On extremal matrices of second largest exponent by Boolean rank

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Abstract

Let \( b = b(A) \) be the Boolean rank of an \( n \times n \) primitive Boolean matrix \( A \) and \( \exp(A) \) be the exponent of \( A \). Then \( \exp(A) \leq (b - 1)^2 + 2 \), and the matrices for which equality occurs have been determined in [D.A. Gregory, S.J. Kirkland, N.J. Pullman, A bound on the exponent of a primitive matrix using Boolean rank, Linear Algebra Appl. 217 (1995) 101–116]. In this paper, we show that for each \( 3 \leq b \leq n - 1 \), there are \( n \times n \) primitive Boolean matrices \( A \) with \( b(A) = b \) such that \( \exp(A) = (b - 1)^2 + 1 \), and we explicitly describe all such matrices.

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1. Introduction

A Boolean matrix is a matrix over the binary Boolean algebra \((0, 1)\). Given an \( m \times n \) Boolean matrix \( A \), we define its Boolean rank \( b(A) \) to be the smallest integer \( k \) such that for some \( m \times k \) Boolean matrix \( F \) and \( k \times n \) Boolean matrix \( G \), \( A = FG \). If \( A = 0 \), \( b(A) \) is defined to be 0. \( A = FG \) is called a Boolean rank factorization of \( A \).

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A real $n \times n$ Boolean matrix $A$ is primitive if one of its powers, $A^k$, has all positive entries for some integer $k \geq 1$. The smallest such $k$ is called the (primitive) exponent of $A$, and is denoted by $\exp(A)$. We will confine our investigation of the exponent to Boolean matrices throughout the sequel.

Two $n \times n$ Boolean matrices $A$ and $B$ are called permutation similar if there is an $n \times n$ permutation matrix $P$ such that $A = PBP^t$ and it is denoted by $A \approx B$.

A number of authors have worked on obtaining upper bounds on the exponent of a primitive (Boolean) matrix. For an $n \times n$ primitive Boolean matrix $A$, Wielandt [5] obtained the following results in terms of $n$.

**Theorem 1.A** [3,5]. If $A$ is an $n \times n$ primitive Boolean matrix, then

$$\exp(A) \leq (n - 1)^2 + 1.$$  

(1.1)

Equality holds in (1.1) if and only if $A \approx W_n$, where

$$W_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$W_n = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 \\
1 & 0 & \cdots & 0 & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix} \quad \text{for } n \geq 3.$$ 

**Theorem 1.B** [3,5]. If $A$ is an $n \times n$ primitive Boolean matrix, then

$$\exp(A) = (n - 1)^2$$  

(1.2)

holds if and only if $A \approx B_n$, where

$$B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and

$$B_n = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 \\
1 & 0 & \cdots & 0 & 0 & 1 \\
1 & 1 & \cdots & 0 & 0 & 0
\end{bmatrix} \quad \text{for } n \geq 3.$$ 

Gregory et al. [1] obtained a new upper bound on the exponents of primitive Boolean matrices in terms of Boolean rank.

**Theorem 1.C** [1]. Suppose $n \geq 2$. If $A$ is an $n \times n$ primitive Boolean matrix with $b(A) = b$, then

$$\exp(A) \leq (b - 1)^2 + 2.$$  

(1.3)
This bound in terms of Boolean rank can also be obtained as a consequence of an early paper of Kim [2]. In [1], Gregory et al. proved that the bound (1.3) can be obtained for each \( b \) with \( 2 \leq b \leq n - 1 \). They also gave a characterization of the matrices for which equality holds in (1.3) and obtained all extremal matrices such that equality holds in (1.3).

To state their characterization, we introduce some notations first. Given an \( m \times n \) matrix \( A \), we will denote by \( A_{ij} \) its entry in the \((i, j)\) position, \( A_i \) its \( i\)th row, and \( A.j \) its \( j\)th column. We will use \( A^t \) to denote the transpose of the matrix \( A \). If \( B \) is another \( m \times n \) matrix, we say that \( B \) is dominated by \( A \), and write \( B \leq A \), if \( B_{ij} \leq A_{ij} \) for all \( i \in \{1, 2, \ldots, m\} \) and \( j \in \{1, 2, \ldots, n\} \).

We denote by \( J_{m,n} \) (and by \( J_n \) if \( m = n \)) the \( m \times n \) all ones matrix, by \( 0_{m,n} \) the \( m \times n \) all zeros matrix, and by \( j_n \) the all ones \( n\)-vector. We denote by \( I_n \) the \( n \times n \) identity matrix and by \( e_i(n) \) its \( i\)th column. The subscripts \( m \) and \( n \) will be omitted whenever their values are clear from the context.

**Theorem 1.D** [1]. If \( A \) is an \( n \times n \) Boolean matrix with \( 2 \leq b = b(A) \leq n - 1 \), then \( A \) is primitive with \( \exp(A) = (b - 1)^2 + 2 \) if and only if \( A \) has a Boolean rank factorization \( A = XY \), where \( X, Y \) have the following properties:

(i) \( YX = W_b \), and
(ii) some row of \( X \) is \( e_1^b(b) \) and some column of \( Y \) is \( e_b(b) \).

In this paper, we characterize all \( n \times n \) primitive Boolean matrices with \( 3 \leq b = b(A) \leq n - 1 \) and \( \exp(A) = (b - 1)^2 + 1 \), obtain the following theorem.

**Theorem.** Suppose \( A \) is an \( n \times n \) Boolean matrix with \( 3 \leq b = b(A) \leq n - 1 \), then \( A \) is primitive with \( \exp(A) = (b - 1)^2 + 1 \) if and only if \( A \) has a Boolean rank factorization \( A = XY \), where \( X, Y \) satisfying one of the following conditions:

(i) \( YX = W_b \), some row of \( X \) is \( e_1^b(b) \), some column of \( Y \) is \( e_b(b) \), and no row of \( X \) is \( e_1^b(b) \);  
(ii) \( YX = W_b \), some row of \( X \) is \( e_1^b(b) \), some column of \( Y \) is \( e_{b-1}(b) \), and no column of \( Y \) is \( e_b(b) \);  
(iii) \( YX = B_b \), some row of \( X \) is \( e_1^b(b) \), and some column of \( Y \) is \( e_b(b) \).

Based on this characterization, we explicitly describe all 201 such matrices in Section 3 as well.

### 2. A characterization of the matrices with second largest exponent by Boolean rank

We start with a basic result on the primitive Boolean matrices.

**Lemma 2.A** [1,4]. Suppose that \( X \) and \( Y \) are \( n \times m \) and \( m \times n \) Boolean matrices respectively, and that neither has a zero line (i.e. row and column). Then

(a) \( XY \) is primitive if and only if \( YX \) is primitive, and  
(b) if \( XY \) and \( YX \) are primitive, then  
\[ |\exp(XY) - \exp(YX)| \leq 1. \] (2.1)
Suppose that $A$ is an $n \times n$ primitive Boolean matrix, we have the following lemma about a Boolean rank factorization of $A$.

**Lemma 2.1.** Suppose $n \times n$ matrix $A$ is a primitive Boolean matrix, $A = XY$ is a Boolean rank factorization of $A$ where $b(A) = b$. Then $B = YX$ is primitive and $X, Y$ neither has a zero line (i.e. row and column).

**Proof.** Since $A$ is primitive, $X$ has no zero rows and $Y$ has no zero columns. Suppose that $X$ has a zero column, and without loss of generality, let it be the $i$th column. Let $X'$ be the matrix obtained from $X$ by deleting the $i$th column and $Y'$ be the matrix obtained from $Y$ by deleting the $i$th row. Then $X'$ is an $n \times (b - 1)$ matrix, $Y'$ is a $(b - 1) \times n$ matrix, and $X'Y' = A$. Therefore the Boolean rank of $A$ is at most $b - 1$, a contradiction. Hence $X$ has no zero columns.

Similarly, $Y$ has no zero rows. Lemma 2.A yields that $B = YX$ is primitive. □

By Lemma 2.1, if neither $X_{n \times b}$ nor $Y_{b \times n}$ has a zero line, then both $A = XY$ and $B = YX$ are primitive.

Suppose that $A$ is an $n \times n$ Boolean matrix with $1 \leq b = b(A) \leq n$. If $b = n$, then $\exp(A) = (b - 1)^2 + 1$ if and only if $A \approx W_n$ by Theorem 1.A. If $b = 1$, then $\exp(A) = (b - 1)^2 + 1 = 1$ if and only if $A = J_n$. Thus we may assume that $2 \leq b \leq n - 1$.

To give the characterization, we first make some observations about the matrix $B_n$ of Theorem 1.B.

**Lemma 2.2.** If $n \geq 3$, then the only zero entry in $B_n^{(n-1)^2 - 1}$ occurs in the $(1, n)$ position.

**Proof.** For $n \geq 3$, a straightforward proof by induction on $k$ shows that for $1 \leq k \leq n - 2$,

$$B_n^k = \begin{bmatrix} U & I_{n-k} \\ V & E_{k \times (n-k)} \end{bmatrix},$$

where the $(n - k) \times k$ matrix $U$ is all 0 except for a 1 in the $(n - k, 1)$ position, and the $k \times k$ matrix $V$ has 1’s on the diagonal and superdiagonal, and 0’s elsewhere, and the $E_{k \times (n-k)}$ matrix is all 0 except for a 1 in the $(k, 1)$ position.

Then

$$B_n^n = B_n^{n-2}B_n^2 = B_n + I_n.$$  

Thus we obtain

$$B_n^{(n-1)^2 - 1} = B_n^{n(n-2)} = (B_n + I_n)^{n-2} = I_n + B_n + B_n^2 + \cdots + B_n^{n-2}$$


$$= \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}. \quad \Box$$

We are ready to give the characterization now.
**Theorem 2.1.** Suppose $A$ is an $n \times n$ Boolean matrix with $3 \leq b = b(A) \leq n - 1$, then $A$ is primitive with $\exp(A) = (b - 1)^2 + 1$ if and only if $A$ has a Boolean rank factorization $A = XY$, where $X, Y$ satisfying one of the following conditions:

(i) $YX = W_b$, some row of $X$ is $e_1(b)$, some column of $Y$ is $e_b(b)$, and no row of $X$ is $e_b(b)^T$;

(ii) $YX = W_b$, some row of $X$ is $e_b(b)$, some column of $Y$ is $e_{b-1}(b)$, and no column of $Y$ is $e_{b-1}(b)$;

(iii) $YX = B_b$, some row of $X$ is $e_1(b)$, and some column of $Y$ is $e_b(b)$.

**Proof.** *Necessity.* Suppose that $A$ is primitive with $\exp(A) = (b - 1)^2 + 1$, and $A = X_1Y_1$ is a Boolean rank factorization of $A$. Then $X_1, Y_1$ neither has a zero line and $Y_1X_1$ is primitive by Lemma 2.1. So $\exp(Y_1X_1) \geq (b - 1)^2$ by Lemma 2.A.

On the other hand, $\exp(Y_1X_1) \leq (b - 1)^2 + 1$ by Theorem 1.A, thus $\exp(Y_1X_1) = (b - 1)^2 + 1$ or $\exp(Y_1X_1) = (b - 1)^2$. Say, there is a permutation matrix $P$ such that $P(Y_1X_1)P^T = W_b$ or $P(Y_1X_1)P^T = B_b$ by Theorem 1.A and Theorem 1.B.

Let $X = X_1P^T$ and $Y = PY_1$. Then neither $X$ nor $Y$ has a zero line and $A = XY$ is a rank factorization of $A$. Thus $YX = W_b$ or $YX = B_b$.

**CASE 1.** $YX = W_b$. Then

$$A^{(b-1)^2} = X(YX)^{(b-1)^2-1}Y = X(W_b)^{b(b-2)}Y = X(W_b + I_b)^{b-2}Y = X \left( \sum_{i=0}^{b-2} W^i_b \right) Y.$$ 

Note that while $1 \leq k \leq n - 1$,

$$W^k_n = \begin{bmatrix} U & I_{n-k} \\ V & 0_{k \times (n-k)} \end{bmatrix},$$

where the $(n - k) \times k$ matrix $U$ is all 0 except for a 1 in the $(n - k, 1)$ position, and the $k \times k$ matrix $V$ has 1’s on the diagonal and superdiagonal, and 0’s elsewhere.

Then

$$\sum_{i=0}^{b-2} W^i_b = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 & 1 \end{bmatrix}.$$ 

Thus

$$A^{(b-1)^2} = X \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 & 1 \end{bmatrix} Y.$$
\[
\begin{align*}
&= \left( J_{n, b-2} \begin{array}{c|c}
\sum_{i=1}^{b-1} X_i & \sum_{i=2}^{b} X_i \\
\end{array} \right) Y \\
&= j_n \left( \sum_{i=1}^{b-2} Y_i \right) + \left( \sum_{i=1}^{b-1} X_i \right) Y_{(b-1)} + \left( \sum_{i=2}^{b} X_i \right) Y_{b}. \\
\end{align*}
\]

Since \( \exp(A) = (b - 1)^2 + 1 > (b - 1)^2 \), the matrix \( A^{(b-1)^2} \) must have a zero entry. Say in the \((p, q)\) position, thus

\[
\sum_{i=1}^{b-2} Y_{iq} + \left( \sum_{i=1}^{b-1} X_{pi} \right) Y_{(b-1)q} + \left( \sum_{i=2}^{b} X_{pi} \right) Y_{bq} = 0.
\]

So the first \( b - 2 \) entries in \( Y_q \) are zero, and either \( Y_{(b-1)q} \) or \( Y_{bq} \) is zero, otherwise \( X_p \) will be zero.

If \( Y_{(b-1)q} = 0 \) and \( Y_{bq} \neq 0 \), then \( Y_q = (0, 0, \ldots, 1)^t = e_b(b) \) and \( \sum_{i=2}^{b} X_{pi} = 0 \), that is, \( X_p = (1, 0, \ldots, 0) = e_1^t(b) \). Therefore \( X \) and \( Y \) satisfy (i) by Theorem 1.D.

If \( Y_{(b-1)q} \neq 0 \) and \( Y_{bq} = 0 \), then \( Y_q = (0, 0, \ldots, 1, 0)^t = e_{b-1}(b) \), and \( \sum_{i=1}^{b-1} X_{pi} = 0 \), that is, \( X_p = (0, 0, \ldots, 1) = e_b(b) \). \( X \) and \( Y \) satisfy (ii) by Theorem 1.D.

**CASE 2.** \( XY = B_b \). Then

\[
A^{(b-1)^2} = (XY)^{(b-1)^2} = X(YX)^{(b-1)^2-1} Y = XB_b^{(b-1)^2-1} Y
\]

Since \( \exp(A) = (b - 1)^2 + 1 > (b - 1)^2 \), the matrix \( A^{(b-1)^2} \) must have a zero entry. Say, in the \((p, q)\) position. Thus we have

\[
\sum_{i=1}^{b-1} X_{iq} + \sum_{i=2}^{b} X_{pi} Y_{bq} = 0.
\]

So the first \( b - 1 \) entries in \( Y_q \) are zero, and since \( Y \) has no zero lines, we see that \( Y_q \) must equal \( e_b(b) \). Hence the last \( b - 1 \) entries of \( X_p \) are zero also, and we find that \( X_p \) must equal \( e_1^t(b) \). Consequently, \( X \) and \( Y \) satisfy (iii).

**Sufficiency.** Suppose that \( A = XY \) is a rank factorization of \( A \), where \( X \) and \( Y \) satisfy the condition in the theorem.

If \( X \) and \( Y \) satisfy (i), then \( A \) is primitive by (a) of Lemma 2.A, and

\[
(b - 1)^2 \leq \exp(A) \leq (b - 1)^2 + 2
\]
by Lemma 2.1 and Theorem 1.A. Further,

\[ A^{(b-1)^2} = (XY)^{(b-1)^2} = X(YX)^{(b-1)^2-1}Y = X(W_b)^{b(b-2)}Y \]

has a zero entry, so we obtain that \( \exp(A) \geq (b - 1)^2 + 1 \).

Note that no row of \( X \) is \( e^t \) \( b \), we can conclude that \( \exp(A) \neq (b - 1)^2 + 2 \) by Theorem 1.D.

Thus \( \exp(A) = (b - 1)^2 + 1 \).

If \( X \) and \( Y \) satisfy \( (ii) \), the proof is similar to \( (i) \).

If \( X \) and \( Y \) satisfy \( (iii) \), then we have

\[ (b - 1)^2 - 1 \leq \exp(A) \leq (b - 1)^2 + 1 \]

by Lemma 2.1, Lemma 2.1 and Theorem 1.B. It follows from Lemma 2.2 that

\[ A^{(b-1)^2} = (XY)^{(b-1)^2} = X(YX)^{(b-1)^2-1}Y = X(B_b)^{b(b-2)}Y \]

has a zero entry, so we obtain

\[ \exp(A) \geq (b - 1)^2 + 1 \]

Thus \( \exp(A) = (b - 1)^2 + 1 \) by combining the above two inequalities. \( \Box \)

3. Description of the matrices with second largest exponent by Boolean rank

In this section, we will reinterpret condition \((i), (ii), \) or \((iii)\) of Theorem 2.1 to show that if \( A \) satisfies \( \exp(A) = (b - 1)^2 + 1 \), then \( A \) is one of 201 basic types of matrices.

**Theorem 3.1.** Suppose \( A \) is an \( n \times n \) primitive Boolean matrix with \( \exp(A) = (b - 1)^2 + 1 \), \( A = XY \) is a Boolean rank factorization of \( A \) with \( 3 \leq b(A) = b \leq n - 1 \), and where \( X, Y \) satisfying condition \((i)\) of Theorem 2.1, then \( A \) is permutation similar to one of \( M_1, M_2, M_3 \) and \( M_4 \) which we will describe later.

**Proof.** Since \( A \) is primitive, \( X \) has no zero line, and each column of \( Y \) is dominated by a column of \( W_b \). Similarly, each row of \( X \) is dominated by a row of \( W_b \). Thus, each column of \( Y \) is in the set \( F_1 = \{ e_1(b), e_2(b), \ldots, e_b(b), u \} \), where \( u = e_{b-1}(b) + e_b(b) \), and each row of \( X \) is in the set \( F_2 = \{ e_1^t(b), e_2^t(b), \ldots, e_{b-1}^t(b), v^t \} \), where \( v = e_1(b) + e_b(b) \).

Next, we note that for each \( 1 \leq i \leq b \), the product \( Y_iX_i \) is dominated by \( W_b \). Since each such \( Y_i \) and \( X_i \) must be in \( F_1 \) and \( F_2 \), respectively, we find that \( (Y_i, X_i) \) must be one of the following pairs: \((e_i, e_{i+1}^t), 1 \leq i \leq b - 2); (e_{b-1}, e_1^t), (e_b, e_1^t), (u, e_1^t), (e_{b-1}, v^t)\), where \( e_i = e_i(b) \) for any \( i \in \{1, 2, \ldots, b\} \).

Thus, for each \( 1 \leq i \leq b - 2 \), take \( (e_i, e_{i+1}^t) = (Y_{k_i}, X_{k_i}) \) for some \( k_i \).

Furthermore, some \( Y_j \) is \( e_b(b) \), so take \( (e_b, e_1^t) = (Y_{k_{b-1}}, X_{k_{b-1}}) \) for some \( k_{b-1} \).

Similarly, some \( X_j \) is \( e_1^t(b) \), so that for some \( k_b \), \( (Y_{k_b}, X_{k_b}) \) is one of \( (e_{b-1}, e_1^t), (e_b, e_1^t) \), and \((u, e_1^t)\).

Finally, some product \( (Y_{k_{b+1}}, X_{k_{b+1}}) \) must have a \( 1 \) in \((b - 1, b)\) position, and hence take \( (e_{b-1}, v^t) = (Y_{k_{b+1}}, X_{k_{b+1}}) \) for some \( k_{b+1} \). Similarly, some product \( (Y_{k_{b+2}}, X_{k_{b+2}}) \) must have a \( 1 \) in \((b - 1, 1)\) position, hence for some \( k_{b+2} \), \( (Y_{k_{b+2}}, X_{k_{b+2}}) \) is one of \( (e_{b-1}, v^t), (e_{b-1}, e_1^t) \), and \((u, e_1^t)\).

From the above considerations, there is an \( n \times n \) permutation matrix \( Q \) such that

\[ YQ = [\tilde{Y} | \tilde{\bar{Y}}] \quad \text{and} \quad QX = \begin{bmatrix} \tilde{X} \\ \bar{X} \end{bmatrix} , \]
where
\[
\bar{Y} = [e_1 j_{n1}^t | e_2 j_{n2}^t | e_3 j_{n3}^t | \ldots | e_{b-2} j_{n_{b-2}}^t | e_b j_{n_{b-1}}^t] \quad \text{and} \quad \bar{X} = \begin{bmatrix}
  j_{n1} e_2^t \\
  j_{n2} e_3^t \\
  \vdots \\
  j_{nb-2} e_{b-1}^t \\
  j_{nb-1} e_1^t
\end{bmatrix}
\]

for some \( n_1, \ldots, n_{b-1} \geq 1 \), and where the pair \( \bar{Y} \) and \( \bar{X} \) can be taken to be one of the following pairs of matrices:

\[
\begin{align*}
\bar{Y}_1 &= e_{b-1} j_{m1}^t, \quad \bar{X}_1 = j_{m1} v^t \quad \text{for some} \ m_1 \geq 1; \\
\bar{Y}_2 &= [e_{b-1} j_{m2}^t | e_{b-1} j_{m3}^t], \quad \bar{X}_2 = \begin{bmatrix} j_{m2} v^t \\ j_{m3} e_1^t \end{bmatrix} \quad \text{for some} \ m_2, m_3 \geq 1; \\
\bar{Y}_3 &= [e_{b-1} j_{m4}^t | u j_{m5}^t], \quad \bar{X}_3 = \begin{bmatrix} j_{m4} v^t \\ j_{m5} e_1^t \end{bmatrix} \quad \text{for some} \ m_4, m_5 \geq 1; \\
\bar{Y}_4 &= [e_{b-1} j_{m6}^t | e_{b-1} j_{m7}^t | u j_{m8}^t], \quad \bar{X}_4 = \begin{bmatrix} j_{m6} v^t \\ j_{m7} e_1^t \\ j_{m8} e_1^t \end{bmatrix} \quad \text{for some} \ m_6, m_7, m_8 \geq 1.
\end{align*}
\]

We take
\[
M_j = \left[ \begin{array}{c}
\bar{X} \\
\bar{X}_j
\end{array} \right] [\bar{Y} \bar{Y}_j], \quad 1 \leq j \leq 4,
\]

then \( QA Q^t = (QX)(Y Q^t) = M_j, j = 1, 2, 3, 4. \)

\[ \square \]

**Theorem 3.2.** Suppose \( A \) is an \( n \times n \) primitive Boolean matrix with \( \exp(A) = (b - 1)^2 + 1 \), \( A = XY \) is a Boolean rank factorization of \( A \) with \( 3 \leq b(A) = b \leq n - 1 \), and where \( X, Y \) satisfying condition (ii) of Theorem 2.1, then \( A \) is permutation similar to one of \( M_5, M_6, M_7 \) and \( M_8 \) which we will describe later.

**Proof.** Since \( A \) is primitive, \( X \) has no zero line, and each column of \( Y \) is dominated by a column of \( W_b \). Similarly, each row of \( X \) is dominated by a row of \( W_b \). Thus, each column of \( Y \) is in the set \( F_1 = \{ e_1(b), e_2(b), \ldots, e_{b-1}(b), u \} \), where \( u = e_{b-1}(b) + e_b(b) \), and each row of \( X \) is in the set \( F_2 = \{ e_i^t(b), e_i^t(b), \ldots, e_i^t(b), v^t \} \), where \( v = e_1(b) + e_b(b) \).

Next, we note that for each \( 1 \leq i \leq b \), the product \( Y_iX_i \) is dominated by \( W_b \). Since each such \( Y_i \) and \( X_i \) must be in \( F_1 \) and \( F_2 \), respectively, we find that \((Y_i, X_i)\) must be one of the following pairs: \((e_i, e_{i+1})^t, 1 \leq i \leq b - 1\); \((e_{b-1}, e_i^t), (u, e_i^t), (e_{b-1}, v^t)\).

Thus, for each \( 1 \leq i \leq b - 2 \), take \((e_i, e_{i+1}) = (Y_{k_i}, X_{k_i})\) for some \( k_i \).

Furthermore, some \( X_j \) is \( e_i^t(b) \), so take \((e_{b-1}, e_i^t) = (Y_{k_{b-1}} b, X_{k_{b-1}})\) for some \( k_{b-1} \).

Similarly, some \( Y_j \) is \( e_{b-1}(b) \), so that for some \( k_b, Y_{k_b}, X_{k_b} \) is one of \((e_{b-1}, e_i^t), (e_{b-1}, e_i^t), \) and \((e_{b-1}, v^t)\).
Finally, some product \((Y_j, X_j)\) must have a 1 in \((b, 1)\) position, and hence take \((u, e_1^i) = (Y_{kb+1}, X_{kb+1})\) for some \(kb+1\). Similarly, some product \((Y_j, X_j)\) must have a 1 in \((b-1, 1)\) position, hence for some \(kb+2, (Y_{kb+2}, X_{kb+2})\) is one of \((u, e_1^i), (e_{b-1}, e_1^i), (e_b, v^i_1), (u_1, e_1^i), (e_{b-1}, v^i_1), (v_1, e_2^i)\).

From the above considerations, there is an \(n \times n\) permutation matrix \(Q\) such that

\[
YQ^i = [\bar{Y} | \tilde{Y}] \quad \text{and} \quad QX = \begin{bmatrix} \bar{X} \\ \tilde{X} \end{bmatrix},
\]

where

\[
\bar{Y} = [e_1j^i_{n_1} | e_2j^i_{n_2} | e_3j^i_{n_3} | \cdots | e_{b-2}j^i_{n_{b-2}} | e_{b-1}j^i_{n_{b-1}}] \quad \text{and} \quad \bar{X} = \begin{bmatrix} f_{n_1}^i e_2^i \\ f_{n_2}^i e_3^i \\ f_{n_3}^i e_4^i \\ \vdots \\ f_{n_{b-2}}^i e_{b-1}^i \\ f_{n_{b-1}}^i e_b^i \end{bmatrix}
\]

for some \(n_1, \ldots, n_{b-1} \geq 1\), and where the pair \(\bar{Y}\) and \(\bar{X}\) can be taken to be one of the following pairs of matrices:

\[
\bar{Y}_1 = u_jm_1^i, \quad \bar{X}_1 = j_m e_1^i \quad \text{for some } m_1 \geq 1;
\]

\[
\bar{Y}_2 = [u_jm_2^i | e_{b-1}j_m^i], \quad \bar{X}_2 = \begin{bmatrix} f_{m_2}^i e_2^i \\ f_{m_3}^i e_3^i \end{bmatrix} \quad \text{for some } m_2, m_3 \geq 1;
\]

\[
\bar{Y}_3 = [u_jm_4^i | e_{b-1}j_m^i], \quad \bar{X}_3 = \begin{bmatrix} f_{m_4}^i e_2^i \\ f_{m_5}^i v^i_1 \end{bmatrix} \quad \text{for some } m_4, m_5 \geq 1;
\]

\[
\bar{Y}_4 = [u_jm_6^i | e_{b-1}j_m^i | e_{b-1}j_m^i], \quad \bar{X}_4 = \begin{bmatrix} f_{m_6}^i e_2^i \\ f_{m_7}^i v^i_1 \end{bmatrix} \quad \text{for some } m_6, m_7, m_8 \geq 1.
\]

Let

\[
M_j = \begin{bmatrix} \bar{X} \\ \tilde{X} \end{bmatrix} [\bar{Y} | \tilde{Y}], \quad 5 \leq j \leq 8,
\]

where \(1 \leq i \leq 4\), then \(QAQ^i = (QX)(YQ^i) = M_j, j = 5, 6, 7, 8\). □

**Theorem 3.3.** Suppose \(A\) is an \(n \times n\) primitive Boolean matrix with \(\exp(A) = (b-1)^2 + 1\), \(A = XY\) is a Boolean rank factorization of \(A\) with \(3 \leq b(A) = b \leq n - 1\), and where \(X, Y\) satisfying condition (iii) of Theorem 2.1, then \(A\) is permutation similar to one of \(M_j, (9 \leq j \leq 201)\) which we will describe later.

**Proof.** Since \(A = XY\) is primitive, \(X\) has no zero line and so each column of \(Y\) is dominated by a column of \(B_b\). Similarly, each row of \(X\) is dominated by a row of \(B_b\). Thus, each column of \(Y\) is in the set \(F_1 = \{e_1(b), e_2(b), \ldots, e_b(b), u_1, v_1\}\), where \(u_1 = e_{b-1}(b) + e_b(b), v_1 = e_1(b) + e_b(b)\), and each row of \(X\) is in the set \(F_2 = \{e_1^i(b), e_2^i(b), \ldots, e_b^i(b), v_1^i, v_2^i\}\), where \(v_2 = e_1(b) + e_2(b)\).

Next, we note that for each \(1 \leq i \leq b\), the product \(Y_iX_i\) is dominated by \(B_b\). Since each such \(Y_i\) and \(X_i\) must be in \(F_1\) and \(F_2\), respectively, we find that \((Y_i, X_i)\) must be one of the following pairs:

\[
(e_i, e_{i+1}^i), 1 \leq i \leq b - 1; (e_{b-1}, e_1^i), (e_b, e_1^i), (e_b, e_2^i), (e_b, v_1^i), (u_1, e_1^i), (e_{b-1}, v_1^i), (v_1, e_2^i).
\]
Clearly, (1, 2) position of $B_b(=YX)$ must have a 1, so for some $k_1$, $(Y_{k_1}, X_{k_1})$ is one of $(e_1, e_2), (v_1, e_2)$. Similarly, for each $2 \leq i \leq b - 2$, take $(e_i, e_{i+1}) = (Y_{k_i}, X_{k_i})$ for some $k_i$.

Furthermore, since some $X_j$ is $e_i^1(b)$, so for some $k_{b-1}$, $(Y_{k_{b-1}}, X_{k_{b-1}})$ is one of $(e_{b-1}, e_1^1), (e_b, e_1^1)$, and $(u_1, e_1^1)$. Similarly, some $Y_j$ is $e_b(b)$, so that for some $k_b$, $(Y_{k_b}, X_{k_b})$ is one of $(e_b, e_1^1), (e_b, e_2^1)$, and $(e_b, v_2^1)$.

Finally, some product $(Y_j, X_j)$ must have a 1 in $(b - 1, b)$ position, and hence for some $k_{b+1}$, $(Y_{k_{b+1}}, X_{k_{b+1}})$ is one of $(e_{b-1}, e_1^1)$ and $(e_{b-1}, v_1^2)$; similarly, some product $(Y_j, X_j)$ must have a 1 in $(b - 1, 1)$ position, and hence for some $k_{b+2}$, $(Y_{k_{b+2}}, X_{k_{b+2}})$ is one of $(e_{b-1}, e_1^1), (e_{b-1}, v_1^2)$ and $(u_1, e_1^1)$; some product $(Y_j, X_j)$ must have a 1 in $(b, 1)$ position, and hence for some $k_{b+3}$, $(Y_{k_{b+3}}, X_{k_{b+3}})$ is one of $(e_b, e_1^1), (e_b, v_2^1)$, and $(u_1, e_1^1)$; some product $(Y_j, X_j)$ must have a 1 in $(b, 2)$ position, and hence for some $k_{b+4}$, $(Y_{k_{b+4}}, X_{k_{b+4}})$ is one of $(e_b, e_1^1), (e_b, v_2^1)$, and $(v_1, e_2^1)$.

From the above considerations, there is an $n \times n$ permutation matrix $Q$ such that

$$YQ^1 = [\tilde{Y}]\bar{Y} \quad \text{and} \quad QX = \begin{bmatrix} \bar{X} \\
X \end{bmatrix},$$

where

$$\tilde{Y} = [x|e_2j_{n_2}^1|e_3j_{n_3}^1|\cdots|e_{b-2}j_{n_{b-2}}^1|y|e_bj_{n_b}^1] \quad \text{and} \quad \bar{X} = \begin{bmatrix}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
j_1e_1^2 \\
j_2e_1^3 \\
j_3e_1^4 \\
\vdots \\
j_{b-2}e_b^1 - 1 \\
j_{b-1}e_b^1 \\
z
\end{array}
\end{array}
\end{array}
\end{array}
\end{bmatrix}$$

for some $n_1, \ldots, n_b \geq 1$, where $x \in \{e_1j_{n_1}^1, v_1j_{n_1}^1, [e_1j_{n_1}^1, v_1j_{n_1}^1]\}$, $y \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, $z \in \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7\}$, and where for some $n_{(b-1)1}, n_{(b-1)2}, n_{(b-1)3} \geq 1$, and $n_{(b-1)1} + n_{(b-1)2} = n_{b-1}$, or $n_{(b-1)1} + n_{(b-1)2} + n_{(b-1)3} = n_{b-1}$, let

$$\alpha_1 = e_bj_{n_b}^1,$$
$$\alpha_2 = e_{b-1}j_{n_{b-1}}^1,$$
$$\alpha_3 = u_1j_{n_{b-1}}^1,$$
$$\alpha_4 = [e_{b-1}f_{n_{(b-1)1}}^1|e_bf_{n_{(b-1)2}}^1],$$
$$\alpha_5 = [e_{b-1}f_{n_{(b-1)1}}^1|u_1f_{n_{(b-1)2}}^1],$$
$$\alpha_6 = [e_bf_{n_{(b-1)1}}^1|u_1f_{n_{(b-1)2}}^1],$$
$$\alpha_7 = [e_{b-1}f_{n_{(b-1)1}}^1|e_{b}f_{n_{(b-1)2}}^1|u_1f_{n_{(b-1)3}}^1];$$

and for some $n_{b_1}, n_{b_2}, n_{b_3} \geq 1, n_{b_1} + n_{b_2} = n_b$, or $n_{b_1} + n_{b_2} + n_{b_3} = n_b$, let

$$\beta_1 = j_{n_b}e_1^1,$$
$$\beta_2 = j_{n_b}e_2^1,$$
$$\beta_3 = j_{n_b}v_1^2,$$
$$\beta_4 = [j_{n_{b_1}}e_1^1|j_{n_{b_2}}e_2^1].$$
\begin{align*}
\beta_5 &= \begin{bmatrix} j_{n_1} e^i_1 | j_{n_2} v^i_2 \end{bmatrix}, \\
\beta_6 &= \begin{bmatrix} j_{n_1} e^i_1 | j_{n_2} v^i_2 \end{bmatrix}, \\
\beta_7 &= \begin{bmatrix} j_{n_1} e^i_1 | j_{n_2} e^i_1 | j_{n_3} v^i_2 \end{bmatrix}.
\end{align*}

Thus the pair \( \tilde{Y} \) and \( \tilde{X} \) can be taken to be one of the following pairs of matrices:

\begin{align*}
\tilde{Y}_1 &= e_{b-1} j^i_{n_1}, \quad \tilde{X}_1 = j_m e^i_b \quad \text{for some } m_1 \geq 1; \\
\tilde{Y}_2 &= e_{b-1} j^i_{n_2}, \quad \tilde{X}_2 = j_{m_2} v^i_1 \quad \text{for some } m_2 \geq 1; \\
\tilde{Y}_3 &= \left[ e_{b-1} j^i_{m_3} | e_{b-1} j^i_{m_4} \right], \quad \tilde{X}_3 = \begin{bmatrix} j_m e^i_b \\ j_{m_4} v^i_1 \end{bmatrix}, \quad \text{for some } m_3, m_4 \geq 1.
\end{align*}

**CASE 1.** \( y \in \{\alpha_2, \alpha_3, \alpha_5\}, \ z \in \{\beta_2, \beta_3, \beta_5\} \).

In this case neither \( (y, j_{n-1} e^i_1) \) nor \( (e_b j^i_{n_b}, z) \) contain \( (e_b j^i_{n_m}, j_m e^i_1) \) for some \( j_m \), and if \( y = \alpha_2, \ z = \beta_2 \), then \( (b, 1) \) position of \( Y X (= B_b) \) will be a zero entry, it is a contradiction. So we let

\[ M_j = \begin{bmatrix} \tilde{X} \\ \tilde{X}_i \end{bmatrix} [\tilde{Y} | \tilde{Y}_i], \quad 9 \leq j \leq 80, \]

where \( 1 \leq i \leq 3, \ x \in \{e_1 j^i_{n_1}, v_1 j^i_{n_1}, [e_1 j^i_{n_1} | v_1 j^i_{n_1}]\} \), and \( (y, z) \in \{(\alpha_2, \beta_3), (\alpha_2, \beta_5), (\alpha_3, \beta_2), (\alpha_3, \beta_3), (\alpha_3, \beta_5), (\alpha_5, \beta_2), (\alpha_5, \beta_3), (\alpha_5, \beta_5)\} \).

**CASE 2.** \( y \in \{\alpha_1, \alpha_4, \alpha_6, \alpha_7\}, \ z \in \{\beta_1, \beta_4, \beta_6, \beta_7\} \).

In this case both \( (y, j_{n-1} e^i_1) \) and \( (e_b j^i_{n_b}, z) \) contain \( (e_b j^i_{n_m}, j_m e^i_1) \) for some \( j_m \).

If \( y = \alpha_1, \ z = \beta_1 \), then take

\[ M_j = \begin{bmatrix} \tilde{X} \\ \tilde{X}_i \end{bmatrix} [\tilde{Y} | \tilde{Y}_i], \quad 81 \leq j \leq 84, \]

where \( 2 \leq i \leq 3, \ x \in \{v_1 j^i_{n_1}, [e_1 j^i_{n_1} | v_1 j^i_{n_1}]\} \).

If \( y \in \{\alpha_4, \alpha_6, \alpha_7\}, \ z = \beta_1 \), then take

\[ M_j = \begin{bmatrix} \tilde{X} \\ \tilde{X}_i \end{bmatrix} [\tilde{Y} | \tilde{Y}_i], \quad 85 \leq j \leq 102, \]

where \( 1 \leq i \leq 3, \ x \in \{v_1 j^i_{n_1}, [e_1 j^i_{n_1} | v_1 j^i_{n_1}]\} \).

If \( y = \alpha_1, \ z \in \{\beta_4, \beta_6, \beta_7\} \), then take

\[ M_j = \begin{bmatrix} \tilde{X} \\ \tilde{X}_i \end{bmatrix} [\tilde{Y} | \tilde{Y}_i], \quad 103 \leq j \leq 120, \]

where \( 2 \leq i \leq 3, \ x \in \{e_1 j^i_{n_1}, v_1 j^i_{n_1}, [e_1 j^i_{n_1} | v_1 j^i_{n_1}]\} \).

If \( y \in \{\alpha_4, \alpha_6, \alpha_7\}, \ z \in \{\beta_4, \beta_6, \beta_7\} \), then take

\[ M_j = \begin{bmatrix} \tilde{X} \\ \tilde{X}_i \end{bmatrix} [\tilde{Y} | \tilde{Y}_i], \quad 121 \leq j \leq 201, \]

where \( 1 \leq i \leq 3, \ x \in \{e_1 j^i_{n_1}, v_1 j^i_{n_1}, [e_1 j^i_{n_1} | v_1 j^i_{n_1}]\} \).

**CASE 3.** \( y \in \{\alpha_1, \alpha_4, \alpha_6, \alpha_7\}, \ z \in \{\beta_2, \beta_3, \beta_5\} \).

For any \( i \in \{1, 4, 6, 7\} \), the effect of \( (\alpha_i, \beta_2) \) is equal to the effect of \( (\alpha_i, \beta_4) \). Similarly, the effect of \( (\alpha_i, \beta_3) \) is equal to the effect of \( (\alpha_i, \beta_6) \), and the effect of \( (\alpha_i, \beta_5) \) is equal to the effect of \( (\alpha_i, \beta_7) \), thus this case is concluded to Case 2, we need not to consider it again.
CASE 4. \( y \in \{\alpha_2, \alpha_3, \alpha_5\}, z \in \{\beta_1, \beta_4, \beta_6, \beta_7\} \).

Similar to Case 3, this case is concluded to Case 2, because for any \( i \in \{1, 4, 6, 7\} \), the effect of \((\alpha_2, \beta_i)\) is equal to the effect of \((\alpha_4, \beta_i)\), the effect of \((\alpha_3, \beta_i)\) is equal to the effect of \((\alpha_6, \beta_i)\), and the effect of \((\alpha_5, \beta_i)\) is equal to the effect of \((\alpha_7, \beta_i)\).

So \( QAQ^t = (QX)(YQ^t) = M_j \) \((9 \leq j \leq 201)\) follows from the above four cases. \(\square\)

Our last result follows from Theorems 2.1 and 3.1–3.3.

**Theorem 3.4.** Suppose \( A \) is an \( n \times n \) Boolean matrix with \( 3 \leq b = b(A) \leq n - 1 \), then \( A \) is primitive with \( \exp(A) = (b - 1)^2 + 1 \) if and only if there is a permutation matrix \( Q \) such that \( QAQ^t \) has one of the forms of \( M_j \) \((1 \leq j \leq 201)\) which defined as above.

A similar argument applies to the case \( b(A) = b = 2 \), so we ignore it.

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**References**


