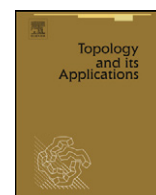




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Dynamics and topology of S -gap shifts

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ABSTRACT

Let $S = \{s_i \in \mathbb{N} \cup \{0\} : 0 \leq s_i < s_{i+1}, i \in \mathbb{N} \cup \{0\}\}$ and let $d_0 = s_0$ and $\Delta(S) = \{d_n\}_n$ where $d_n = s_n - s_{n-1}$. In this note, we show that an S -gap shift is subshift of finite type (SFT) if and only if S is finite or cofinite, is almost-finite-type (AFT) if and only if $\Delta(S)$ is eventually constant and is sofic if and only if $\Delta(S)$ is eventually periodic. We also show that there is a one-to-one correspondence between the set of all S -gap shifts and $\{r \in \mathbb{R} : r \geq 0\} \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$ up to conjugacy. This enables us to induce a topology and measure structure on the set of all S -gaps. By using this, we give the frequency of certain S -gap shifts with respect to their dynamical properties.

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1. Introduction

S -gap shifts are symbolic dynamical systems which have many applications in practice, in particular, for coding of data [13] and in theory for its simplicity of producing different classes of dynamical systems.

To define an S -gap shift $X(S)$, fix $S = \{s_i \in \mathbb{N} \cup \{0\} : 0 \leq s_i < s_{i+1}, i \in \mathbb{N} \cup \{0\}\}$. If S is finite, define $X(S)$ to be the set of all binary sequences for which 1's occur infinitely often in each direction and such that the number of 0's between successive occurrences of a 1 is in S . When S is infinite, we need to allow points that begin or end with an infinite string of 0's. We also need a sequence obtained from the difference of two successive s_n in S . That is, let $d_0 = s_0$ and $\Delta(S) = \{d_n\}_n$ where $d_n = s_n - s_{n-1}$. S -gap shifts are already well known, see for instance the classical book of Lind and Marcus [13]. These systems are highly chaotic, for they are transitive with dense periodic points and have positive entropy or comprise a single periodic orbit. These systems are coded system: there is a countable collection of words such that the sequences which are concatenations of these words are a dense subset in $X(S)$ [7, §3.5.4]. Also in [8], Climenhaga and Thompson showed that every subshift factor of an S -gap shift is intrinsically ergodic: there is a unique measure of maximal entropy. Jung [12] proved an S -gap shift has specification with variable gap length (there was called almost specification) if and only if $\sup_n |s_{n+1} - s_n| < \infty$, it is mixing if and only if $\gcd\{s+1 : s \in S\} = 1$ and it has the specification property if and only if it has specification with variable gap length and it is mixing.

A main task in this note is to give a picture of the dynamical properties of $X(S)$ with respect to $\Delta(S)$. Similar studies have been carried out for β -shifts with respect to β -expansion of 1 [4,3,20].

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To summarize the results in this paper, suppose S, S' are different subsets of \mathbb{N}_0 and neither is $\{0, n\}$ for some $n \in \mathbb{N}$. Then in Theorem 4.1, it is proved that $X(S)$ and $X(S')$ are not conjugate which gives a one-to-one correspondence between the space of all S -gap shifts and

$$\mathcal{R} = \mathbb{R}^{\geq 0} \setminus \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \{r \in \mathbb{R} : r \geq 0\} \setminus \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

up to conjugacy. This endows S -gap shifts with a natural topological and measure structure and we will give some classifications of the dynamical properties with respect to these structures. In particular, Theorem 4.4 shows that mixing is a generic stable phenomenon in S -gap shifts. This follows from Theorem 3.1 which states that an S -gap shift is mixing if and only if it is *totally transitive*. By totally transitive, we mean that σ_S^n is transitive for all $n \in \mathbb{N}$ where σ_S is the shift map defined on $X(S)$; or equivalently in symbolic dynamics, for every ordered pair of blocks $u, v \in \mathcal{B}(X)$ and for $n \in \mathbb{N}$ there is a word $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$ and $|w| = kn$ for some $k \in \mathbb{N}$. Therefore, most of the S -gap shifts enjoy having rich dynamics.

In Theorem 3.3, it is shown that $X(S)$ is a subshift of finite type (SFT) if and only if S is finite or cofinite. Other dynamical properties of S -gap shifts are easy to state when one uses $\Delta(S)$. For example, it is shown that $X(S)$ is almost-finite-type (AFT) if and only if $\Delta(S)$ is eventually constant (Theorem 3.6) and it is sofic if and only if $\Delta(S)$ is eventually periodic (Theorem 3.4). Also a non-SFT S -gap shift is proper periodic-finite-type (PFT) if and only if it is AFT and non-mixing (Theorem 3.7).

2. Background and notations

This section is devoted to the very basic definitions in symbolic dynamics. The notations has been taken from [13] and the proofs of the relevant claims in this section can be found there. Let \mathcal{A} be an alphabet, that is a non-empty finite set. The full \mathcal{A} -shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in \mathcal{A} . A block (or word) over \mathcal{A} is a finite sequence of symbols from \mathcal{A} . It is convenient to include the sequence of no symbols, called the empty word and denoted by ε . If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a word of length $j - i$ by $x_{[i, j]} = x_i x_{i+1} \cdots x_j$. If $n \geq 1$, then u^n denotes the concatenation of n copies of u , and put $u^0 = \varepsilon$. The shift map σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i th coordinate is $y_i = x_{i+1}$.

Let \mathcal{F} be the collection of all forbidden blocks over \mathcal{A} . The complement of \mathcal{F} is the set of *admissible blocks*. For any such $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any word in \mathcal{F} . A shift space is a closed subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection \mathcal{F} of forbidden words over \mathcal{A} .

Let $\mathcal{B}_n(X)$ denote the set of all admissible n blocks. The *Language* of X is the collection $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks $u, v \in \mathcal{B}(X)$ there is a word $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$. It is called *weak mixing* if for every ordered pair $u, v \in \mathcal{B}(X)$, there is a thick set (a subset of integers containing arbitrarily long blocks of consecutive integers) P such that for every $n \in P$, there is a word $w \in \mathcal{B}_n(X)$ such that $uwv \in \mathcal{B}(X)$. It is *mixing* if for every ordered pair $u, v \in \mathcal{B}(X)$, there is an N such that for each $n \geq N$ there is a word $w \in \mathcal{B}_n(X)$ such that $uwv \in \mathcal{B}(X)$. A word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever uv and vw are in $\mathcal{B}(X)$, we have $uvw \in \mathcal{B}(X)$. An irreducible shift space X is a *synchronized system* if it has a synchronizing word [6].

Let \mathcal{A} and \mathcal{D} be alphabets and X a shift space over \mathcal{A} . Fix integers m and n with $m \leq n$. Define the $(m + n + 1)$ -block map $\Phi : \mathcal{B}_{m+n+1}(X) \rightarrow \mathcal{D}$ by

$$y_i = \Phi(x_{-m} x_{-m+1} \cdots x_{i+n}) = \Phi(x_{[i-m, i+n]}) \tag{2.1}$$

where y_i is a symbol in \mathcal{D} . The map $\phi : X \rightarrow \mathcal{D}^{\mathbb{Z}}$ defined by $y = \phi(x)$ with y_i given by (2.1) is called the *sliding block code* with *memory* m and *anticipation* n induced by Φ . An onto sliding block code $\phi : X \rightarrow Y$ is called a *factor code*. In this case, we say that Y is a *factor* of X . The map ϕ is a *conjugacy*, if it is invertible.

A shift space X is called a *shift of finite type* (SFT) if there is a finite set \mathcal{F} of forbidden words such that $X = X_{\mathcal{F}}$. An *edge shift*, denoted by X_G , is a shift space which consist of all bi-infinite walks in a directed graph G . Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A *labeled graph* \mathcal{G} is a pair (G, \mathcal{L}) where G is a graph with edge set \mathcal{E} , and the labeling $\mathcal{L} : \mathcal{E} \rightarrow \mathcal{A}$. A *sofic shift* X_G is the set of sequences obtained by reading the labels of walks on G ,

$$X_G = \{ \mathcal{L}_{\infty}(\xi) : \xi \in X_G \} = \mathcal{L}_{\infty}(X_G). \tag{2.2}$$

We say \mathcal{G} is a *presentation* of X_G . Every SFT is sofic [13, Theorem 3.1.5], but the converse is not true.

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex l of G the edges starting at l carry different labels. A *minimal right-resolving presentation* of a sofic shift X is a right-resolving presentation of X having the least vertices among all right-resolving presentations of X .

Let X be a shift space and $w \in \mathcal{B}(X)$. The *follower set* $F(w)$ of w is defined by $F(w) = \{v \in \mathcal{B}(X) : wv \in \mathcal{B}(X)\}$. A shift space X is sofic if and only if it has a finite number of follower sets [13, Theorem 3.2.10]. In this case, we have a labeled graph $\mathcal{G} = (G, \mathcal{L})$ called the *follower set graph* of X . The vertices of G are the follower sets and if $wa \in \mathcal{B}(X)$, then draw an edge labeled a from $F(w)$ to $F(wa)$. If $wa \notin \mathcal{B}(X)$ then do nothing.

A labeled graph is *right-closing* with delay D if whenever two paths of length $D + 1$ start at the same vertex and have the same label, then they must have the same initial edge. Similarly, left-closing will be defined. A labeled graph is bi-closing, if it is simultaneously right-closing and left-closing.

Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph, and l be a vertex of \mathcal{G} . The follower set $F_{\mathcal{G}}(l)$ of l in \mathcal{G} is the collection of labels of paths starting at l . Set A to be a non-negative matrix. The *period* of a state l , denoted by $\text{per}(l)$, is the greatest common divisor of those integers $n \geq 1$ for which $(A^n)_{ll} > 0$. The *period* $\text{per}(A)$ of the matrix A is the greatest common divisor of the numbers $\text{per}(l)$. If A is irreducible, then all states have the same period. Let $X_{\mathcal{G}}$ be an irreducible edge shift and $p = \text{per}(A_{\mathcal{G}})$ where $A_{\mathcal{G}}$ is the adjacency matrix of \mathcal{G} . Then there exists a unique partition $\{D_0, D_1, \dots, D_{p-1}\}$ of the vertices of \mathcal{G} , called *period classes*, so that every edge that starts in D_i terminates in D_{i+1} (or in D_0 if $i = p - 1$).

The entropy of a shift space X is defined by $h(X) = \lim_{n \rightarrow \infty} (1/n) \log |\mathcal{B}_n(X)|$ where $\mathcal{B}_n(X)$ is the set of all admissible n blocks.

Let $T : X \rightarrow X$ be a (surjective) homeomorphism of a metric space and d be a compatible metric for X . A sequence of points $\{x_i : i \in (a, b)\}$ ($-\infty \leq a < b \leq \infty$) is called a δ -pseudo-orbit for T if $d(T(x_i), x_{i+1}) < \delta$ for $i \in (a, b - 1)$. A sequence $\{x_i\}$ is called to be ε -traced by $x \in X$ if $d(T^i(x), x_i) < \varepsilon$ holds for $i \in (a, b)$. A homeomorphism is said to have the *pseudo-orbit-tracing property* (POTP) if for every $\varepsilon > 0$ there is $\delta > 0$ such that each δ -pseudo-orbit for T is ε -traced by some point of X [14]. The map T is said to have *pointwise pseudo-orbit tracing property* (PPOTP), if for any $\varepsilon > 0$, there is $\delta > 0$, such that for any δ -pseudo-orbit $\{x_0, x_1, \dots\}$ of T , there is non-negative integer N , such that $\{x_N, x_{N+1}, \dots\}$ can be ε -traced by the orbit of T on some point in X . By definition, if T has POTP $\Rightarrow T$ has PPOTP, but the converse is false [18].

A *continued fraction* representation of a real number x (here $x \geq 0$) is a formal expression of the form

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}, \tag{2.3}$$

which we will also denote it by $x = [a_0; a_1, a_2, \dots]$, $a_n \in \mathbb{N}$ for $n \geq 1$ and $a_0 \in \mathbb{N}_0$. The numbers a_n are called the *partial quotients* of the continued fraction.

The continued fraction for a number is finite (that is the sequence of partial quotients is finite) if and only if the number is rational. Any positive rational number has exactly two continued fraction expansions $[a_0; a_1, \dots, a_{n-1}, a_n]$ and $[a_0; a_1, \dots, a_{n-1}, a_n - 1, 1]$ where by convention the former is chosen but the continued fraction representation of an irrational number is unique [9].

3. On dynamical properties of S-gap shifts

All S-gap shifts are transitive and have a dense set of periodic points. The following two theorems exhibit two more properties showing the richness of the dynamics of S-gap shifts.

Theorem 3.1. *An S-gap shift $X(S)$ is mixing if and only if it is totally transitive.*

Proof. Clearly every mixing shift space is weak mixing. Furthermore, any weak mixing shift space is totally transitive [10].

To prove the necessity, notice that if X is totally transitive and has a dense set of periodic points, it is weak mixing [2]. The proof follows if we show that any weak mixing S-gap shift is mixing. Recall that an S-gap shift is mixing if and only if $\text{gcd}\{s + 1 : s \in S\} = 1$. So it suffices to show that for a weak mixing S-gap shift, $\text{gcd}\{s + 1 : s \in S\} = 1$. So let $X(S)$ be weak mixing. Therefore, there is a thick set P such that for every $n \in P$ there exists a word $w \in \mathcal{B}_n(X)$ with $1w1 \in \mathcal{B}(X)$. Thus there are words of length $m, m + 1 \in P$ of the form $10^{i_1}10^{i_2} \dots 10^{i_k}$ with $i_l \in S$, implying that $\text{gcd}\{s + 1 : s \in S\} = 1$. \square

A topological dynamical system has *completely positive entropy* (c.p.e.) if every non-trivial topological factor has positive entropy.

Later in Theorem 3.3, we will show that if $|S| < \infty$, then $X(S)$ is SFT. Now we use that fact in the following theorem.

Theorem 3.2.

- (1) If $|S| = \infty$, then every non-trivial factor shift of $X(S)$ has positive entropy.
- (2) If $|S| < \infty$, then $X(S)$ has c.p.e. if and only if $X(S)$ is mixing.

Proof.

- (1) In [8], Climenhaga and Thompson proved that any shift with *uniform CGC-decomposition*, either has positive entropy or comprises a single periodic orbit and they also proved that this property is preserved under the operation of taking factor shifts. Moreover, they showed that any S-gap shift has a uniform CGC-decomposition. So it is sufficient to prove that any non-trivial factor shift of $X(S)$ is not a periodic orbit. Let Y be a shift space over the alphabet \mathcal{A} and let

$\phi : X(S) \rightarrow Y$ be a non-trivial factor code with memory m and anticipation n induced by the block map Φ . Then $\Phi(0^{m+n+1})$ is a symbol in \mathcal{A} , say a . Since $|S| = \infty$, the language of Y must contain a^k for any $k \in \mathbb{N}$. This means Y is not a non-trivial periodic orbit.

(2) $|S| < \infty$ implies $X(S)$ is SFT (Theorem 3.3). But any SFT shift has pseudo-orbit tracing property (POTP) and POTP implies pointwise pseudo-orbit tracing property (PPOTP) [18]. On the other hand, a factor which has PPOTP property, is topologically c.p.e. if and only if it is topologically mixing [18, Theorem 6]. \square

Therefore, from the proof of the above theorem, we see that if an S -gap shift has a non-trivial factor shift with zero entropy, then $|S| < \infty$ and $X(S)$ is non-mixing. Also, when $|S| < \infty$ and $X(S)$ is mixing, then $X(S)$ has other equivalent stronger properties [18, Theorem 6]. It is not clear for us what conditions for the case $|S| = \infty$ is needed to have at least c.p.e.

Theorem 3.3. *An S -gap shift is SFT if and only if S is finite or cofinite.*

Proof. If S is finite, then the set of forbidden words is

$$\mathcal{F} = \{10^n 1 : n \in \{0, 1, \dots, \max S\} \setminus S\} \cup \{0^{1+\max S}\}. \tag{3.1}$$

Otherwise,

$$\mathcal{F} = \{10^n 1 : n \in \mathbb{N}_0 \setminus S\}. \quad \square \tag{3.2}$$

Theorem 3.4. *An S -gap shift is sofic if and only if $\Delta(S)$ is eventually periodic.*

Proof. If S is finite, then $X(S)$ is SFT and the statement is obvious. Thus suppose $S = \{s_0, s_1, \dots\}$ is an infinite subset of \mathbb{N}_0 . First we suppose $\Delta(S)$ is eventually periodic. Then

$$\Delta(S) = \{d_0, d_1, \dots, d_{k-1}, \overline{m_1, m_2, \dots, m_l}\} \tag{3.3}$$

where $m_i = s_{k+i-1} - s_{k+i-2}$, $1 \leq i \leq l$. The follower sets are among

$$F(0), \quad F(10^n), \quad 0 \leq n < s_{k-1}, \quad F(10^{s_{k+i-2}+j_i}), \quad 0 \leq j_i \leq m_i - 1, \quad 1 \leq i \leq l; \tag{3.4}$$

which are finite and thus $X(S)$ is sofic. Note that some of the elements in (3.4) may be equal.

Now we suppose $X(S)$ is sofic. So the number of follower sets is finite. On the other hand, a follower set of an S -gap shift is $F(0)$ or $F(10^n)$ for some $n \in \mathbb{N}_0$. Thus there exist p, q ; $p < q$ such that

$$F(10^p) = F(10^q), \quad p < q \tag{3.5}$$

and let q be the first incident that (3.5) holds. Now if we have $\{p + n_1, p + n_2, \dots, p + n_t\} \subseteq S$, $0 \leq n_1 < \dots < n_t \leq q - p - 1$ and $s_m \leq p$ where $s_m = \max\{s \in S : s \leq p\}$, then $\Delta(S)$ is

$$\{s_0, s_1 - s_0, \dots, s_m - s_{m-1}, p + n_1 - s_m, \overline{n_2 - n_1, \dots, n_t - n_{t-1}, q - p + n_1 - n_t}\}. \quad \square \tag{3.6}$$

A shift space X is called *almost sofic* if $h(X) = \sup\{h(Y) : Y \subseteq X \text{ is a sofic subshift}\}$ [19, Definition 6.8].

Theorem 3.5. *Every S -gap shift is almost sofic.*

Proof. If $X(S)$ is sofic, the statement is obvious. Thus we suppose $S = \{n_0, n_1, \dots\}$ is an infinite subset of \mathbb{N}_0 such that $X(S)$ is not sofic. For every $k \geq 0$, we define $S_k = \{n_0, n_1, \dots, n_k\}$. Then for all k , $X(S_k)$ is a sofic subsystem of $X(S)$ and $\{h(X(S_k))\}_{k \geq 0}$ is an increasing sequence.

On the other hand, the entropy of an S -gap shift is $\log \lambda$ where λ is a non-negative solution of the $\sum_{n \in S} \lambda^{-(n+1)} = 1$ [19]. Thus $h(X(S_k)) \nearrow h(X(S))$ which implies $X(S)$ is almost sofic. \square

An irreducible sofic shift is called *almost-finite-type* (AFT) if it has a bi-closing presentation [13]. Since any sofic shift is a factor of an SFT, it is clear that an AFT is sofic. Nasu [16] showed that an irreducible sofic shift is AFT if and only if its minimal right-resolving presentation is left-closing.

Theorem 3.6. *An S -gap shift is AFT if and only if $\Delta(S)$ is eventually constant.*

Proof. Suppose $\Delta(S)$ is not eventually constant which implies $S = \{s_0, s_1, \dots\}$ is an infinite subset of \mathbb{N}_0 . We show that $X(S)$ is not AFT by showing that the minimal right-resolving presentation of $X(S)$ is not left-closing.

We may assume that $X(S)$ is sofic, then the number of follower sets is finite and by Theorem 3.4, $\Delta(S)$ is eventually periodic. Suppose

$$\Delta(S) = \{d_0, d_1, \dots, d_{k-1}, \overline{m_1, m_2, \dots, m_l}\} \tag{3.7}$$

where $l \geq 2$ and $m_i = s_{k+i-1} - s_{k+i-2}$, $1 \leq i \leq l$. So,

$$\mathcal{C} = \{F(0), F(1), F(10), \dots, F(10^{s_{k-1}}), F(10^{s_{k-1}+1}), \dots, F(10^{s_{k+l-2}+r})\} \tag{3.8}$$

is the set of follower sets of $X(S)$ where $0 \leq r < m_l$. (Some of the elements of \mathcal{C} may be equal where without loss of generality we assume they are all different.) It is an easy exercise to see that the minimal right-resolving presentation of $X(S)$ is the labeled subgraph of the follower set graph consisting of only the follower sets of synchronizing words [13, Problem 3.3.4]. Note that the minimal right-resolving presentation is a labeled graph $\mathcal{G} = (G, \mathcal{L})$ with vertices $\mathcal{C} \setminus \{F(0)\}$ and the edges and labels defined by

$$\mathcal{L}(e(v_1, v_2)) = \begin{cases} 1 & v_1 = F(10^s), v_2 = F(1), & s \in S; \\ 0 & v_1 = F(10^j), v_2 = F(10^{j+1}), & 0 \leq j \leq s_{k+l-2} + r - 1; \\ 0 & v_1 = F(10^{s_{k+l-2}+r}), v_2 = F(10^{u_0}), & u_0 \in [0, s_{k-1}]; \end{cases} \tag{3.9}$$

where $e(v_1, v_2)$ is an edge with $i(e(v_1, v_2)) = v_1$ and $t(e(v_1, v_2)) = v_2$. Now consider any two disjoint points x and y in X_G with $x_{[0,\infty)} = y_{[0,\infty)}$, $t(x_{-1}) = t(y_{-1}) = F(1)$ and $x_{(-\infty,-1]} = (\dots, e(F(10^{s_{k-1}}), F(1)))$, $y_{(-\infty,-1]} = (\dots, e(F(10^{s_k}), F(1)))$ so that $\mathcal{L}_\infty(x) = \mathcal{L}_\infty(y)$. (A choice for x and y can have $\mathcal{L}_\infty(x) = \mathcal{L}_\infty(y) = 0^\infty 10^\infty$.) This implies that the minimal right-resolving presentation of $X(S)$ is not left-closing which is absurd. So $l = 1$.

Now suppose $X(S)$ is not SFT and $\Delta(S)$ is eventually constant. The sets $\Delta(S)$, \mathcal{C} and the minimal right-resolving presentation will be as above for $l = 1$. Hence there is only one edge with label 1, say e , such that $F(10^{u_0}) \leq i(e) \leq F(10^{s_{k-1}+r})$.

To prove the theorem, we show that the minimal right-resolving presentation is left-closing by giving a delay. There exists only one inner edge corresponding to all vertices except $F(1)$ and $F(10^{u_0})$. Therefore, it is sufficient to consider these vertices.

The label of a path of length $u_0 + 1$ ending at $F(10^{u_0})$ determines its terminal edge. On the other hand, $F(1)$ has k inner edges such that $F(10^{s_{n-1}})$ is the initial vertex of the n th edge, $1 \leq n \leq k$. The label of a path of length $s_{k-1} + 2$ ending at $F(1)$ determines its terminal edge. So the minimal right-resolving presentation has $D = \max\{u_0, s_{k-1} + 1\}$ as its delay and the proof is complete. \square

Set \mathcal{F} to be a finite collection of words over a finite alphabet \mathcal{A} where each $w_j \in \mathcal{F}$ is associated with a non-negative integer index n_j . Write

$$\mathcal{F} = \{w_1^{(n_1)}, w_2^{(n_2)}, \dots, w_{|\mathcal{F}|}^{(n_{|\mathcal{F}|})}\} \tag{3.10}$$

and associate with the indexed list \mathcal{F} a period T , where T is a positive integer satisfying $T \geq \max\{n_1, n_2, \dots, n_{|\mathcal{F}|}\} + 1$.

A shift space X is a shift of *periodic-finite-type* (PFT) if there exists a pair $\{\mathcal{F}, T\}$ with $|\mathcal{F}|$ and T finite so that $X = X_{\{\mathcal{F}, T\}}$ is the set of bi-infinite sequences that can be shifted such that the shifted sequence does not contain a word $w_j^{n_j} \in \mathcal{F}$ starting at any index m with $m \bmod T = n_j$. We call a PFT *proper* when it cannot be represented as an SFT.

Let \mathcal{G} be the minimal right-resolving presentation of an irreducible sofic shift, $p = \text{per}(A_G)$ and D_0, D_1, \dots, D_{p-1} the period classes of \mathcal{G} . An indexed word $w^{(n)} = (w_0, w_1, \dots, w_{l-1})^{(n)}$ is a *periodic first offender of period class n* if $w \notin \bigcup_{I \in D_n} F_G(I)$ but for all $i, j \in [0, l-1]$ with $i \leq j$ and $w_{[i,j]} \neq w$, $w_{[i,j]} \in \bigcup_{I \in D_{(n+i) \bmod p}} F_G(I)$. An irreducible sofic shift is PFT if and only if the set of periodic first offenders is finite [5, Corollary 14].

Theorem 3.7. *Suppose $X(S)$ is not SFT. Then it is a proper PFT if and only if it is AFT and non-mixing.*

Proof. Suppose $X(S)$ is proper PFT. Since it is irreducible; it must be AFT [15, Theorem 2]. If it is mixing, then $\text{per}(A_G) = 1$ [13, Exercise 4.5.16] where \mathcal{G} is the minimal right-resolving presentation of $X(S)$ and this contradicts [15, Proposition 1] which states that if \mathcal{G} is an irreducible presentation of X_G and X_G be a proper periodic-finite-type shift with $X_G = X_{\{\mathcal{F}, T\}}$, then $\text{gcd}(\text{per}(A_G), T) \neq 1$.

Now suppose $X(S)$ is AFT and non-mixing. Consider its minimal right-resolving presentation as in the second part of the proof in Theorem 3.6. Let $p = \text{per}(A_G)$ and let D_0, D_1, \dots, D_{p-1} be the period classes of \mathcal{G} . Since every edge with label 1 ends at $F(1)$, so the initial vertices of these edges belong to a period class. Without loss of generality, we assume that it is D_{p-1} . Then it is obvious that $\{1^{(0)}, 1^{(1)}, \dots, 1^{(p-2)}\} \subseteq \mathcal{O}$ where \mathcal{O} is the collection of all periodic first offenders. To prove the theorem, we show that $\mathcal{O} = \{1^{(0)}, 1^{(1)}, \dots, 1^{(p-2)}\}$ and hence finite. Suppose $w = (w_0, w_1, \dots, w_{l-1}) \in \mathcal{O} \setminus \{1^{(0)}, 1^{(1)}, \dots, 1^{(p-2)}\}$. Then $w \notin \bigcup_{I \in D_n} F_G(I)$ for some $0 \leq n \leq p-1$. So for all m , $w \neq 0^m$. Now if $w_i = 1$ where $0 \leq i \leq l-1$, then $1 \notin D_{(n+i) \bmod p}$ which implies $w \notin \mathcal{O}$. \square

We end this section notifying that S -gap shifts are all synchronized. This can be deduced directly by showing that 1 is a synchronizing word or as a result from the fact that if a system has countable follower sets then it is synchronized [11].

4. A topology on the set of all S-gap shifts

The following theorem provides a necessary and sufficient condition for two S-gap shifts being conjugate.

Theorem 4.1. *Let S and S' be two different subsets of \mathbb{N}_0 . Then $X(S)$ and $X(S')$ are conjugate if and only if one of the S and S' is $\{0, n\}$ and the other $\{n, n + 1, n + 2, \dots\}$ for some $n \in \mathbb{N}$.*

Proof. Suppose $S = \{0, n\}$ and $S' = \{n, n + 1, n + 2, \dots\}$ for some $n \in \mathbb{N}$. Define $\Phi : \mathcal{B}_n(X(S)) \rightarrow \{0, 1\}$,

$$\Phi(w) = \begin{cases} 1 & w = 0^n, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Now let $\phi : X(S) \rightarrow X(S')$ be the sliding block code with memory 0 and anticipation $n - 1$ induced by Φ . Then ϕ defines a conjugacy map.

To prove the necessity, consider two cases.

1. Both of S and S' are finite or infinite and suppose that $S \neq S'$. Set $s_0 = \min\{s \in \mathbb{N} : s \in S \Delta S'\}$. Then $q_{s_0+1}(X(S)) \neq q_{s_0+1}(X(S'))$ (q_n denotes the number of points of least period n) which contradicts the conjugacy of two systems.

2. The set S is finite and S' is infinite. We have $0^\infty \in X(S')$ and hence by conjugacy, $0 \in S$. But then $0 \notin S'$; otherwise, 1^∞ is also in $X(S')$ which in turn implies $1 = q_1(X(S)) \neq q_1(X(S')) = 2$. If $S \cap S' = \emptyset$, then a proof as in the case 1 shows that $X(S)$ and $X(S')$ cannot be conjugate. Set $r_0 = \min\{s \in \mathbb{N} : s \in S \cap S'\}$. Then $(110^{r_0})^\infty$ is a point of period $r_0 + 2$ in $X(S) \setminus X(S')$. To have a point in S' with the same period, that is $r_0 + 2$, we must have $(10^{r_0+1})^\infty \in X(S')$ which implies $r_0 + 1 \in S' \setminus S$.

By the same reasoning, $(1^r 0^{r_0})^\infty \in X(S) \setminus X(S')$ where $r \geq 3$ and hence $r_0 + r - 1 \in S' \setminus S$. By an induction argument $\{r_0 + 1, r_0 + 2, \dots\} \subseteq S' \setminus S$. Again by conjugacy and considering periodic points, $(S \Delta S') \cap \{1, 2, \dots, r_0 - 1\} = \emptyset$. \square

For a dynamical system (X, T) , let p_n be the number of periodic points in X having period n . When $p_n < \infty$, the zeta function $\zeta_T(t)$ is defined as

$$\zeta_T(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n} t^n\right). \tag{4.2}$$

In particular, $\zeta_{\sigma_S}(t) = \frac{1}{1-2t}$ when $S = \mathbb{N}_0$ or equivalently when $X(S)$ is full shift on $\{0, 1\}$. Therefore, for any S-gap shift the radius of convergence in (4.2) is positive and hence

$$\left. \frac{d^n}{dt^n} \log \zeta_T(t) \right|_{t=0} = n! \frac{p_n}{n} = (n-1)! p_n. \tag{4.3}$$

Corollary 4.2. *Suppose S and S' are two different non-empty subset of \mathbb{N}_0 . Then S and S' are conjugate if and only if they have the same zeta function.*

Proof. Suppose $S = \{0, n\}$ and $S' = \{n, n + 1, n + 2, \dots\}$ for some $n \in \mathbb{N}$. Then a direct computation shows that

$$\zeta_{\sigma_S}(t) = \zeta_{\sigma_{S'}}(t) = t^{n+1} - t^n - 1. \tag{4.4}$$

Hence by above theorem it suffices to assume $S \neq S'$ and neither is $\{0, n\}$. Let $r = \min\{s \in \mathbb{N} : s \in S \Delta S'\}$, then $r \neq 0$ and $p_{r+1}(X(S)) \neq p_{r+1}(X(S'))$ and by (4.3), they have different zeta functions. \square

Let $S = \{s_i \in \mathbb{N} \cup \{0\} : 0 \leq s_i < s_{i+1}, i \in \mathbb{N} \cup \{0\}\}$ and let $\Delta(S) = \{d_n\}_n$ where $d_0 = s_0$ and $d_n = s_n - s_{n-1}$. Assign to $X(S)$ the real number $x_S = [d_0; d_1, d_2, \dots]$. So by Theorem 4.1, there exists a one-to-one correspondence between the S-gap shifts and $\mathcal{R} = \mathbb{R}^{\geq 0} \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$ up to conjugacy. Equip the collection of S-gaps with the subspace topology and the Lebesgue measure induced from \mathbb{R} . Then an easy observation is that rationals in \mathcal{R} will represent S-gap shifts which are SFT (Theorem 3.3) while quadratic irrationals represent sofic ones (Lagrange's Theorem [9] and Theorem 3.4).

Remark 4.3. We have chosen $\Delta(S)$ (not directly S) to define the topology. For by earlier results, $\Delta(S)$ is more compatible with the dynamics of $X(S)$. Moreover, S is a set and $\Delta(S)$ is a sequence and writing x_S in terms of S , say $x_S = [s_0; s_1, s_2, \dots]$, then we will miss all those real numbers which have some equal partial quotients in their continued fraction expansions.

Note that relative to the topology defined for the collection of the S-gap shifts, the mixing S-gaps are dense in that space. We have actually more:

Theorem 4.4. *The set of mixing S -gap shifts is an open dense subset of the space of S -gap shifts.*

Proof. Let $x_0 \in \mathbb{R}$ and suppose $x_0 = [a_0; a_1, a_2, \dots]$. Then the subset of \mathbb{N}_0 representing x_0 is

$$S = \{a_0; a_0 + a_1, a_0 + a_1 + a_2, \dots\}. \quad (4.5)$$

If $X(S)$ is mixing, then $\gcd\{(\sum_{i=0}^k a_i) + 1: k \in \mathbb{N}_0\} = 1$ [12, Example 3.4]. On the other hand, two real numbers are close if sufficiently large number of their partial quotients are equal. So we can select $\epsilon > 0$, so that for every $x \in B(x_0, \epsilon)$, the S -gap shift representing x is mixing. \square

Theorem 4.5. *The set of non-mixing S -gaps is a Cantor dust (a nowhere dense perfect set).*

Proof. It is fairly easy to see that an S -gap shift is non-mixing if and only if $\gcd\{s + 1: s \in S\} > 1$ [12, Example 3.4]. Suppose $x_S = [d_0; d_1, \dots]$ corresponds to a non-mixing S -gap and the set U is a neighborhood of x_S . We have $x_{S'} = [d_0; d_1, \dots, d_{N-1}, d_{N+1}, \dots] \in U$ for sufficiently large N and $X(S')$ is non-mixing. So all points of the set of non-mixing S -gaps are limit points of themselves which shows the set of non-mixing S -gaps is a perfect set. By Theorem 4.4, the proof is complete. \square

The entropy of an S -gap shift is $\log \lambda$ where λ is a non-negative solution of the $\sum_{n \in S} x^{-(n+1)} = 1$ [19].

Theorem 4.6. *The map assigning to an S -gap shift its entropy is continuous.*

Proof. Let $S = \{s_0, s_1, s_2, \dots\} \subseteq \mathbb{N}$ and $\{S^{(n)}\}_n$ be a sequence in the space of all S -gap shifts such that $S^{(n)} \rightarrow S$. By definition of x_S , two S -gap shifts are close if their respective x_S 's are close. Then $x_{S^{(n)}} = [s_0^{(n)}; s_1^{(n)} - s_0^{(n)}, \dots] \rightarrow x_S = [s_0; s_1 - s_0, \dots]$ and

$$\sum_{s^{(n)} \in S^{(n)}} x^{-(s^{(n)}+1)} \rightarrow \sum_{s \in S} x^{-(s+1)}. \quad \square$$

An irrational number $x = [a_0; a_1, \dots]$ is called *badly approximable* if there is some bound M with the property that $a_n \leq M$ for all $n \geq 0$. We will denote the set of badly approximable numbers by \mathcal{B} . These numbers cannot be approximated very well by rationals [9, Proposition 3.10]. The set \mathcal{B} is uncountable and has measure Lebesgue zero and Hausdorff dimension 1 on the real line [1].

Theorems 4.4 and 4.5 show the frequency of mixing and non-mixing S -gap shifts. In the next theorem, we bring about frequencies of other classes. Before that we recall that a shift space X has the *specification with variable gap length* if there exists $N \in \mathbb{N}$ such that for all $u, v \in \mathcal{B}(X)$, there exists $w \in \mathcal{B}(X)$ with $uwv \in \mathcal{B}(X)$ and $|w| \leq N$ [12].

Remark 4.7. As we mentioned in the introduction, this property was named almost specification in [12]. Our terminology was proposed by referee who brought to our attention that “almost specification” is used for other meanings in the literature [21,22]. For instance, based on the definition of almost product specification given in [17], Climenhaga and Thompson give a symbolic version of that definition and call it almost specification [8]. Then they show that there is an S -gap shift which does not have his version of almost specification property.

Theorem 4.8. *In the space of all S -gap shifts,*

- (1) *The SFT S -gap shifts are dense.*
- (2) *The AFT S -gap shifts which are not SFT, are dense.*
- (3) *The sofic S -gap shifts which are not AFT, are dense.*
- (4) *An S -gap shift has specification with variable gap length if and only if $x_S \in \mathcal{B}$. (S -gap shifts having this property are uncountably dense with measure zero.) Here x_S is the real number assigned to $X(S)$ ($S \neq \{0, n\}$, $n \in \mathbb{N}$).*

Proof. The proof for (1) is obvious and (2) follows from the fact that between any two rationals, one can find an irrational whose partial quotients $\{d_n\}_n$ are constant when n is sufficiently large. Proof of (3) is similar. (4) follows from [12, Example 3.4]. There it has been shown that an S -gap has specification with variable gap length if and only if $\sup_n |s_{n+1} - s_n| < \infty$. \square

Briefly, a rational number, quadratic irrational and a badly approximable real number in \mathcal{R} represents an SFT, a sofic and an specification with variable gap length S -gap shift respectively.

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