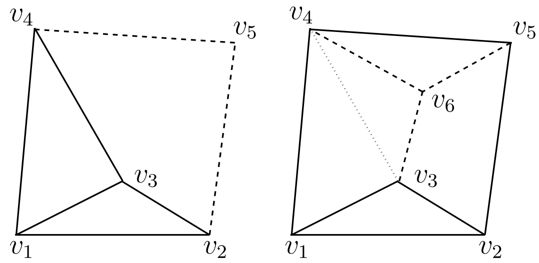

Fig. 1. A Henneberg I and a Henneberg II step. New edges are dashed and the deleted edge is pointed.

Determining the maximal number of embeddings (modulo rigid motions) for a given Laman graph is an open problem. The best upper bounds are due to Borcea and Streinu (see [4,5]) who show that the number of embeddings is bounded by $\binom{2n-4}{n-2} \approx \frac{4^{n-2}}{\sqrt{n-2}}$. Their bounds are based on degree results of determinantal varieties.

A general method to study the number of (complex) solutions of systems of polynomial equations is to use Bernstein's Theorem [2] for sparse polynomial systems. This theorem provides bounds on the number of solutions in terms of the mixed volume of the underlying Newton polytopes. Since the systems of polynomial equations describing the Laman embeddings are sparse, the question arose how good these Bernstein bounds are for the Laman embedding problem. While for concrete systems of equations, the mixed volume can be computed algorithmically, studying the mixed volume for *classes of polytopes* is connected with a variety of issues in convex geometry (such as understanding the Minkowski sum of the polytopes).

In this paper, we study the quality of the Bernstein bound on the Laman embedding problem and provide methods to handle the resulting convex geometric problems. In most cases, our bounds are worse than the bounds in [5]. However, we think that the general methodology of studying Bernstein bounds nevertheless provides an interesting technique, and we see the main contribution of this paper in providing the technical tools (such as achieving to determine the mixed volume) to compute these bounds for whole classes of graphs. It is particularly interesting that for some classes of graphs, the mixed volume bound is tight.

To use these algebraic tools for the embedding problem we formulate that problem as a system of polynomial equations in the $2n$ unknowns $(x_1, y_1, \dots, x_n, y_n)$ where (x_i, y_i) denote the coordinates of the embedding of the vertex v_i. Each prescribed edge length translates into a polynomial equation. I.e. if $e_k := [v_i, v_j] \in E$ with length $l_{i,j}$, we require $h_k(x) := (x_i - x_j)^2 + (y_i - y_j)^2 - l_{i,j}^2 = 0$. Thus we obtain a system of $|E|$ quadratic equations whose solutions represent the embeddings of our framework. To get rid of translations and rotations we fix the points $(x_1, y_1) = (c_1, c_2)$ and $(x_2, y_2) = (c_3, c_4)$. (Here we assume without loss of generality that there is an edge between v_1 and v_2.) For practical reasons we choose $c_i \neq 0$ and c_3, c_4 are chosen such that the embedded points (x_1, y_1) and (x_2, y_2) have distance $l_{1,2}$. Hence we want to study the solutions to the following system of $2n$ equations.

$$\left\{ \begin{array}{l} h_1(x) := x_1 - c_1 = 0 \\ h_2(x) := y_1 - c_2 = 0 \\ h_3(x) := x_2 - c_3 = 0 \\ h_4(x) := y_2 - c_4 = 0 \\ h_k(x) := (x_i - x_j)^2 + (y_i - y_j)^2 - l_{i,j}^2 = 0 \quad \forall e_k = [v_i, v_j] \in E - \{[v_1, v_2]\} \end{array} \right\}. \tag{1}$$

The paper is structured as follows. In Section 2 we review the concepts of mixed volumes and Bernstein's Theorem. In Section 3 we present some technical tools to simplify mixed volume calculation. Then, in Section 4 we discuss the quality of the Bernstein bounds on the Laman embedding problem.

2. Preliminaries

2.1. Mixed volumes and mixed subdivisions

The *Minkowski sum* of two sets $A_1, A_2 \subset \mathbb{R}^k$ is defined as

$$A_1 + A_2 = \{a_1 + a_2 \mid a_1 \in A_1, a_2 \in A_2\}.$$

Let $P_1, \dots, P_k$ be k polytopes in $\mathbb{R}^k$. For non-negative parameters $\lambda_1, \dots, \lambda_k$ the k-dimensional Euclidean volume $\mathrm{vol}_k(\lambda_1 P_1 + \dots + \lambda_k P_k)$ of the scaled Minkowski sum is, as a function of $\lambda_1, \dots, \lambda_k$, a homogeneous polynomial of degree k with non-negative coefficients (see e.g. [24,30]). The coefficient of the monomial $\lambda_1 \cdots \lambda_k$ is called the *mixed volume of* $P_1, \dots, P_k$ and is denoted by $\mathrm{MV}_k(P_1, \dots, P_k)$.

We denote by $\mathrm{MV}_k(P_1, d_1; \dots; P_r, d_r)$ the mixed volume where P_i is taken d_i times and $\sum_{i=1}^r d_i = k$. The mixed volume is invariant under permutation of its arguments, it is linear in each argument, i.e.

$$\mathrm{MV}_k(\dots, \alpha P_i + \beta P_i', \dots) = \alpha\, \mathrm{MV}_k(\dots, P_i, \dots) + \beta\, \mathrm{MV}_k(\dots, P_i', \dots) \tag{2}$$

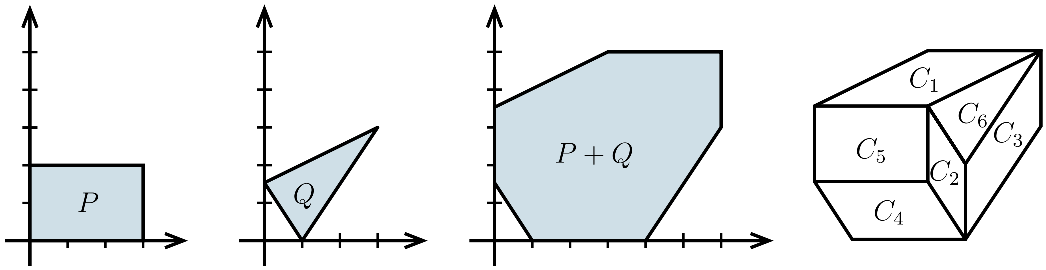

Fig. 2. From left to right: P, Q, the Minkowski sum of P and Q and a mixed subdivision Γ of $P+Q$.

and it generalizes the usual volume in the sense that

$$\mathrm{MV}_k(P,\dots,P) = k!\,\mathrm{vol}_k(P) \tag{3}$$

holds (see [24]).

Let $P = P_1 + \cdots + P_r \subset \mathbb{R}^k$ be a Minkowski sum of polytopes that affinely spans $\mathbb{R}^k$. A sum $C = F_1 + \cdots + F_r$ of faces $F_i \subset P_i$ is called *cell* of P. A *subdivision* of P is a collection $\Gamma = \{C_1,\dots,C_m\}$ of cells such that each cell is of full dimension, the intersection of two cells is a face of both and the union of all cells covers P. Each cell is given a type $type(C) = (\dim(F_1),\dots,\dim(F_r))$. Clearly the entries in the type vector sum up to at least the dimension of the cell C. A subdivision is called *mixed* if for each cell $C \in \Gamma$ we have that $\sum d_i = k$ where $type(C) = (d_1,\dots,d_r)$. Cells of type $(d_1,\dots,d_r)$ with $d_i \geqslant 1$ for each i will be called *mixed cells*.

With this terminology the mixed volume can be calculated by

$$\mathrm{MV}_k(P_1,d_1;\dots;P_r,d_r) = \sum_C d_1!\cdots d_r!\,\mathrm{vol}_k(C) \tag{4}$$

where the sum is over all cells C of type $(d_1,\dots,d_r)$ in an arbitrary mixed subdivision of $P_1 + \cdots + P_r$ (see [17]).

To construct mixed subdivisions we proceed as in [17]. Not every subdivision can be constructed in this way but since we only need one arbitrary mixed subdivision this simple construction can be used. For each polytope P_i choose a linear lifting function $\mu_i : \mathbb{R}^k \to \mathbb{R}$. In the following μ_i is identified with the vector of $\mathbb{R}^k$ that defines it. By $\hat{P}_i$ we denote the lifted polytopes $\mathrm{conv}\{(q, \langle \mu_i, q\rangle):\ q \in P_i\} \subset \mathbb{R}^{k+1}$, where $\langle \cdot,\cdot \rangle$ denotes the Euclidean scalar product and $\mathrm{conv}\,A$ denotes the convex hull of a point set A.

The set of those facets, i.e. codimension 1 faces, of $\hat{P} := \hat{P}_1 + \cdots + \hat{P}_r$ which have an inward pointing normal with a positive last coordinate is called *the lower hull* of $\hat{P}$. Projecting down this lower hull back to $\mathbb{R}^k$ by forgetting the last coordinate yields a subdivision of $P_1 + \cdots + P_r$. Such a subdivision is called *coherent* and is said to be *induced by* $\mu = (\mu_1,\dots,\mu_r)$.

Example 1. Let

$$P = \mathrm{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}\right\}, \qquad Q = \mathrm{conv}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\\frac{3}{2}\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}\right\}.$$

The Minkowski sum of P and Q is depicted in Fig. 2 together with one of the possible coherent mixed subdivisions. The cells $C_1,\dots,C_4$ are mixed cells of this subdivision.

There exist several software packages to compute mixed volumes. For the results in this work the `PHCpack` by Jan Verschelde [29] was employed.

2.2. BKK theory

The main tool in this work is the following theorem that provides a connection between solutions to systems of polynomial equations and discrete geometry. For a polynomial $f = \sum_{\alpha \in A} c_\alpha x^\alpha \in \mathbb{C}[x_1,\dots,x_k]$ the Newton polytope $\mathrm{NP}(f) \subset \mathbb{R}^k$ is the convex hull of the monomial exponent vectors, i.e. $\mathrm{NP}(f) = \mathrm{conv}\,A$. Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Theorem 2 *(Bernstein [2]). Given polynomials $f_1,\dots,f_k \in \mathbb{C}[x_1,\dots,x_k]$ with finitely many common zeroes in $(\mathbb{C}^*)^k$ and let $\mathrm{NP}(f_i)$ denote the Newton polytope of f_i. Then the number of common zeroes of the f_i in $(\mathbb{C}^*)^k$ is bounded above by the mixed volume $\mathrm{MV}_k(\mathrm{NP}(f_1),\dots,\mathrm{NP}(f_k))$. Moreover for generic choices of the coefficients in the f_i, the number of common solutions is exactly $\mathrm{MV}_k(\mathrm{NP}(f_1),\dots,\mathrm{NP}(f_k))$.*

Remark 3. Here, and throughout this work *generic* is interpreted as follows. A subset A of $\mathbb{C}^m$ is called *Zariski open* if there is an algebraic variety V, i.e. a solution set to a system of algebraic equations, such that $A = \mathbb{C}^m \setminus V$. We say that a statement is true for a *generic choice* in $\mathbb{C}^m$ if it is true for a non-empty Zariski open subset of $\mathbb{C}^m$. This implies that the statement is true "almost everywhere" in a measure theoretic sense.

Various attempts have been made to generalize these results to count all common roots in $\mathbb{C}^k$ (see for example [12,18, 22]). The easiest, but sometimes not the best bound is $\mathrm{MV}_k(\mathrm{conv}(\mathrm{NP}(f_1) \cup 0), \dots, \mathrm{conv}(\mathrm{NP}(f_k) \cup 0))$ which is shown in [22]. Since the Newton polytopes of system (1) all contain the point 0 as a vertex, the mixed volume of (1) yields a bound on the number of solutions in $\mathbb{C}$ rather then only on those in $\mathbb{C}^*$.

The bound on the number of solutions of a polynomial system arising from Bernstein's Theorem is also often referred to as the *BKK bound* due to the work of Bernstein, Khovanskii and Kushnirenko. The BKK bound generalizes the Bézout bound (see [8, Chapter 7]) and for sparse polynomial systems it is often significantly better.

Bernstein also gives an explicit condition that characterizes when a choice of coefficients is generic. Let w be a non-zero vector and let $\partial_w P$ denote the face of a polytope P which is minimal with respect to the direction w, i.e. $\partial_w P := \{p \in P\colon \langle w, p\rangle = \min_{q \in P} \langle w, q\rangle\}$. For a given $f = \sum_{\alpha \in A} c_\alpha x^\alpha$ we set $\partial_w f = \sum_\alpha c_\alpha x^\alpha$ to be the *face equation* with respect to w, where the sum is over all integer points $\alpha \in \partial_w \mathrm{NP}(f)$.

Theorem 4 *(Bernstein's Second Theorem [2]). If for all $w \neq 0$, the face system $\partial_w f_1 = 0, \dots, \partial_w f_k = 0$ has no solution in $(\mathbb{C}^*)^k$, then the mixed volume of the Newton polytopes of the f_i gives the exact number of common zeros in $(\mathbb{C}^*)^k$ and all solutions are isolated. Otherwise it is a strict upper bound.*

Remark 5. If a direction w is a witness of the degeneracy, then it lies on the so-called tropical prevariety (see [23]) of the polynomials $f_1, \dots, f_k$.

3. New technical tools to simplify mixed volume calculation

In the special case of Henneberg I graphs, system (1) is of a shape that allows to separate the mixed volume calculation into smaller pieces. The main tool to do this is the following lemma. An equivalent decomposition result was already mentioned in [6] in which the authors refer to [14] (in Russian) for the proof. For the convenience of the reader we provide here a proof based on the properties of symmetric multilinear functions.

Lemma 6. *Let $P_1, \dots, P_k$ be polytopes in $\mathbb{R}^{m+k}$ and $Q_1, \dots, Q_m$ be polytopes in $\mathbb{R}^m \subset \mathbb{R}^{m+k}$. Then*

$$\mathrm{MV}_{m+k}(Q_1, \dots, Q_m, P_1, \dots, P_k) = \mathrm{MV}_m(Q_1, \dots, Q_m) \cdot \mathrm{MV}_k\big(\pi(P_1), \dots, \pi(P_k)\big) \tag{5}$$

where $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ denotes the projection on the last k coordinates.

Proof. First we show the lemma in the *semimixed case* where $Q_1 = \dots = Q_m =: Q$ and $P_1 = \dots = P_k =: P$, then we use properties of symmetric multilinear functions to reduce the general situation to the semimixed case.

By (3) we have to show first that

$$\mathrm{MV}_{m+k}(Q, \dots, Q, P, \dots, P) = m!k!\,\mathrm{vol}_m(Q) \cdot \mathrm{vol}_k\big(\pi(P)\big) \tag{6}$$

where Q is taken m times and P is taken k times. But this formula for semimixed systems is a special case of [13, Chapter IV, Lemma 4.9] or also of [3, Theorem 1].

Let $\mathcal{P}^m$ (resp. $\mathcal{P}^{m+k}$) be the set of all m-dimensional (resp. $(m+k)$-dimensional) polytopes and define two functions g_1 and g_2 on $(\mathcal{P}^m)^m \times (\mathcal{P}^{m+k})^k$ via

$$g_1(Q_1, \dots, Q_m, P_1, \dots, P_k) := \mathrm{MV}_{m+k}(Q_1, \dots, Q_m, P_1, \dots, P_k),$$
$$g_2(Q_1, \dots, Q_m, P_1, \dots, P_k) := \mathrm{MV}_m(Q_1, \dots, Q_m) \cdot \mathrm{MV}_k\big(\pi(P_1), \dots, \pi(P_k)\big).$$

Due to the properties of mixed volumes (see Section 2.1) it is easy to see that g_1 and g_2 are invariant under changing the order of the Q_i and under changing the order of the P_j. Furthermore it follows from (2) that both functions are linear in each argument.

Hence, for fixed $P_1, \dots, P_k$ the induced mappings

$$\tilde{g}_i^{(P_1,\dots,P_k)}(Q_1, \dots, Q_m) := g_i(Q_1, \dots, Q_m, P_1, \dots, P_k) \quad (i = 1, 2)$$

are symmetric and multilinear, and analogously, for fixed Q, the mappings

$$\bar{g}_i^{(Q)}(P_1, \dots, P_k) := g_i(Q, \dots, Q, P_1, \dots, P_k) \quad (i = 1, 2)$$

are symmetric and multilinear. For any semigroups A, B and any symmetric multilinear function $f : A^n \to B$, it follows from an inclusion-exclusion argument (see [13, Theorem 3.7]) that

$$f(a_1, \dots, a_n) = \frac{1}{n!} \sum_{1 \leqslant i_1 < \dots < i_q \leqslant n} (-1)^{n-q} f(a_{i_1} + \dots + a_{i_q}, \dots, a_{i_1} + \dots + a_{i_q}). \tag{7}$$

Hence we have for $i = 1, 2$ that

$$\begin{aligned} &g_i(Q_1, \dots, Q_m, P_1, \dots, P_k) \\ &\quad = \tilde{g}_i^{(P_1,\dots,P_k)}(Q_1, \dots, Q_m) \\ &\quad = \frac{1}{m!} \sum_{1 \leqslant i_1 < \dots < i_q \leqslant m} (-1)^{m-q} \tilde{g}_i^{(P_1,\dots,P_k)}(Q_{i_1} + \dots + Q_{i_q}, \dots, Q_{i_1} + \dots + Q_{i_q}) \\ &\quad = \frac{1}{m!} \sum_{1 \leqslant i_1 < \dots < i_q \leqslant m} (-1)^{m-q} \bar{g}_i^{(Q_{i_1}+\dots+Q_{i_q})}(P_1, \dots, P_k). \end{aligned}$$

Since we can expand $\bar{g}_i^{(Q_{i_1}+\dots+Q_{i_q})}(P_1, \dots, P_k)$ by using (7) as well, we see that both functions g_1 and g_2 are fully determined by their images of tuples of polytopes where $Q_1 = \dots = Q_m = Q$ and $P_1 = \dots = P_k = P$. This proves the lemma. $\square$

Another technical tool which is employed in a subsequent proof is the following lemma. This goes back to an idea of Emiris and Canny [10] to use linear programming and formula (4) to compute the mixed volume.

Lemma 7. *Given polytopes $P_1, \dots, P_k \subset \mathbb{R}^k$ and lifting vectors $\mu_1, \dots, \mu_k \in \mathbb{R}^k_{\geqslant 0}$. Denote the vertices of P_i by $v_1^{(i)}, \dots, v_{r_i}^{(i)}$ and choose one edge $e_i = [v_{t_i}^{(i)}, v_{l_i}^{(i)}]$ from each P_i. Then $C := e_1 + \dots + e_k$ is a mixed cell of the mixed subdivision induced by the liftings μ_i if and only if*

i) *The edge matrix $E := V_a - V_b$ is non-singular (where $V_a := (v_{t_1}^{(1)}, \dots, v_{t_k}^{(k)})$ and $V_b := (v_{l_1}^{(1)}, \dots, v_{l_k}^{(k)})$) and*

ii) *For all polytopes P_i and all vertices $v_s^{(i)}$ of P_i which are not in e_i we have:*

$$\big(\langle \mu_1 - \mu_i, \vec{e}_1 \rangle, \dots, \langle \mu_k - \mu_i, \vec{e}_k \rangle\big) \cdot E^{-1} \cdot \big(v_{l_i}^{(i)} - v_s^{(i)}\big) \geqslant 0 \tag{8}$$

where $\vec{e}_i = v_{t_i}^{(i)} - v_{l_i}^{(i)}$.

Before beginning with the proof we start with some auxiliary considerations about how to apply linear programming here. In [10] it is shown that the test, if a cell lies on the lower envelope of the lifted Minkowski sum can be formulated as a linear program. Let $\hat{m}_i \in \mathbb{R}^{k+1}$ denote the midpoint of the lifted edge $\hat{e}_i$ of $\hat{P}_i$ such that $\hat{m} = \hat{m}_1 + \dots + \hat{m}_k$ is an interior point of the Minkowski sum $\hat{e}_1 + \dots + \hat{e}_k$. Consider the linear program

$$\begin{aligned} &\text{maximize } s \in \mathbb{R}_{\geqslant 0} \\ &\quad \text{s.t. } \hat{m} - (0, \dots, 0, s) \in \hat{P}_1 + \dots + \hat{P}_k. \end{aligned} \tag{9}$$

If we denote the vertices of P_i by $v_1^{(i)}, \dots, v_{r_i}^{(i)}$ this can be written as

$$\begin{aligned} &\text{maximize } s \in \mathbb{R}_{\geqslant 0} \\ &\quad \text{s.t. } \hat{m} - (0, \dots, 0, s) = \sum_{i=1}^{k} \sum_{j=1}^{r_i} \lambda_j^{(i)} \hat{v}_j^{(i)} \\ &\quad \sum_{j=1}^{r_i} \lambda_j^{(i)} = 1 \quad \forall i = 1, \dots, n \\ &\quad \lambda_j^{(i)} \geqslant 0 \quad \forall i, j. \end{aligned} \tag{10}$$

s measures the vertical distance of $\hat{m}$ to the lower envelope of the Minkowski sum. Hence $\hat{m}$ lies on the lower envelope of $\hat{P}_1 + \dots + \hat{P}_k$ if and only if the optimal value of (9) is zero.

Setting $x^T = (\lambda_1^{(1)}, \dots, \lambda_{r_1}^{(1)}, \dots, \lambda_1^{(k)}, \dots, \lambda_{r_k}^{(k)}, s) \in \mathbb{R}^{r_1+\dots+r_k+1}$, the linear program (10) can be written in standard matrix form $\max\{c^T x\colon\ Ax = b, x \geqslant 0\}$ with

$$A = \begin{pmatrix} v_1^{(1)} & \dots & v_{r_1}^{(1)} & \dots & \dots & v_1^{(k)} & \dots & v_{r_k}^{(k)} & \mathbf{0}_k \\ \langle \mu_1, v_1^{(1)} \rangle & \dots & \langle \mu_1, v_{r_1}^{(1)} \rangle & \dots & \dots & \langle \mu_k, v_1^{(k)} \rangle & \dots & \langle \mu_k, v_{r_k}^{(k)} \rangle & 1 \\ & \mathbf{1}_{r_1}^T & & \mathbf{0}_{r_2}^T & \dots & & \mathbf{0}_{r_k}^T & & 0 \\ & \mathbf{0}_{r_1}^T & & \mathbf{1}_{r_2}^T & \dots & & \mathbf{0}_{r_k}^T & & 0 \\ & \vdots & & & \ddots & & \vdots & & \vdots \\ & \mathbf{0}_{r_1}^T & & \mathbf{0}_{r_2}^T & \dots & & \mathbf{1}_{r_k}^T & & 0 \end{pmatrix},$$

$$b^T = (\hat{m}, \mathbf{1}_k^T) \in \mathbb{R}^{2k+1},$$
$$c^T = (\mathbf{0}^T_{r_1+\cdots+r_k}, 1) \in \mathbb{R}^{r_1+\cdots+r_k+1}.$$

Here $\mathbf{0}_k$ and $\mathbf{1}_k$ denote the all-0-vector and the all-1-vector in $\mathbb{R}^k$, respectively. In this notation the point $\hat{m}$ from (9) corresponds to $\bar{x} = (\lambda_1^{(1)}, \ldots, \lambda_{r_k}^{(k)}, s)$ where $s = 0$ and $\lambda_j^{(i)} = \frac{1}{2}$ if the edge $\hat{e}_i$ contains the vertex $\hat{v}_j^{(i)}$ and $\lambda_j^{(i)} = 0$ otherwise.

Assume a feasible vertex $\bar{x} \geqslant 0$ of the linear program (10) is given. For a subset $S \subset \{1, \ldots, r_1 + \cdots + r_k + 1\}$ let A_S be the submatrix of A that consists of the columns with indices in S. If v is a vector, then v_S is understood as the vector where all entries with indices which are not in S are deleted. Now let B be a (not necessarily unique) choice of $2k + 1$ indices such that $A_B^{-1} \cdot b = \bar{x}_B$ and denote by N those indices which are not in B. By linear programming duality (see, e.g. [15]) $\bar{x}$ is optimal if and only if

$$c_N^T - c_B^T \cdot A_B^{-1} \cdot A_N \leqslant 0, \tag{11}$$

where the inequality is understood componentwise, i.e. each component of the vector on the left hand side is non-positive.

To prove Lemma 7 we assume that $\bar{x}$ is optimal and deduce conditions on the lifting vectors μ_i by using inequality (11).

Proof of Lemma 7. Note that C is full-dimensional if and only if E is non-singular. In the following only this full-dimensional case will be considered. To simplify the notation write $\mu(V)$ to denote $(\langle \mu_1, v_1 \rangle, \ldots, \langle \mu_k, v_k \rangle)$.

We know that C is a mixed cell if and only if the following $\bar{x}$ is the optimal solution to the linear program defined above:

$$\bar{x} = (\lambda_1^{(1)}, \ldots, \lambda_{r_k}^{(k)}, 0) \quad \text{where } \lambda_j^{(i)} = \begin{cases} \frac{1}{2}, & j \in \{t_i, l_i\}, \\ 0, & \text{else}. \end{cases}$$

The submatrices of A corresponding to $\bar{x}$ are

$$A_B = \begin{pmatrix} V_a & V_b & \mathbf{0}_k \\ \mu(V_a) & \mu(V_b) & 1 \\ \mathrm{Id}_k & \mathrm{Id}_k & \mathbf{0}_k \end{pmatrix} \quad \text{and} \quad A_N = \begin{pmatrix} v_s^{(i)} \\ \langle \mu_r, v_s^{(i)} \rangle \\ \xi_i \end{pmatrix}_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant s \leqslant r_i \\ s \neq t_i, l_i}}$$

where ξ_i denotes the ith unit vector. Since

$$A_B^{-1} = \begin{pmatrix} E^{-1} & \mathbf{0}_k & -E^{-1} \cdot V_b \\ -E^{-1} & \mathbf{0}_k & E^{-1} \cdot V_a \\ -\mu(E) \cdot E^{-1} & 1 & \mu(E) \cdot E^{-1} \cdot V_b - \mu(V_b) \end{pmatrix}$$

and $c_N = (0, \ldots, 0)$ the criterion (11) implies that $\bar{x}$ is optimal if and only if

$$(0, \ldots, 0, 1) \cdot A_B^{-1} \cdot A_N \geqslant 0 \quad \text{(componentwise)}.$$

But the ith component of the vector on the left can be explicitly computed as

$$-(\mu(E) \cdot E^{-1}) \cdot v_s^{(i)} + \langle \mu_r, v_s^{(i)} \rangle + (\mu(E) \cdot E^{-1} \cdot V_b - \mu(V_b)) \cdot \xi_i$$

which equals the left hand side of (8) since $\langle \mu_i, v_s^{(i)} \rangle = (\langle \mu_i, \vec{e}_1 \rangle, \ldots, \langle \mu_i, \vec{e}_n \rangle) \cdot E^{-1} \cdot v_s^{(i)}$ and $\mu(V_b) \cdot \xi_i = \langle \mu_i, v_{l_i}^{(i)} \rangle$. □

Note that (8) is linear in the μ_j. Hence, for a given choice of edges this condition defines a cone of lifting vectors which induce a mixed subdivision that contains our chosen cell as a mixed cell.

4. Application of the BKK theory on the graph embedding problem

Our goal is to apply Bernstein's results to give bounds on the number of embeddings of Laman graphs. A first observation shows that for the formulation (1) the Bernstein bound is not tight. Namely, the system (1) allows to choose a direction w that satisfies the conditions of Bernstein's Second Theorem 4. The choice $w = (0, 0, 0, 0, -1, -1, \ldots, -1)$ yields the face system

$$\left\{ \begin{array}{l} \partial_w h_1 = x_1 - c_1 = 0 \\ \partial_w h_2 = y_1 - c_2 = 0 \\ \partial_w h_3 = x_2 - c_3 = 0 \\ \partial_w h_4 = y_2 - c_4 = 0 \\ \partial_w h_k = x_i^2 + y_i^2 = 0 \quad \forall e_k = [v_1, v_i], [v_2, v_i] \in E \\ \partial_w h_k = (x_i - x_j)^2 + (y_i - y_j)^2 = 0 \quad \forall e_k = [v_i, v_j] \in E \text{ with } i, j \neq 1, 2 \end{array} \right\}$$

which has $(x_1, y_1, \ldots, x_n, y_n) = (c_1, c_2, c_3, c_4, 1, i, 1, i, \ldots, 1, i)$ as a solution with non-zero complex entries. So the mixed volume of the system in (1) is a strict upper bound on the number of graph embeddings.

To decrease this degeneracy we apply an idea of Ioannis Emiris[1] (see [9]). Surprisingly the introduction of new variables for common subexpressions, which increases the Bézout bound, can decrease the BKK bound. To the best of our knowledge it is an open problem to characterize in general when substitutions can be applied to remove degeneracies and reduce the mixed volume.

Here we introduce for every $i = 1, \dots, n$ the variable s_i together with the new equation $s_i = x_i^2 + y_i^2$. This leads to the following system of equations.

$$\left\{\begin{array}{l} x_1 - c_1 = 0 \\ y_1 - c_2 = 0 \\ x_2 - c_3 = 0 \\ y_2 - c_4 = 0 \\ s_i + s_j - 2x_i x_j - 2y_i y_j - l_{i,j}^2 = 0 \quad \forall [v_i, v_j] \in E - \{[v_1, v_2]\} \\ s_i - x_i^2 - y_i^2 = 0 \quad \forall i = 1, \dots, n \end{array}\right\}. \tag{12}$$

Experiments show that the system (12) is still not generic in the sense of Theorem 4 for every underlying minimally rigid graph. Hence the upper bound on the number of embeddings given by the mixed volume might not be tight in every case.

4.1. Henneberg I graphs

For this simple class of Laman graphs the mixed volume bound is tight as we will demonstrate below. Our proof exploits the inductive structure of Henneberg I graphs which is why it cannot be used for Henneberg II graphs.

Lemma 8. *For a Henneberg I graph on n vertices, the mixed volume of system* (12) *equals* 2^{n-2}.

Proof. Each Henneberg sequence starts with a triangle for which system (12) has mixed volume 2 which can be computed using the software from [29]. Starting from the triangle we consider a sequence of Henneberg I steps and show that the mixed volume doubles in each of these steps.

In a Henneberg I step we add one vertex v_{n+1} and two edges $[v_r, v_{n+1}]$, $[v_q, v_{n+1}]$ with lengths $l_{r,n+1}$ and $l_{q,n+1}$. So our system of equations (12) gets three new equations, namely

$$s_{n+1} - x_{n+1}^2 - y_{n+1}^2 = 0, \tag{13}$$

$$s_r + s_{n+1} - 2x_r x_{n+1} - 2y_r y_{n+1} - l_{r,n+1}^2 = 0, \tag{14}$$

$$s_q + s_{n+1} - 2x_q x_{n+1} - 2y_q y_{n+1} - l_{q,n+1}^2 = 0. \tag{15}$$

In the new system of equations these three are the only polynomials involving x_{n+1}, y_{n+1} and s_{n+1}, so Lemma 6 can be used to calculate the mixed volume separately. The projections of the Newton polytopes of Eqs. (13), (14) and (15) to the coordinates x_{n+1}, y_{n+1} and s_{n+1} are

$$\operatorname{conv}\{(2\ \ 0\ \ 0)^T, (0\ \ 2\ \ 0)^T, (0\ \ 0\ \ 1)^T\}$$

and twice

$$\operatorname{conv}\{(1\ \ 0\ \ 0)^T, (0\ \ 1\ \ 0)^T, (0\ \ 0\ \ 1)^T, (0\ \ 0\ \ 0)^T\}.$$

The mixed volume of these equals 2. So by Lemma 6 the mixed volume of the new system is twice the mixed volume of the system before the Henneberg I step. □

To get two new embeddings in every Henneberg I step we choose the new edge lengths to be almost equal to each other and much larger then all previous edge lengths (larger then the sum of all previous is certainly enough).

Corollary 9 *(Borcea and Streinu [5]). The number of embeddings of Henneberg I graph frameworks is less than or equal to* 2^{n-2} *and this bound is sharp.*

Of course the elementary proof described in [5] of this statement does not need such heavy machinery as Bernstein's Theorem. The purpose of Lemma 8 is to show that the techniques described in this work apply here and that the BKK bound is tight in this case.

4.2. Laman graphs on 6 vertices

The first Laman graphs which are not constructable using only Henneberg I steps arise on 6 vertices. A simple case analysis shows that up to isomorphisms there are only two such graphs, the Desargues graph and $K_{3,3}$ (see Fig. 3).

[1] Personal communication at EuroCG 2008, Nancy.

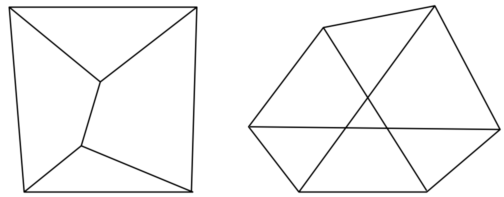

Fig. 3. Left: Desargues graph. Right: $K_{3,3}$.

The number of embeddings of both graphs has been studied in detail. The Desargues graph is studied in [5] where the authors show that there can only be 24 embeddings and that there exists a choice of edge lengths giving 24 different embeddings. This is obtained by investigating the curve that is traced out by one of the vertices after one incident edge is removed. Husty and Walter [19] apply resultants to show that $K_{3,3}$ can have up to 16 embeddings and give as well specific edge lengths leading to 16 different embeddings.

Both approaches rely on the special combinatorial structure of the specific graphs. The general bound in [5] for the number of embeddings of a graph with 6 vertices yields $\binom{2\cdot(6-2)}{6-2} = 70$. In this case the BKK bound gives a closer estimate. Namely the mixed volume of the system (12) (which uses the substitution trick to remove degeneracies) can be shown to be 32 for both graphs. As before we used the PHCpack [29] for the mixed volume calculations.

4.3. General case

For the classes discussed above (Henneberg I, graphs on six vertices) as well as some other special cases, the BKK bound on the number of embeddings resembles or even improves the general bound of $\binom{2n-4}{n-2}$. For the general case, the mixed volume approach for the system (1) without the substitutions suggested by Emiris provides a simple, but very weak bound. However, it may be of independent interest that the mixed volume can be exactly determined as a function of n and that in particular the value is independent of the structure of the Laman graph.

Theorem 10. *For any Laman graph on n vertices, the mixed volume of the initial system* (1) *is exactly* 4^{n-2}.

Proof. The mixed volume of (1) is at most the product of the degrees 2^{2n-4} of the polynomial equations because it is less than or equal to the Bézout bound (see [26]). To show that the mixed volume is at least this number we will use Lemma 7 to give a lifting that induces a mixed cell of volume 4^{n-2}.

For $i \in \{1, \dots, 4\}$ the Newton polytope $\mathrm{NP}(h_i)$ is a segment. We claim that the polynomials h_i can be ordered in a way such that for $i \geqslant 5$, $\mathrm{NP}(h_i)$ contains the edge $[0, 2\xi_i]$ where ξ_i denotes the ith unit vector. To see this, note first that every polynomial h_j $(1 \leqslant j \leqslant 2n)$ has a non-vanishing constant term and therefore $0 \in \mathrm{NP}(h_j)$. For $i \in \{1, \dots, n\}$, each of the monomials x_i^2 and y_i^2 occurs in h_j (for $j \geqslant 5$) if and only if the edge which is modeled by h_j is incident to v_i.

Let $E' := E \setminus \{[v_1, v_2]\}$. The Henneberg construction of a Laman graph allows to orient the edges such that in the graph (V, E') each vertex in $V \setminus \{v_1, v_2\}$ has exactly two incoming edges (see [1,21]). Namely, in a Henneberg I step the two new edges point to the new vertex. For a Henneberg II step we remember the direction of the deleted edge $[\overrightarrow{v_r, v_s}]$ and let the new edge, which connects the new vertex to v_s, point to v_s. The other two new edges point to the new vertex. (Fig. 4 depicts this in an example where $v_r = v_3$ and $v_s = v_4$.)

This orientation shows how to order the polynomials $h_5, \dots, h_{2n}$ in such a way that the polynomials h_{2i-1} and h_{2i} model edges which are incoming edges of the vertex v_i within the directed graph. Remembering that the order of the variables was $(x_1, y_1, \dots, x_n, y_n)$ this implies that $2\xi_{2i-1} \in \mathrm{NP}(h_{2i-1})$ and $2\xi_{2i} \in \mathrm{NP}(h_{2i})$ (for $i \geqslant 3$).

Now Lemma 7 can be used to describe a lifting that induces a subdivision that has

$$[\xi_1, 0] + \dots + [\xi_4, 0] + [2\xi_5, 0] + \dots + [2\xi_{2n}, 0] \tag{16}$$

as a mixed cell. In the notation of Lemma 7 the chosen edges give rise to the edge matrix

$$E = \begin{pmatrix} \mathrm{Id}_4 & \mathbf{0} \\ \mathbf{0} & 2\,\mathrm{Id}_{2n-4} \end{pmatrix},$$

where Id_k denotes the $k \times k$ identity matrix. Substituting this into the second condition (8) of Lemma 7 we get that for $i \geqslant 5$ and each Newton polytope $\mathrm{NP}(h_i)$ all vertices $v_s^{(i)}$ of $\mathrm{NP}(h_i)$ which are not 0 or $2\xi_i$ have to satisfy

$$(\mu_{1_1} - \mu_{i_1}, \dots, \mu_{2n_{2n}} - \mu_{i_{2n}}) \cdot v_s^{(i)} \leqslant 0,$$

where we denote by $\mu_j = (\mu_{j_1}, \dots, \mu_{j_{2n}}) \in \mathbb{Q}^{2n}$ the lifting vector for $\mathrm{NP}(h_j)$. Since all the entries of each $v_s^{(i)}$ are non-negative this can easily be done by choosing the vectors μ_j such that their jth entry is sufficiently small and all other entries are sufficiently large. Note that for $i < 5$ the Newton polytope $\mathrm{NP}(h_i)$ is an edge and therefore is part of any full-dimensional cell.

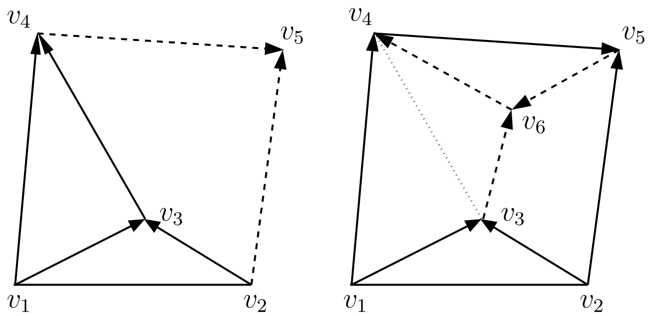

Fig. 4. A Henneberg I and a Henneberg II step with directed edges.

Since the cell (16) has volume $2^{2n-4} = 4^{n-2}$, this proves the theorem. $\square$

As stated in the preliminary remarks at the beginning of this section the Bernstein bound is never tight for the system (1) and hence we have:

Corollary 11. *The number of embeddings of a Laman graph framework with generic edge lengths is strictly less then* 4^{n-2}.

4.4. Open problems and future prospects

We have presented techniques to study the embedding problem of Laman graph frameworks using the BKK theory. As already mentioned in Section 4 it is an open question whether the Bernstein bounds can be improved by applying suitable transformations (such as substitutions) on the system of equations. Examples like the case study of Laman graph frameworks on 6 vertices in Section 4.2 suggest that the mixed volume of the system (12) gives a significantly better bound on the number of embeddings than the one analyzed in Theorem 10. However it also remains open to compute the mixed volume of the system (12) as a function of n like it was done for the system (1) in Theorem 10.

The focus of our paper was on embeddings in the plane. Borcea and Streinu [5] as well as Emiris and Varvitsiotis [11] gave also bounds for embeddings into 3-dimensional and general n-dimensional spaces. Since for these 3- and n-dimensional problems the resulting polynomial equations are sparse as well, the BKK techniques are also applicable. With regard to the Bernstein bounds there are straightforward analogs of Lemma 8 and Theorem 10 to higher dimensions.

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