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Finite groups of $G_2(3)$-type

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Dedicated to J.G. Thompson on his 70th birthday

A finite group G is said to be of $G_2(3)$-*type* if G has subgroups H and M such that

(G1) H has normal subgroups H_1 and H_2 with $H_1 \cong H_2 \cong \mathrm{SL}_2(3)$, $|H : H_1H_2| = 2$, $\mathbf{Z}_2 \cong H_1 \cap H_2$, and $H = C_G(H_1 \cap H_2)$; and
(G2) $H_1 \cap H_2 \leqslant V \trianglelefteq M$ with $C_M(V) = V \cong E_8$ and $M/V \cong L_3(2)$.

Our main theorem is:

Main Theorem. *If G is of $G_2(3)$-type then $G \cong G_2(3)$.*

See [1] for the definition of basic notation and terminology. The group $G_2(3)$ is the Chevalley group of type G_2 over the field of order 3.

In the proof of the classification of the finite simple groups, the group $G_2(3)$ arises as a quasithin group of characteristic 2. This class of groups is treated in [3], where $G_2(3)$ is identified using the Main Theorem. Our definition of "$G_2(3)$-type" is chosen to provide a characterization of $G_2(3)$ convenient for the purposes of [3]. The important condition is (G1), which gives the general structure of the centralizer of an involution, but some extra condition such as (G2) is necessary to rule out examples which are not simple. Two other such conditions are:

(G2′) $H_1 \cap H_2$ is not weakly closed in H with respect to G.
(G2″) G has no subgroup of index 2.

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In Section 6 we sketch a proof that in a group G satisfying (G1), hypotheses (G2), (G2′), and (G2″) are equivalent.

There are existing characterizations of $G_2(3)$ in the literature which we will discuss in a moment. Our purpose here is to obtain a much shorter and simpler treatment for purposes of the classification, using modern methods which are more conceptual, avoid character theory, and minimize detailed computation. In the existing treatments, as in ours, the proof divides into two cases:

Case I: H is not strongly 3-embedded in G. Case II: H is strongly 3-embedded in G.

Thompson established the first characterization of $G_2(3)$ in terms of local information in the N-group paper [10]. His hypotheses involve restrictions on both 2-locals and 3-locals, and implicitly exclude Case II. The first characterization of $G_2(3)$ via the centralizer of an involution is due to Janko in [9]; he essentially assumes Hypotheses (G1) and (G2″). In [8] and [7], Fong and Wong characterize groups with more general, but related centralizers; in the special case of $G_2(3)$ they appeal to Janko's paper to handle Case II. On the other hand Janko appeals to Thompson's work to handle Case I. Janko shows Case II leads to a contradiction using exceptional character theory. Both Fong–Wong and Thompson identify G as $G_2(3)$ in Case I by constructing a BN-pair for G.

We identify G in Case I: first by constructing a pair of 3-locals resembling the maximal parabolics in $G_2(3)$; then by appealing to work of Delgado and Stellmacher in [6] to conclude the amalgam determined by the 3-locals is unique up to isomorphism; and finally by an appeal to Corollary F.4.21 in [3] to identify G. In Case II we calculate the order of G by counting involutions, using an approach of Bender in [4]. This leads to an immediate contradiction via Sylow's Theorem.

1. A preliminary lemma

1.1. *Let G be a group such that $G = QL$ where $Q = O_3(G) \cong 3^{1+2}$ and $L \cong \mathrm{GL}_2(3)$ acts faithfully on $Q/Z(Q)$. Let $P \in \mathrm{Syl}_3(G)$. Then*

(1) $P \cong \mathbf{Z}_3 \text{ wr } \mathbf{Z}_3$.

(2) $J(P) \cong E_{27}$.

(3) *$J(P)$ is inverted by an involution in $L - Z(L)$.*

(4) $P \cap L \leqslant J(P)$.

Proof. First $P = XQ$, where $X = P \cap L$ is of order 3 and $N_L(X) = XF$, where $F = \langle t, z\rangle$, $Z(L) = \langle z\rangle$, and t is an involution inverting X. Let $Z = Z(Q)$; as L acts naturally on Q/Z, $C_{Q/Z}(X) = E/Z$ is of order 3. Now $E = [E, z] \times C_E(z)$ with $Z = C_E(z)$, so as X centralizes z, X centralizes $[E, z]$ and Z. Therefore $A = EX \cong E_{27}$. Further t inverts Z and replacing t by tz if necessary, we may assume t inverts $[E, z]$, so t inverts A. As $|P : A| = 3$, $A \trianglelefteq P$. Let $y \in Q - E$;

then y is of order 3 and as $[X, Q/Z] = E/Z$ and $[E, y] = Z$, y acts on A with one Jordon block of size 3. We conclude that $A = J(P)$ and the lemma holds.

2. 2-local structure

In this section we assume G is of $G_2(3)$-type and let $Z = H_1 \cap H_2$, z a generator of Z, $U = O_2(H)$, $\widetilde{H} = H/Z$, and $H^* = H/U$. Let $\overline{M} = M/V$.

2.1. (1) *V is the natural module for $\overline{M} \cong L_3(2)$.*
(2) *$H = C_G(z)$ and $M = N_G(V)$.*
(3) *$H \cap M = C_M(z) = N_H(V)$ is of index 3 in H and index 7 in M.*
(4) *$[H_1, H_2] = 1$, so $O^2(H) \cong \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$.*
(5) *$U = F^*(H) = O_2(H_1)O_2(H_2) \cong Q_8^2$.*
(6) *$U = O_2(H \cap M)$ and $(H \cap M)^* \cong S_3$.*
(7) *A Sylow 2-subgroup T of $H \cap M$ is Sylow in G.*
(8) *Let $X_H \in \mathrm{Syl}_3(H)$. Then $N_{\widetilde{H}}(\widetilde{X}_H) = \widetilde{X}_H\langle\tilde{t}\rangle$, where $\tilde{t}$ is an involution inverting $\widetilde{X}_H$ and $T = U\langle t\rangle$.*

Proof. By (G2), $E_8 \cong V = C_M(V)$. Thus $M/V \leqslant \mathrm{GL}(V)$, so as $M/V \cong \mathrm{GL}(V)$, (1) holds.

By (G1), $H \cap M = C_M(z)$. Then by (1), $|M : H \cap M| = 7$ and $|H \cap M| = 2^6 \cdot 3$. By (G1):

$$|H| = 2|H_1H_2| = 2|H_1||H_2|/|H_1 \cap H_2| = |H_1|H_2| = 2^6 \cdot 3^2,$$

so $|H : H \cap M| = 3$. Thus a Sylow 2-subgroup T of $H \cap M$ is Sylow in H and M, so $U = O_2(H) \leqslant O_2(H \cap M)$. By (G1), $|U| \geqslant O_2(H_1)O_2(H_2) = 2^5$, while by (1) and the action of $\mathrm{GL}(V)$ on V, $|O_2(C_M(z))| = 2^5$ and $C_M(z)/O_2(C_M(z)) \cong S_3$. Thus $U = O_2(H_1)O_2(H_2)$ and (6) holds.

Next as $H_i \trianglelefteq H$, $[H_1, H_2] \leqslant H_1 \cap H_2 = Z \leqslant Z(H)$, so as $H_i = O^2(H_i)$ and Z is of order 2, (4) holds. By (4), $F^*(H) = U \cong Q_8^2$, completing the proof of (5). As $F^*(H) = U$, $Z(T) \leqslant Z(U)$, so $Z(T) = Z$ by (5). Thus $T \in \mathrm{Syl}_2(G)$ as $H = C_G(z)$ and $T \in \mathrm{Syl}_2(H)$, so (7) holds.

As $C_H(E)$ is a 2-group for each elementary abelian subgroup E of U properly containing Z, $C_G(V)$ is a 2-group. But $V = C_M(V)$, so by (7), V is Sylow in $C_G(V)$ and hence $V = C_G(V)$. Then as $\mathrm{Aut}_M(V) = \mathrm{GL}(V)$, $M = N_G(V)$, completing the proof of (2) and (3).

Let $X \in \mathrm{Syl}_3(H \cap M)$ and $X \leqslant X_H \in \mathrm{Syl}_3(H)$. Then $X \cong \mathbf{Z}_3$ and $X_H \cong E_9$ is Sylow in H_1H_2. From the structure of M, $XZ = C_M(X)$ and X is inverted by some $t \in T$. Thus $Z = C_U(X)$, so $C_{\widetilde{U}}(\widetilde{X}) = 1$ and hence $\widetilde{X}$ is diagonally embedded in $\widetilde{H}_1 \times \widetilde{H}_2$ and $\widetilde{X}_H = C_{\widetilde{H}_1\widetilde{H}_2}(\widetilde{X})$. Thus $\widetilde{X}_H$ is $\tilde{t}$-invariant and then $N_{\widetilde{H}}(\widetilde{X}_H) = \widetilde{X}_H\langle\tilde{t}\rangle$. As $H_i \trianglelefteq H$ and $\tilde{t}$ inverts $\widetilde{X}$, $\tilde{t}$ inverts $\widetilde{X}_H$, establishing (8).

2.2. (1) *For each $g \in G - M$ with $Z \leqslant V^g$, $U = VV^g$ and $V \cap V^g = Z$.*
(2) $|V \cap V^g| \leqslant 2$ *for all* $V^g \neq V$.

Proof. By 2.1.6, $V \leqslant O_2(H \cap M) = U$. As V is the natural module for M/V, M is transitive on $V^{\#}$, so $H = C_G(z)$ is transitive on $\{V^g\colon z \in V^g\}$. (Cf. A.1.7.1 in [3].)

Suppose $g \in G - M$ and $Z \leqslant V^g$. Then by the previous paragraph, $V^g \in V^H$, so $V^g \leqslant U$, and as $O^2(H \cap M) \trianglelefteq H$, $O^2(Hm)$ acts on V^g. Then as $O^2(H \cap M)$ is irreducible on $\overline{U}$ and $\widetilde{V}$, $U = VV^g$ and $V \cap V^g = Z$, establishing (1). As M is transitive on $V^{\#}$, (1) implies (2).

As V is the natural module for $\overline{M}$, there is a unique T-invariant 4-subgroup V_2 of V. Let $I_2 = N_M(V_2)$.

Identify Z with $\mathbf{F}_2$. As U is extraspecial, H preserves the bilinear form $(\,,)$ on $\widetilde{U}$ and the associated quadratic form q defined by $(\tilde{u}, \tilde{v}) = [u, v]$ and $q(\tilde{u}) = u^2$; cf. 23.10 in [1]. Thus $H^* \leqslant O(\widetilde{U}, q)$. We use this fact throughout the paper, usually without further comment.

2.3. (1) *H is transitive on the* 18 *involutions in $U - Z$ and the* 12 *elements of order* 4 *in U.*
(2) *If i is an involution in $U - V_2$ then $C_T(i) = C_U(i) \cong \mathbf{Z}_2 \times D_8$.*
(3) *M has two orbits on its involutions: $V^{\#}$ and the involutions in $M - V$. For i an involution with $\langle \bar{i} \rangle = Z(\overline{T})$, $C_M(i) = C_U(i) \cong \mathbf{Z}_2 \times D_8$.*
(4) *$\widetilde{H}$ is transitive on involutions in $\widetilde{H} - \widetilde{U}$; each such element lifts to an involution.*
(5) *H is transitive on involutions in $H - U$. For j an involution in $T - U$, $C_H(j) = \langle j \rangle V_2 \cong E_8$.*

Proof. By 2.1.8, $\widetilde{T} = \widetilde{U}\langle \tilde{t} \rangle$, where $\tilde{t}$ is an involution inverting $\widetilde{X}_H \in \mathrm{Syl}_3(\widetilde{H})$. It follows that $C_{\widetilde{U}}(T) = \langle \tilde{u}_1, \tilde{u}_2 \rangle$, where $u_i \in U \cap H_i$ for $i = 1, 2$. As $U \cap H_i \cong Q_8$ and $[H_1, H_2] = 1$, $F = \langle u_1, u_2 \rangle \cong \mathbf{Z}_2 \times \mathbf{Z}_4$ and $\langle \tilde{u}_1 \tilde{u}_2 \rangle$ is the unique singular point in $\widetilde{F}$. As $\widetilde{V}_2$ is a T-invariant singular point, it follows that $V_2 = \langle u_1 u_2, z \rangle = \Omega_1(F)$.

Next there are involutions in $U - V$, and each such involution is fused into $T - U$ under M. Thus there is an involution j in $T - U$. For each such involution, $\tilde{j}\tilde{u}_1 \in \tilde{j}^U$, so ju_1 is an involution and hence j inverts u_1. Thus $C_U(j) = V_2$, so as j^* is selfcentralizing in H^* by 2.1.8, $C_H(j) = \langle j \rangle V_2 \cong E_8$.

As $|H^*|_2 = 2$, H^* is transitive on its involutions, and then as $C_{\widetilde{U}}(j) = [\widetilde{U}, j]$, $\widetilde{H}$ is transitive on its involutions by Exercise 2.8 in [2]. Thus (4) holds. As $|U : C_U(j)| = 8 = |F|$, U is transitive on jF, so H is transitive on involutions in $H - U$, completing the proof of (5).

Part (1) is a standard fact about the orthogonal space $\widetilde{U}$, as is the fact that $C_U(i) \cong \mathbf{Z}_2 \times D_8$ for i an involution in $U - Z$. By paragraph one, $C_{\widetilde{T}}(\tilde{i}) = \widetilde{U}$ if $i \notin V_2$, so (2) holds.

As V is the natural module for $\overline{M}$, M is transitive on $V^{\#}$. For each involution $x \in M - V$, $\bar{x}$ is fused to a generator of $Z(\overline{T})$. Further if $\bar{x} \in Z(\overline{T})$ then $\overline{T} = C_{\overline{H}}(\bar{x})$, so $C_M(x) \leqslant T$. But by (2), $C_T(x) = C_U(x) \cong \mathbf{Z}_2 \times D_8$, so (3) holds.

2.4. *G has one class of involutions.*

Proof. By 2.3.3, each involution in M is fused into U. Also z is fused into $V - Z \subseteq U$ in M. Then the lemma follows from 2.3.1.

2.5. (1) $I_2/V_2 \cong \mathbf{Z}_2 \times S_4$.
(2) $O_2(O^2(I_2)) \cong \mathbf{Z}_4^2$.
(3) *V is the unique normal E_8-subgroup of I_2.*

Proof. First $\bar{I}_2 \cong S_4$ and there are involutions in $T - O_2(I_2)$, so either $I_2/V_2 \cong \mathbf{Z}_2 \times S_4$ or $O^2(I_2/V_2) \cong \mathrm{SL}_2(3)$. But $\Phi(U/V_2) = 1$, so $(U \cap O_2(I_2))/V_2 - V/V_2$ contains involutions, and hence (1) holds. Let $R = O_2(O^2(I_2))$. As I_2 is transitive on $(R/V_2)^{\#}$, either $V_2 = \Omega_1(R)$ or $R \cong E_{16}$. But by 2.3, $m_2(T) = 3$, so $V_2 = \Omega_1(R)$. Next $U \cap R \cong \mathbf{Z}_4 \times \mathbf{Z}_2$ and for $u \in U \cap R - V_2$ and $v \in V - V_2$, $[u, v] = z$ generates $\Phi(U \cap R)$ as $\Phi(U) = \langle z \rangle$. Therefore v inverts $U \cap R$, so as $C_{I_2}(v)$ is irreducible on R/V_2, v inverts R. Therefore (2) and (3) hold.

3. 3-local structure

In this section we continue to assume G is of $G_2(3)$-type and continue the notation from the previous section. In addition let $X_H \in \mathrm{Syl}_3(H)$, $X_i = X_H \cap H_i$ for $i = 1, 2$, and X_3 and X_4 the remaining subgroups of X_H of order 3. Let $Q_i = O(N_G(X_i))$.

3.1. (1) *$N_H(X_H) = X_H\langle t, z\rangle$, where t is an involution inverting X_H.*
(2) *For $i = 1, 2$, $N_H(X_i) = K_iX_i$, where $K_i = H_{3-i}\langle t\rangle \cong \mathrm{GL}_2(3)$.*
(3) *For $k = 3, 4$, $N_H(X_k) = N_H(X_H)$.*
(4) *For $i = 1, 2$, $N_G(X_i) = Q_iK_i$.*
(5) *For each j, $1 \leqslant j \leqslant 4$, $z^G \cap C_G(X_j) = z^{C_G(X_j)}$.*
(6) *For $r \neq s$, $X_s \notin X_r^G$.*

Proof. By 2.1.8, $N_H(X_H) = X_H\langle t, z\rangle$, where $\tilde{t}$ is an involution inverting $\widetilde{X}_H$, and by 2.3.4, t is an involution. Thus (1) holds. Similarly for $k = 3, 4$, $N_{H^*}(X_k^*) = N_H(X_H)^*$ and $C_{\widetilde{U}}(X_k) = 1$, so (3) holds. On the other hand for $i = 1, 2$, $X_iH_{3-i} = N_{H_1H_2}(X_i)$, so as t inverts X_H, (2) holds.

By (2) and (3), $C_H(X_j)$ has a Sylow 2-subgroup T_j isomorphic to Q_8 or $\mathbf{Z}_2$, so Z char T_j and hence $T_j \in \mathrm{Syl}_2(C_G(X_j))$. Then (4) holds by Brauer–Suzuki [5]. Also Z is weakly closed in T_j, so (5) holds and then as $X_s \notin X_r^H$, (5) implies (6).

3.2. *For $i = 1, 2$:*

(1) *z inverts Q_i/X_i.*

(2) *$Q_i = X_i C_{Q_i}(t) C_{Q_i}(tz)$, with $C_{Q_i}(tz) = C_{Q_i}(t)^h$ for $h \in K_i$ with $t^h = tz$.*

(3) *$\Phi(Q_i) \leqslant X_i$ and Q_i is of exponent* 3.

(4) *$|Q_i| = 3$, 3^3, or 3^5, and $|N_G(X_i)|_3 = 3^2$, 3^4, 3^6, respectively.*

Proof. By 3.1.2, $X_i = O(N_H(X_i))$, so (1) holds. By (1), Q_i/X_i is abelian, so by Exercise 8.1 in [1], $Q_i = C_{Q_i}(z)C_{Q_i}(t)C_{Q_i}(tz)$, and hence (2) holds. Next by 2.4, $t \in z^G$, so $C_G(t)$ is a $\{2,3\}$-group and hence $C_{Q_i}(t)$ is contained in a Sylow 3-group of $C_G(t)$, which is isomorphic to E_9. Thus $C_{Q_i}(t)$ is of exponent 3 and order at most 9, so by (2), Q_i/X_I is an elementary abelian 3-group of order 1, 3^2, or 3^4. Thus (4) holds, $\Phi(Q_i) \leqslant X_i$, and Q_i is generated by elements of order 3. As $\Phi(Q_i) \leqslant X_i \leqslant Z(Q_i)$, Q_i is of class at most 2, so as $Q_i = \Omega_1(Q_i)$, Q_i is of exponent 3 by 23.11 in [1]. Thus (3) holds.

3.3. (1) *For $k = 3, 4$, $N_G(X_k) = O_3(N_G(X_j))\langle t, z\rangle$ with $|O_3(N_G(X_k))| \leqslant 3^6$.*

(2) *$|N_G(X_j)|_3 \leqslant 3^6$ for all j, $1 \leqslant j \leqslant 4$.*

Proof. Let $k = 3$ or 4, $I = N_G(X_k)$, and $Y = O(I)$. By 3.1.3 and Thompson transfer, $I = Y\langle t, z\rangle$. If p is a prime divisor of $|Y|$ then by 18.7 in [1] there is a $\langle t, z\rangle$-invariant Sylow p-subgroup P of Y, and by Exercise 8.1 in [1], $Y = \langle C_Y(z), C_Y(t), C_Y(tz)\rangle$. Therefore Y is a 3-group by 2.4. Then using Exercise 8.1 in [1] and inducting on the order of Y, $Y = C_Y(z)C_Y(t)C_Y(tz)$, with $|C_Y(i)| \leqslant 9$ for $i \in \langle t, z\rangle^\#$. Thus (1) holds, and (1) and 3.2.4 imply (2).

In the remainder of this section we assume $Q_i \neq X_i$ for $i = 1$ or 2, and set $X = X_i$, $Q = Q_i$, $I = N_G(X)$, $K = K_i$, and $P_i = X_H Q$. Thus $P_i \in \mathrm{Syl}_3(N_G(X_i))$ and $|P_i| = 3|Q|$. Changing notation if necessary, we may take $i = 1$.

3.4. *Q is not isomorphic to* 3^{1+2}.

Proof. Assume $Q \cong 3^{1+2}$ and let $P = P_1$. By 3.1, $I = KQ$ with $K \cong \mathrm{GL}_2(3)$ and by 3.2, z inverts Q/X. Thus $P \cong \mathbf{Z}_3$ wr $\mathbf{Z}_3$ and $X_H \leqslant A = J(P) \cong E_{27}$ by 1.1. Further $X = Z(P)$, so $P \in \mathrm{Syl}_3(G)$. As $X_2 \leqslant X_H \leqslant A$, $|N_G(X_2)|_3 \geqslant 3^3$, so $|N_G(X_2)|_3 \geqslant 3^4$ by 3.2.4. Thus X_2 is in the center of some Sylow 3-subgroup of G, impossible as $X = Z(P)$ and $X_2 \notin X^G$ by 3.1.6.

3.5. $|Q| = 3^5$.

Proof. Assume otherwise; then by 3.2.4, $|Q| = 3^3$. By 3.2.1, z inverts $Q/X \cong E_9$ and by 3.2.2, Q is of exponent 3 with $\Phi(Q) \leqslant X$. Thus by 3.4, $Q \cong E_{27}$, so $Q = X \times E$, where $E = [Q, z] \cong E_9$ and K acts faithfully as $\mathrm{GL}(E)$ on E. Thus $P_1 = X_2E \times X \cong 3^{1+2} \times \mathbf{Z}_3$, so $D = C_E(P_1)$ is of order 3, and we may choose notation so that $D = C_E(t)$. Hence D is fused to X_j for some $1 \leqslant j \leqslant 4$.

Suppose X is weakly closed in $Z(P_1)$ with respect to G. Then $P_1 \in \mathrm{Syl}_3(G)$ and $N_G(P_1) = N_I(P_1) = P_1\langle t, z\rangle$. Also $|N_{P_1}(X_2)| = 3^3$, so $|N_G(X_2)|_3 \geqslant 3^4$ by 3.2.4, and hence X_2 is in the center of some Sylow 3-subgroup of G. Thus by symmetry between X_1 and X_2, $D \neq X_2^g \trianglelefteq N_G(P_1)$ for some $g \in G$, so as X and D are the only normal subgroups of order 3, $X = X_2^g$, contrary to 3.1.6.

Therefore X is not weakly closed in $Z(P_1)$, so as $P_1 \in \mathrm{Syl}_3(I)$, $N_G(P_1) \nleqslant I$. Then as $D = \Phi(P_1)$ and $N_G(P_1)$ acts on $Z(P_1) = XD$ with $P_1 = C_I(XD) = C_G(XD)$, $N_G(P_1)/P_1 \cong \mathbf{Z}_2 \times S_3$. Then as tz inverts $Z(P_1)$, $N_G(P_1) = P_1(C_G(tz) \cap N_G(P_1))$ and $C_G(tz) \cap N_G(P_1)$ acts on $Z(P_1)C_{P_1}(tz) = Q$, so $Q \trianglelefteq N_G(P_1)$. Now K has orbits $\{X\}$, D^K, X_0^K of order 1, 4, 8 on the set Δ of points of Q. Thus $|X^{N_G(Q)}| = 13$, 5, or 9. As 5 does not divide $|\mathrm{GL}_3(3)|$, the second case is impossible. As $\mathrm{GL}_3(3)$ has no subgroup of order $13 \cdot |I : Q| = 13 \cdot |\mathrm{GL}_2(3)|$, the first case is out. Thus $X^{N_G(Q)}$ is the set of 9 points in $Q - E$ and $E \trianglelefteq N_G(Q)$. Therefore $\mathrm{Aut}_G(Q)$ is the stabilizer in $\mathrm{SL}(Q)$ of the hyperplane E of Q, so $N_G(Q) = RK$, with $|R| = 3^5$, $P = RX_H \in \mathrm{Syl}_3(N_G(Q))$, and $|P| = 3^6$. As $D \leqslant Z(P)$ and $D \in X_j^G$ for some j, $P \in \mathrm{Syl}_3(G)$ by 3.3.2.

As K is irreducible on R/Q, $R/E \cong 3^{1+2}$ or $R/E = [R/E, z] \times Q/E$. Assume the latter. Then $R_0 = [R, z] = C_{R_0}(t) \times C_{R_0}(tz) \cong E_{81}$. But there is $y \in G$ with $X^y \leqslant C_{R_0}(t)$, so $m_3(I) \geqslant 4$, impossible as $m_3(P_1) = 3$.

Therefore $R/E \cong 3^{1+2}$. By 1.1, $P/E \cong \mathbf{Z}_3$ wr $\mathbf{Z}_3$ and $S/E = J(P/E) \cong E_{27}$ is inverted by $s = t$ or tz. In particular, $Q/E = Z(P/E)$, so $Z(P) = C_Q(P) = D$. Next $R \cap S \cong 3^{1+2} \times \mathbf{Z}_3$ with s inverting $(R \cap S)/E$, so s centralizes $\Phi(R \cap S)$. As $R \cap S \trianglelefteq P$, $\Phi(R \cap S) \leqslant Z(P) = D$, so $D = \Phi(R \cap S)$. Thus as s centralizes $\Phi(R \cap S)$, $s = t$. Therefore $D = C_S(t)$, so t inverts S/D. As usual $S = C_S(z)C_S(tz)C_S(t)$, so $\Phi(S) = D$ and S is of exponent 3.

Let $\widehat{S} = S/D$ and Y of order 3 in $C_R(t) - S$. Then $\widehat{S}$ is a 4-dimensional $\mathbf{F}_3Y$-module, so $m_3(C_{\widehat{S}}(Y)) \geqslant 2$. Therefore as $Q/E = C_{P/E}(Y)$, $C_{\widehat{S}}(Y) = \widehat{Q}$. This is impossible as $[R, \widehat{X}] = \widehat{E}$ and $\widehat{Q} = \widehat{E}\widehat{X}$. Thus the proof of 3.5 is complete.

3.6. $Q_1 \cong Q_2 \cong 3^{1+2} \times E_9$ *and* $|N_G(X_1)|_3 = |N_G(X_2)|_3 = 3^6$.

Proof. By 3.2.3, $\Phi(Q) \leqslant X$ and Q is of exponent 3, while by 3.5, $|Q| = 3^5$. Therefore $Q \cong E_{3^5}$, $3^{1+2} \times E_9$, or 3^{1+4}. Also $C_Q(t) \cong E_9$, so $X_2^g \leqslant X_H^g \leqslant Q$ for some $g \in G$. Then $|C_Q(X_2^g)| \geqslant 3^4$, so $|Q_2| \geqslant 3^3$, and hence $|Q_2| = 3^5$ by 3.5. Thus $|N_G(X_j)|_3 = 3^6$ for $j = 1$ and 2 by 3.2.4.

Assume $Q \cong E_{3^5}$. Then $Q = X \times [Q,z]$ with $C_Q(t) \leqslant [Q,z]$. Thus $X_1^g \leqslant [Q,z]$. As $m(C_Q(X_H)) \leqslant 3$, $Q = J(P_1)$, so $Q = J(P_1^g) = Q^g$. But then $1 = m(C_Q(z)) = m(C_{Q^g}(z)) = 2$, a contradiction.

Therefore we may assume $Q \cong 3^{1+4}$. Then $X = Z(P_1)$, so $P = P_1 \in \mathrm{Syl}_3(G)$. Thus by 3.1.6, $|N_G(X_2)| < 3^6$, contrary to the first paragraph.

Let $G_1 = N_G(X)$, $Y_1 = X$, and $R_1 = Q$. By 3.6, $R_1 \cong 3^{1+2} \times E_9$, so $Z(R_1) = Y_1 \times E_1$, where $E_1 = [Z(R_1), z]$ is the natural module for $L_1 = K$. Let $P = P_1$, $Y_2 = C_{E_1}(P)$, $G_2 = N_G(Y_2)$, and $R_2 = O_3(G_2)$. Observe:

3.7. (1) *$G_1 = R_1L_1$ with $R_1 \cong 3^{1+2} \times E_9$, $L_1 \cong \mathrm{GL}_2(3)$, $Z(R_1) = Y_1 \times E_1$, and E_1 is the natural module for L_1.*

(2) *$F^*(G_1) = R_1$.*

(3) *$P\langle t,z\rangle = N_{G_1}(Y_2) = G_1 \cap G_2$.*

(4) *$Z(P) = Y_1 \times Y_2$.*

3.8. (1) *$N_G(Z(P)) = N_G(P) = P\langle t,z\rangle$.*

(2) *$P \in \mathrm{Syl}_3(G)$.*

Proof. Let $J = N_G(Z(P))$. By 3.7.4, $Z(P) = Y_1Y_2$, so

$$C_G\bigl(Z(P)\bigr) = C_{G_1}\bigl(Z(P)\bigr) = C_{G_1}(Y_2) = P$$

by 3.7.3. As $Z(P) = Y_1Y_2$ we may choose notation so that tz inverts $Z(P)$. Thus as $P = C_G(Z(P))$, by a Frattini argument, $J = PC_J(tz)$ and $P_0 = C_P(tz) \trianglelefteq C_J(tz)$. But $|P_0| = 9$ so $P_0 \in \mathrm{Syl}_3(C_G(tz))$ and hence $P_0\langle t,z\rangle = C_G(tz) \cap N_G(P_0)$ by 2.4 and 3.1.1. Therefore

$$J = PC_J(tz) = PP_0\langle t,z\rangle = P\langle t,z\rangle \leqslant N_G(P),$$

establishing (1). Of course (1) implies (2).

3.9. *$Y_2 \in X_2^G$.*

Proof. By 3.8.2 and 3.6, there is $g \in G$ with $X_2^g \leqslant Z(P)$. By 3.1.6, $X_2^g \neq X$. Now Y_1 and Y_2 are the only $\langle t,z\rangle$-invariant points of $Z(P)$, and hence by 3.8.2 the only points of $Z(P)$ normal in $N_G(P)$. By symmetry between X_1 and X_2, $X_2^g \trianglelefteq N_G(P)$, so $X_2^g = Y_2$.

By 3.9, $Y_2 = X_2^a$ for some $a \in G$. Pick notation so that t centralizes Y_2; thus we may choose a so that $z^a = t$.

3.10. (1) *$R_2 \cong 3^{1+2} \times E_9$ with $Z(R_2) = Y_2 \times E_2$, $E_2 \cong E_9$, and E_2 is the natural module for $L_2 = H_1^a\langle z\rangle \cong \mathrm{GL}_2(3)$.*

(2) *L_2 is a complement to R_2 in G_2.*

(3) *$F^*(G_2) = R_2$.*

Proof. As $Y_2 = X_2^a$, $Y_2 \leqslant H_2^a$ and $G_2 = N_G(X_2)^a$. Then the various remarks follow by symmetry between X_1 and X_2.

3.11. *Let $G_0 = \langle G_1, G_2\rangle$. Then $O_3(G_0) = 1$.*

Proof. Let $R = O_3(G_0)$. By 3.8, $P \in \mathrm{Syl}_3(G_0)$, so $R \leqslant P$ and hence $R \leqslant P \cap O_3(G_j) = R_j$ for $j = 1$ and 2. Thus $R \leqslant S = R_1 \cap R_2$. But $Z(P) \leqslant S$ and $[P,t] \leqslant R_2$, so $E_{81} \cong Z(P)[R_1,t] \leqslant S$. Indeed $C_{R_2}(t) = Y_2$ while $C_{R_1}(t) \cong E_9$, so $R_1 \neq R_2$, and hence $|S| \leqslant 3^4$. Therefore $S = Z(P)[R_1,t]$.

Suppose $R \neq 1$. Then $1 \neq C_R(P) \leqslant Z(P)$ and $C_R(P)$ is $\langle t, z\rangle$-invariant, so $Y_j \leqslant R$ for $j = 1$ or 2. Thus, interchanging the roles of Y_1 and Y_2 if necessary, we may assume $Y_2 \leqslant R$. Thus $E_1 = \langle Y_2^{G_1}\rangle \leqslant R$.

If $E_1 \leqslant Y_2E_2$ then $Y_1E_1 = Y_2E_2$, so $R_1 = C_P(Y_1E_1) = C_P(Y_2E_2) = R_2$, which we saw is not the case. Thus $E_1 \nleqslant Y_2E_2$. But L_2 is irreducible on $R_2/Z(R_2)$, so $R_2 = RZ(R_2) = RE_2$. However as $Y_2 \leqslant Z(P)$, $E_1 \leqslant Z(R)$, so $R \leqslant C_{R_2}(E_1) = E_1E_2$, contradicting $R_2 = RE_2$.

Theorem 3.12. *If $Q_i \neq X_i$ for $i = 1$ or 2, then $G \cong G_2(3)$.*

Proof. Let $\alpha = (G_1, G_{1,2}, G_2)$, where $G_{1,2} = G_1 \cap G_2$. By 3.7, 3.8, 3.10, and 3.11, α is the amalgam of a weak BN-pair, in the sense of Section 4 of the Green Book [6]. Then as $|R_j| = 3^5$ and $G_j/R_j \cong \mathrm{GL}_2(3)$, it follows from Theorem A in the Green Book that α is isomorphic to the amalgam of $G_2(3)$.

Let $F = \langle t, z\rangle$. Then $F \leqslant F_1 \leqslant L_1$, where $F_1 \cong D_8$. Thus $F_1 = F\langle s_1\rangle$, where s_1 is an involution in $G_1 - G_2$. Similarly there is an involution $s_2 \in G_2 - G_1$ with $F\langle s_2\rangle \cong D_8$. Then $[F, s_1] = z$ and $[F, s_2] = t$, so $\langle s_1, s_2\rangle \leqslant N_G(F)$ with $S/C_S(F) \cong S_3$. Therefore $(s_1s_2)^3 \in C_S(F)$. But by 2.3.5, $C_G(F) \cong E_8$, so $C_S(F)$ is of exponent 2. Thus $|s_1s_2| = 3$ or 6.

As α is the $G_2(3)$-amalgam, as G_0 is a faithful completion of α (cf. Section 36 in [2]), and as $|s_1s_2| \leqslant 6$, it follows from Corollary F.4.21 in [3] that $G_0 \cong G_2(3)$. Therefore G_0 has one class of involutions and $|C_{G_0}(z)| = 2^6 \cdot 3^2 = |H|$, so $C_G(z) = H \leqslant G_0$. Thus $N_G(T) \leqslant N_G(Z(T)) = H \leqslant G_0$, so if $G \neq G_0$ then G_0 is strongly embedded in G. Hence by 7.6 in [2], there is a subgroup D of odd order in G_0 transitive on the involutions of G_0. Therefore $|G_0 : H| = 3^6 \cdot 7 \cdot 13$ divides $|D|$, so D contains a Sylow 3-subgroup of G_0. Thus D is contained in a maximal parabolic subgroup of G_0, whereas the maximal parabolics are $\{2,3\}$-groups. Hence $G = G_0 \cong G_2(3)$.

4. The geometry Γ

In this section we continue to assume G is of $G_2(3)$-type and continue the notation from the previous sections. We generate information about the permutation representation of G on G/M by right multiplication, which will be used in the next section to show that H is not strongly 3-embedded in G.

4.1. *Either*
(1) *H is strongly* 3*-embedded in G, or*
(2) $G \cong G_2(3)$.

Proof. Assume (2) fails. We observe first that $N_G(X_i) \leqslant H$ for $i = 1$ and 2. For if not then by 3.1.4, $Q_i \neq X_i$, contrary to Theorem 3.12 and our assumption that (2) fails.

As $N_G(X_1) \leqslant H$, also $N_G(X_H) \leqslant H$ by 3.1.6. Thus $X_H \in \mathrm{Syl}_3(G)$ and if (1) fails then $N_G(X_j) \nleqslant H$ for $j = 3$ or 4. But by 3.3.1, $N_G(X_j) = O_3(N_G(X_j))\langle t, z\rangle$. However as $X_H \in \mathrm{Syl}_3(G)$, $O_3(N_G(X_j)) \leqslant X_H \leqslant H$, so $N_G(X_j) \leqslant H$, completing the proof.

During the remainder of the section assume H is strongly 3-embedded in G.

4.2. *Let $S_M \in \mathrm{Syl}_7(M)$. Then*
(1) *$C_G(S_M)$ is a $\{2,3\}'$-group.*
(2) $|N_G(S_M) : C_G(S_M)| = 3$.

Proof. By 2.4, G has one class of involutions, so as H is a $7'$-group, $C_G(S_M)$ is of odd order. Similarly as H is strongly 3-embedded in G, $C_G(S_M)$ is a $3'$-group, so (1) holds.

Next $N_M(S_M) = S_M X$, where X is of order 3, and of course $\mathrm{Aut}(S_M) \cong \mathbf{Z}_6$. Thus if (2) fails then S_M is inverted by some involution i, and by (1) and a Frattini argument we may take i to centralize X. But as H is strongly 3-embedded in G, X centralizes a unique involution, so $\langle i\rangle = C_V(X)$, impossible as i inverts S_M and S_M acts on V.

See Section 4 in [2] for a discussion of geometries, (in the sense of Tits) including notation and terminology. Let Γ be the rank 2 geometry with point set V^G, line set Z^G, and incidence equal to inclusion. Thus G is represented as a group of automorphisms of Γ by conjugation, and by 2.1.2, $M = N_G(V)$ and $H = N_G(Z)$ are the stabilizers of V and Z, respectively. By construction, G is transitive on the points and lines of Γ, and from 2.1, M is transitive on the set $\Gamma(V)$ of lines through V, so G is flag transitive on Γ. For $\alpha, \gamma \in \Gamma$, let $d(\alpha, \gamma)$ denote the distance of α from γ in Γ and $\Gamma^i(\gamma)$ the set of vertices at distance i from γ in Γ.

4.3. *Distinct lines are incident with at most one point and distinct points are incident with at most one line.*

Proof. By 2.2.2, $|A \cap B| \leqslant 2$ for distinct points A, B.

4.4. (1) *If $\alpha, \beta \in \Gamma$ with $d(\alpha, \beta) \leqslant 2$ then there is a unique geodesic from α to β.*
(2) *G_α is transitive on $\Gamma^2(\alpha)$.*
(3) *$V^g \in \Gamma^2(V)$ iff $V \cap V^g$ is a line, in which case the global stabilizer in G of $\{V, V^g\}$ is the stabilizer of the edge $(V \cap V^g, V^y)$, where V^y is the third point on $V \cap V^g$.*

Proof. Part (1) follows from 4.3. Part (2) holds as M is 2-transitive on $\Gamma(V)$ and H is 2-transitive on $\Gamma(Z)$. By 4.3, $V^g \in \Gamma^2(V)$ iff $V \cap V^g = Z$ for some line Z. Then by (1), $M \cap M^g$ is the stabilizer $O^2(H \cap M)$ in H of V and V^g. As $x \in H \cap M^y - O^2(H \cap M)$ interchanges V and V^g, (3) holds.

4.5. (1) *If $\alpha, \beta \in \Gamma$ with $d(\alpha, \beta) = 3$ then there is a unique geodesic from α to β.*
(2) *G_α is transitive on $\Gamma^3(\alpha)$ for each $\alpha \in \Gamma$.*
(3) *$\Gamma^3(V) = Z^G \cap (M - V)$.*

Proof. Let p be a geodesic of length 3. Replacing p by its inverse if necessary, and conjugating in G, we may take p to be Z, V, Z^g, V^x. By 2.2.1, $U^g = VV^x$ and $V \cap V^x = Z^g$. Thus as $z \in V$, z acts on V^x but $z \notin V^x$. As $[U^g, V^x] = Z^g$ and $V^x = C_{U^g}(V^x)$, $[V^x, Z] = Z^g$, so Z^g is determined by Z and V^x. Thus (1) follows from 4.4.1, while (2) and (3) follow from 2.3.3 and the fact that $z \in M^x - V^x$.

4.6. (1) *If $d(V, V^g) = 4$ then there is a unique geodesic from V to V^g.*
(2) *M is transitive on $\Gamma^4(V)$.*
(3) *$M \cap M^g = V^y$, where $\{V^y\} = \Gamma^2(V) \cap \Gamma^2(V^g)$.*
(4) *The global stabilizer of $\{V, V^g\}$ is isomorphic to $\mathbf{Z}_2 \times D_8$.*

Proof. Suppose $p = V^x, Z, V, Z^y, V^g$ is a geodesic in Γ. By 2.2.1, $U = VV^x$ with $V \cap V^x = Z$, and similarly $U^y = VV^g$ with $V \cap V^g = Z^y$. Therefore $[V^x, Z^y] = Z$ and $[V^g, Z] = Z^y$, so $I_0 = \langle V^x, V^g \rangle \leqslant N_M(ZZ^y)$ and $E_4 \cong ZZ^y \leqslant V$. Thus we may choose notation so that $ZZ^y = V_2$. Therefore $I_0 \leqslant I_2 = N_M(V_2)$. By 2.5, $I_2/V_2 \cong \mathbf{Z}_2 \times S_4$ with $V/V_2 = Z(I_2/V_2)$ and $O_2(O^2(I_2)) \cong \mathbf{Z}_4^2$, so we conclude $I_0 = I_2$. Again by 2.5, V is the unique normal E_8-subgroup of I_2, so it follows that $\{V\} = \Gamma^2(V^x) \cap \Gamma^2(V^g)$, and then (1) follows from 4.4.1, and (3) from 4.4.3.

To prove (2), given 4.4.2, it suffices to show $N_M(V^x)$ is transitive on $\Gamma^2(V) - \Gamma(Z)$. But by 4.4.3, $N_M(V^x) = O^2(H \cap M)$ and from 2.1.3, $O^2(H \cap M)$ is transitive on $V - Z$ with the stabilizer $C_U(Z^y)$ in $O^2(H \cap M)$ of Z^y satisfying $|C_U(Z^y) : V| = 2$ and $C_M(Z^y) = O^2(C_M(Z^y))C_U(Z^y)$. As $C_M(Z^y)$ is transitive on $\Gamma(Z^y) - \{V\}$ with $O^2(C_M(Z^y))$ the kernel of this action, (2) follows. By (1) and (2), the inverse of p is conjugate to p, so the global stabilizer of $\{V^x, V^g\}$

is $V\langle a\rangle$ where $a \in M - V$ with $a^2 \in V$. As $\overline{M}$ is transitive on its involutions we may choose a to be an involution and then (4) holds.

4.7. (1) *If* $d(Z, Z^g) = 4$ *then there is a unique geodesic from* Z *to* Z^g.

(2) H *has three orbits on* $\Gamma^4(Z)$ *and the corresponding orbitals are all selfpaired.*

(3) H *is transitive on* $\Gamma^4(Z) \cap H$ *and* $H \cap H^g \cong E_8$ *for each* $Z^g \in \Gamma^4(Z) \cap H$.

(4) *If* $z^g \notin H$ *then* $\langle z, z^g\rangle \cong D_8$ *and* $H \cap H^g \cong D_8$.

Proof. Suppose $p = Z, V, Z^y, V^x, Z^g$ is a geodesic. Then $Z \leqslant V \leqslant U^y$, $Z^g \leqslant V^x \leqslant U^y$, and by 2.2.1, $U^y = VV^x$ with $V \cap V^x = Z^y$. Thus $[V, Z^g] = Z^y$. If $[Z, Z^g] = 1$ then $z^g \in H$ but as $[V, Z^g] = Z^y$, $z^g \notin U$. Thus $H \cap H^g = C_G(ZZ^g) \cong E_8$ by 2.3.5, so $H \cap H^g = ZZ^gZ^y$. In particular $U \cap U^g = Z^y$, so Z^y is determined and p is determined by 4.4.1. Hence (1) holds in this case, as does (3) by 2.3.5. By (1) and (3), G is transitive on geodesics of length 4 between commuting lines, so p is conjugate to the inverse of p, and hence the orbital $(Z, Z^g)^G$ is selfpaired, establishing (2) in this case.

So assume $[Z, Z^g] \neq 1$; then $[Z, Z^g] = Z^y$, so Z^y is determined, and hence (1) follows from 4.4.1. Further $S = C_{H^y}(Z)$ is of index 2 in the Sylow 2-group $N_{H^y}(ZZ^y)$ and has two orbits on the involutions in $U^y - C_{U^y}(Z)$, so H has two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ on $\Gamma^4(Z) - H$. Now H^* has 9 involutions, each fixing a unique singular point of $\widetilde{U}$ and each with 4 cycles of length 2 on the remaining singular points. Further there are 36 pairs of distinct singular points and at most one involution interchanges two such points, so each pair of points is a cycle in a unique involution. This shows the orbitals determined by $\mathcal{O}_i$ are selfpaired, completing the proof of (2). Finally $H \cap H^g = C_{U^y}(\langle Z, Z^g\rangle) \cong D_8$, so (4) holds.

4.8. (1) *If* $d(V, V^g) = 6$ *then there is a unique geodesic from* V *to* V^g.

(2) $M \cap M^g = Z^y$, *where* $\{Z^y\} = \Gamma^3(V) \cap \Gamma^3(V^g)$.

(3) M *has three orbits on* $\Gamma^6(V)$.

(4) *The global stabilizer of* $\{V, V^g\}$ *is isomorphic to* E_4.

Proof. We first show that $\Gamma^6(V) \neq \emptyset$. For if not

$$|G : M| = \left|V^G\right| = \left|\Gamma^0(V)\right| + \left|\Gamma^2(V)\right| + \left|\Gamma^4(V)\right|.$$

Now by 4.4–4.6, for each $m \leqslant 4$ and each $\alpha \in \Gamma^m(V)$, there is a unique geodesic from V to α. Thus $|\Gamma^m(V)|$ is the number of geodesic of length m with origin V. Further if $V = \alpha_0, \ldots, \alpha_{m-1}$ is a geodesic then there are $|\Gamma(\alpha_{m-1})| - 1$ choices for α_m, so $|\Gamma^m(V)| = 1, 7, 14, 84, 168$, for $m = 0, 1, 2, 3, 4$, respectively. Thus $|G : M| = 183 = 3 \cdot 61$, so

$$|G| = 2^6 \cdot 3^2 \cdot 7 \cdot 61.$$

Let $P \in \mathrm{Syl}_{61}(G)$. As H is strongly 3-embedded in G and of order prime to 61, $C_G(P)$ is a $\{2, 3\}'$-group, so $C_G(P) = P$ or $C_G(P) = PS$ for some $S \in \mathrm{Syl}_7(G)$.

But in the latter case by 4.2, $PS = C_G(S)$ and $N_G(S)$ is of order $3 \cdot 7 \cdot 61$. Then $|G : N_G(S)| = 2^6 \cdot 3 \equiv 3 \bmod 7$, contrary to Sylow's theorem. Thus $P = C_G(P)$, and $|N_G(P) : P|$ divides the order 60 of $\mathrm{Aut}(P)$ and $|G|$, so $|N_G(P) : P|$ is a divisor of 12. Again this contradicts Sylow's theorem. This establishes the claim that $\Gamma^6(V) \neq \emptyset$.

Thus we may suppose $p = V^x, Z^r, V, Z, V^w, Z^s, V^g$ is a geodesic in Γ. Then $d(V^x, Z) = 3$, so $z \in M^x - V^x$ by 4.5.3. Similarly $z \in M^g - V^g$. Further by 4.5.1, the geodesic is determined by V^x, Z, and V^g. Conversely if a is an involution in $D = M^x \cap M^g$, then setting $A = \langle a \rangle$, $d(V^x, A) \leqslant 3 \geqslant d(V^g, A)$ and then as $d(V^x, V^g) = 6$, these inequalities are equalities. Therefore a is not in V^x or V^g and A determines a unique geodesic from V^x to V^y. Thus the map $\mathcal{G} \mapsto a(\mathcal{G})$ is a bijection of the set $\mathcal{G}$ of geodesics from V^x to V^g with the set $\mathcal{A}$ of involutions in D. So to prove (1) it remains to show that $|\mathcal{A}| = 1$.

First if X is of order 3 in D then $C_{V^x}(X) \neq 1 \neq C_{V^g}(X)$. But as H is strongly 3-embedded in G, X centralizes a unique involution, so $C_{V^x}(X) = C_{V^g}(X)$, contradicting $d(V^x, V^g) > 2$. Thus D is a $\{2, 7\}$-group. However all involutions in a $\{2, 7\}$-subgroup of M^x are in V^x, so D is a 2-group. Therefore if $|\mathcal{A}| \neq 1$ then there exist distinct commuting involutions a and z in $\mathcal{A}$. By the previous paragraph, a acts on V^w. Thus by 4.6.3, $a \in M^x \cap M^w = V$. Similarly $a \in M \cap M^g = V^w$, so $a \in V \cap V^w = Z$, contradicting $a \neq z$. This establishes (1) and (2).

Write $p(V^x, V^g)$ for the unique geodesic p from V^x to V^g, and define $q(V^x, V^g) = Z^r, V, Z, V^w, Z^s$. Thus $q = q(V^x, V^g)$ is the geodesic from Z^r to Z^s. By (1), the map $\theta : p(A, B)^G \mapsto q(A, B)^G$ is a well defined function from the set of orbits of G on geodesics of length 6 whose origin is a point, to the set of orbits of G on geodesics of length 4 whose origin is a line. By 4.7, G_q is of order 8. Now G_q acts on $\Delta = (\Gamma(Z^r) - \{V\}) \times (\Gamma(Z^s) - \{V^w\})$ of order 4, so as $|G_p| = 2$, it follows that G_q is transitive on Δ. This shows that the map θ is a bijection. Therefore 4.7.2 implies (3). By 4.7.2, the orbital $(Z^r, Z^s)^G$ is selfpaired so there is $a \in G$ interchanging Z^r and Z^s. Then a also reverses the order of the pairs in Δ, so as G_q is transitive on Δ, the orbital $(V^x, V^g)^G$ is also selfpaired. Thus the global stabilizer A of $\{V^x, V^g\}$ is of order 4 by (2), so $A \cong \mathbf{Z}_4$ or E_4. Now in the former case $A = \langle a \rangle$ with $a^2 \in G_p = Z$, so by 2.3.4, $a \in U$. But then $V^a = V$, impossible as a maps p to its inverse, so $V^a = V^w \neq V$. Thus (4) is established.

5. Counting involutions

In this section we assume G is of $G_2(3)$-type and H is strongly 3-embedded in G. Under these hypotheses we calculate the order of G, and then use Sylow's theorem to obtain a contradiction. We calculate $|G|$ by counting involutions, using an approach of Helmut Bender in [4].

We continue the notation of previous sections.

For $Mg \in G/M$ let $n(Mg) = |z^G \cap Mg|$ be the number of involutions in Mg, and following Bender, define

$$b_m = \bigl|\bigl\{Mg \in G/M - \{M\}\colon n(Mg) = m\bigr\}\bigr| \quad \text{and} \quad f = \frac{|z^G|}{|G:M|} - 1.$$

As $|z^G| = |G:H|$, it follows that

$$f = \frac{|M|}{|H|} - 1 = \frac{7}{3} - 1 = \frac{4}{3}.$$

5.1. *Let $M \neq Mg \in G/M$; then the following are equivalent*:

(1) $n(Mg) > 0$.

(2) $V^g = V^i$ *for some involution* $i \in G$.

(3) *The global stabilizer* $G(\{V, V^g\})$ *in* G *of* $\{V, V^g\}$ *contains an involution not in* M.

Proof. As $M = N_G(V)$, $Mg = Mx$ iff $V^g = V^x$, so the lemma holds.

5.2. *Let i be an involution not in M. Then*

$$n(Mi) = \bigl|\bigl\{x \in M \cap M^i\colon x^i = x^{-1}\bigr\}\bigr| = \bigl|z^G \cap \bigl(G\bigl(\bigl\{V, V^i\bigr\}\bigr) - M\bigr)\bigr|.$$

Proof. The map $x \mapsto xi$ is a bijection of the set of elements of M inverted by i and $z^G \cap Mi$. Further each such x is in $M \cap M^i$, and $(M \cap M^i)\langle i\rangle = G(\{V, V^i\})$, so the lemma holds.

5.3. *Let $d(V, V^g) = d$. Then*

(1) *If* $d = 2$ *then* $n(Mg) = 24$.

(2) *If* $d = 4$ *then* $n(Mg) = 4$.

(3) *If* $d = 6$ *then* $n(Mg) = 2$.

Proof. First suppose $d = 2$. Then by 4.4.3, up to conjugation in G, $g \in H$, $G(\{V, V^g\}) = H \cap M^h$, where $\{V, V^g, V^h\} = \Gamma(Z)$, and $M \cap M^g = O^2(H \cap M)$. By 2.3.5, H is transitive on involutions in $H - U$ and for each such involution j, $C_H(j) = \langle j\rangle C_U(j) \cong E_8$. Conjugating in H, we may take $j^* \in (H \cap M^h)^*$, while if $k \in H$ with $j^k \in M^h$ then $(j^k)^* \in j^{*(H \cap M^h)}$, so as $C_{H^*}(j^*) = \langle j^*\rangle$, $k \in H \cap M^h$. Thus $H \cap M^h$ is transitive on involutions in $H \cap M^h - O^2(H \cap M)$ and $|j^{H \cap M^h}| = |(H \cap M^h) : C_H(j)| = 24$. Thus (1) follows from 5.2.

Next suppose $d = 4$. Then by 4.6, $M \cap M^g \cong E_8$ and $G(\{V, V^g\}) \cong \mathbf{Z}_2 \times D_8$, so (2) follows from 5.2. Finally if $d = 6$ then by 4.8, $M \cap M^g \cong \mathbf{Z}_2$ and $G(\{V, V^g\}) \cong E_4$, so (3) follows from 5.2.

5.4. $n(Mg) > 1$ *iff* $d(V, V^g) \leqslant 6$.

Proof. If $d(V, V^g) \leqslant 6$ then $n(Mg) > 1$ by 5.3. So we may assume $d(V, V^g) > 6$ but $n(Hg) > 1$ and it remains to derive a contradiction. Let $a \in Mg$ be an involution, $A = \langle a \rangle$, and $Y = M \cap M^g$. By 5.2, a inverts some $y \in Y^\#$. By 4.2, y is not of order 7. If y is of order 3, then as H is strongly 3-embedded in G, $a \in N_G(\langle y \rangle) \leqslant N_G(Z^x)$ for some $Z^x \leqslant V$. But by symmetry, $Z^x \leqslant V^g$, so $d(V, V^g) = 1$, contrary to assumption.

Therefore Y is a 2-group, so we may take y to be an involution. Thus $\langle y \rangle = Z^b$ for some $b \in G$. By 4.5.3, $d(Z^b, V) \leqslant 3 \geqslant d(Z^b, V^g)$, so $d(V, V^g) \leqslant 6$, again a contradiction.

5.5. (1) $b_2 = 2^5 \cdot 3^2 \cdot 7 = 2016$.
(2) $b_4 = 2^3 \cdot 3 \cdot 7 = 168$.
(3) $b_{24} = 14$.
(4) *If* $m > 1$ *and* $m \neq 2, 4,$ *or* 24, *then* $b_m = 0$.

Proof. Let $m > 1$. By 5.4, $b_m \neq 0$ iff $m = n(Mg)$ for some $g \in G$ with $d(V, V^g) \leqslant 6$, in which case $n(Mg) = n(d(V, V^g))$, where $n(d) = 24$, 4, or 2 for $d = 2$, 4, or 6, respectively. Let $\mathcal{M}_d = \{Mg\colon d(V, V^g) = d\}$; it follows that $b_m = 0$ unless $m = n(d)$ for $d = 2$, 4, or 6; further $b_{n(d)} = |\mathcal{M}_d|$. In particular (4) holds.

Next by 4.4, M is transitive on $\mathcal{M}_2$ with

$$b_{24} = b_{n(2)} = |\mathcal{M}_2| = \left|M : M \cap M^g\right| = 14,$$

for $Mg \in \mathcal{M}_2$. This establishes (3) . Similarly by 4.6, M is transitive on $\mathcal{M}_4$ and $b_4 = |\mathcal{M}_4| = |M|/8 = 168$, establishing (2). Finally by 4.8, M has three orbits on $\mathcal{M}_6$, each of length $|M|/2 = 672$, so (1) holds.

5.6. $n(M) = |z^G \cap M| = 7 \cdot 13 = 91$.

Proof. By 2.3.3,

$$\left|z^G \cap M\right| = \left|V^\#\right| + \left|z^G \cap (M - V)\right| = 7 + 84 = 91.$$

5.7. $b_1 = 0$.

Proof. By Lemma 1 in [4],

$$b_1 < \sigma = f^{-1}\left(n(M) + \sum_{i>1}(i-1)b_i\right) - 1 - \sum_{i>1} b_i. \qquad (*)$$

As we observed earlier, $f^{-1} = 3/4$. By 5.5,

$$\begin{aligned}\sum_{i>1}(i-1)b_i &= 2^5\cdot 3^2\cdot 7+3\cdot\bigl(2^3\cdot 3\cdot 7\bigr)+23\cdot 14\\ &= 14\cdot\bigl(2^4\cdot 3^2+2^2\cdot 3^2+23\bigr)=14\cdot(144+36+23)\\ &= 14\cdot 203.\end{aligned}$$

Then by 5.6:

$$\begin{aligned}f^{-1}\Bigl(n(M)+\sum_{i>1}(i-1)b_i\Bigr) &= \frac{3\cdot(7\cdot 13+14\cdot 203)}{4}=\frac{21\cdot(13+406)}{4}\\ &= \frac{21\cdot 419}{4}.\end{aligned}$$

Similarly

$$1+\sum_{i>1}b_i=1+2016+168+14=2199=3\cdot 733,$$

so

$$\sigma=\frac{21\cdot 419}{4}-3\cdot 733=\frac{3\cdot(7\cdot 419-2932)}{4}=\frac{3\cdot(2933-2932)}{4}=\frac{3}{4},$$

and hence $b_1<\sigma<3/4$ by $(*)$, so $b_1=0$.

5.8. $|G|=2^6\cdot 3^2\cdot 7\cdot 733.$

Proof. By Lemma 1 in [4],

$$|G:H|=\bigl|z^G\bigr|=n(M)+\sum_{i\geqslant 1}ib_i,$$

so by 5.5, 5.6, and 5.7,

$$\begin{aligned}|G:H| &= 7\cdot 13+2\cdot\bigl(2^5\cdot 3^2\cdot 7\bigr)+4\cdot\bigl(2^3\cdot 3\cdot 7\bigr)+24\cdot 14\\ &= 7\cdot(13+576+96+48)=7\cdot 733.\end{aligned}$$

Therefore as $|H|=2^6\cdot 3^2$, the lemma holds.

Observe next that 733 is a prime, so by 5.8, a Sylow 733-subgroup P of G is of order 733.

5.9. (1) $P=C_G(P)$.
(2) $|N_G(P):P|$ *divides* 12.

Proof. An argument in the first paragraph of the proof of 4.8 establishes (1). Then as $732=2^2\cdot 3\cdot 61$, (1) and 5.8 imply (2).

We are now in a position to obtain a contradiction to the hypotheses of this section, proving:

Theorem 5.9. *If G is of $G_2(3)$-type then H is not strongly 3-embedded in G.*

Namely by Sylow's theorem, $|G : N_G(P)| \equiv 1 \bmod 733$. But by 5.7 and 5.8, $|G : N_G(P)| = 2^{4+a} \cdot 3^{1+b} \cdot 7$, where $0 \leqslant a \leqslant 2$, and $b = 0$ or 1. However none of these integers is congruent to 1 modulo 733. Therefore Theorem 5.9 is established.

Finally observe that 4.1 and Theorem 5.9 imply the Main Theorem.

6. Some equivalent hypotheses

In this section we assume G is a finite group satisfying Hypothesis (G1), set $Z = H_1 \cap H_2$, let z be a generator for Z, and pick $T \in \mathrm{Syl}_2(H)$. We will sketch a proof that hypotheses (G2), (G2′), and (G2″) are equivalent.

Assume Hypothesis (G2). By 2.1.7 we may choose $T \leqslant M$ and T is Sylow in G. Thus as $M = O^2(M)$ and M contains a Sylow 2-subgroup of G, $G = O^2(G)$, so (G2) implies (G2″).

Next assume (G2′) fails; that is z is weakly closed in H with respect to G. Then T is Sylow in G and (cf. 7.7.1 in [2]) H controls fusion of its 2-elements, so by a standard transfer result (cf. 37.4 in [1]) $O^2(G) \cap H = O^2(H)$. Thus as $H \neq O^2(H)$, $G \neq O^2(G)$. Thus (G2″) implies (G2′).

It remains to show (G2′) implies (G2), so assume (G2′). Thus there is $g \in G$ with $z \neq z^g \in H$. Let $Q = O_2(H)$ and $U = O_2(H_1H_2)$. The proof of 2.1.4 uses only (G1) and shows that $O^2(H) = H_1H_2 \cong \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$, so $U \cong Q_8^2$. By (G1), $F^*(H) = Q$ and either $Q = U$ or $Q = T$.

Assume $Q = T$. Then $z^g \in O^{2'}(H) = Q$. Further $\mathrm{Out}(U) \cong O_4^+(2)$ and $H_1H_2/U = F^*(\mathrm{Out}(U))$, so $O_2(\mathrm{Aut}_H(U)) = \mathrm{Inn}(U)$ and hence $Q = UC_Q(U)$. Observe $Z = \Phi(Q) = \Phi(C_Q(z^g))$, whereas $C_Q(z^g) \leqslant O^2(H^g) = Q^g$, so $Z = \Phi(C_Q(z^g)) \leqslant \Phi(Q^g) = Z^g$, contradicting $z \neq z^g$.

Therefore $F^*(G) = U \cong Q_8^2$. Assume next that $z^g \in U$ and set $V = U \cap U^g$. Then (cf. 8.3 in [2]) $\widetilde{U} = U/Z$ is an orthogonal space over $\mathbf{F}_2$ and by 8.15.3 in [2], $\widetilde{V}$ is a totally singular line. From the structure of $\mathrm{Out}(U) = O_4^+(2)$, each totally singular line is stabilized by some subgroup of H of index 3. Thus $N_H(V)/V$ is the stabilizer in $\mathrm{GL}(V)$ of z, so $M = \langle N_H(V), U^g\rangle$ induces $\mathrm{GL}(V)$ on V. Further $C_G(V) = C_H(V) = V$, so (G2) is satisfied in this case.

Thus we may assume z is weakly closed in U with respect to G. Hence $t = z^g \in H - H_1H_2$ and setting $H^* = H/U$, either $H_i^* = [H_i^*, t^*]$ for $i = 1$ and 2, or we may assume t centralizes $\widetilde{U} \cap \widetilde{H}_1$. In the latter case t centralizes an element u of order 4 in $U \cap H_1$, so as $|H : O^2(H)| = 2$, $z = u^2 \in O^{2'}(O^2(H^g)) = U^g$, contradicting z weakly closed in U. Thus we may assume the former case holds. Then arguing as in 2.3.5, H is transitive on involutions in $H - U$, so all such involutions are in z^G, and $C_H(z^g) = Z^g C_U(z^g)$, with $C_U(z^g) \cong E_4$. By symmetry $C_{H^g}(z) = ZC_{U^g}(z)$ so $U \cap U^g = \langle u\rangle$ is of order 2. Thus uz is an involution in $H^g - U^g$, so $uz \in z^G \cap U$, completing the proof.

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