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Logarithmic order and type of indeterminate moment problems II

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ABSTRACT

We show that there exist indeterminate Stieltjes moment problems with prescribed common logarithmic order and type.

1. Introduction and results

The set of solutions to an indeterminate moment problem on the real line can be given via Nevanlinna's parametrization. See [4]. In this parametrization four entire functions (usually denoted A, B, C and D) appear and it is known that these functions have a common order (equal to at most 1) and common type. See [5]. In the case where the common order is equal to 0, a refined growth scale, logarithmic order and logarithmic type, can be applied. It has recently been proved that in this situation, the functions have the same logarithmic order and type. See [1]. This paper is a continuation of [1] and we refer the reader thereto for background material. For the readers' convenience we give the definitions of logarithmic order and type:

For an entire function f of order zero the *logarithmic order* $\rho = \rho_f$ is defined as

$$\rho = \inf\{\alpha > 0 \mid \log M(f, r) \leq (\log r)^{\alpha} \text{ eventually}\},$$

where $M(f, r)$ denotes the maximum modulus of $f(z)$ in the closed ball $|z| \leq r$. When $\rho < \infty$ we define the *logarithmic type* $\tau = \tau_f$ as

$$\tau = \inf\{\beta > 0 \mid \log M(f, r) \leq \beta(\log r)^{\rho} \text{ eventually}\}.$$

Indeterminate moment problems of order 0 are often related to q-special functions. The logarithmic order of the (Stieltjes) moment problem associated with the q-Meixner polynomials is 2 (and the logarithmic type is $-1/(4\log q)$). The moment problem associated with the Continuous q^{-1}-Hermite polynomials has logarithmic order 2 and logarithmic type $-1/\log q$. (See again [1].) These results made us wonder if there is any restriction on the values of the logarithmic order and type of indeterminate moment problems. The goal of this paper is to show that there is an indeterminate moment problem with any prescribed logarithmic order and type.

In Section 3 we remark that also for any prescribed ordinary order in $(0, 1/2)$ and positive and finite ordinary type there is an indeterminate Stieltjes moment problem of this growth.

To formulate the main result the following notation and results are needed. A positive measure μ on the real line is called Nevanlinna extremal if it generates an indeterminate moment problem and if the polynomials are dense in the space $L^2(\mu)$.

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It is known that the support of a Nevanlinna extremal measure is exactly the zero set of a certain linear combination of the functions B and D in Nevanlinna's parametrization. (It was proved in [1] that all linear combinations of B and D have the common logarithmic order and type.)

We denote by δ_λ the point mass at λ. For an increasing sequence $\{\lambda_n\}$ of positive numbers such that $\sum_{n=1}^\infty \lambda_n^{-1}$ converges we consider the measure

$$\mu = \sum_{n=1}^\infty \frac{1}{|f'(\lambda_n)|^2}\delta_{\lambda_n}, \tag{1}$$

where

$$f(z) = \prod_{n=1}^\infty \left(1 - \frac{z}{\lambda_n}\right)$$

is the canonical product having the sequence $\{\lambda_n\}$ as its zero set.

Theorem 1.1. *The following hold:*

(a) *For $\lambda_n = n^{\log n}$ the measure in (1) is indeterminate and Nevanlinna extremal. The moment problem is of order 0 and logarithmic order ∞.*
(b) *For $\lambda_n = a^{n^b}$, with $a > 1$ and $b > 0$, the measure in (1) is indeterminate and Nevanlinna extremal. The moment problem is of logarithmic order $1 + 1/b$ and logarithmic type $(\log a)^{-1/b}/(1 + 1/b)$.*
(c) *For $\lambda_n = e^{e^n}$ the measure in (1) is indeterminate and Nevanlinna extremal. The moment problem is of logarithmic order 1 and infinite logarithmic type.*
(d) *For $\lambda_n = e^{(n\log n)^b}$, where $b > 1$, the measure in (1) is indeterminate and Nevanlinna extremal. The moment problem is of logarithmic order $1 + 1/b$ and zero logarithmic type.*
(e) *For $\lambda_n = e^{(n/\log n)^b}$, where $b > 1$, the measure in (1) is indeterminate and Nevanlinna extremal. The moment problem is of logarithmic order $1 + 1/b$ and infinite logarithmic type.*

In paper [2] necessary and sufficient conditions on a positive measure are given in order that it be indeterminate and Nevanlinna extremal. We have based our results on their result below (Theorem 1.2) providing sufficient conditions for Nevanlinna extremality.

The *Hamburger class* $\mathcal{H}$ consists of all real transcendental entire functions of zero exponential type with only real and simple zeros Λ such that

$$\lim_{|\lambda|\to\infty, \lambda\in\Lambda} \frac{|\lambda|^n}{|f'(\lambda)|} = 0$$

for any $n \geq 0$.

Theorem 1.2 (*Theorem C in [2]*). *Let f be a function of the Hamburger class with zero set Λ such that $\sum_{\lambda\in\Lambda} 1/|\lambda| < \infty$ and such that for some constant $M > 0$ there is $C > 0$ such that*

$$|\lambda - \lambda'| \geq C(1 + |\lambda|)^{-M},$$

for all $\lambda, \lambda' \in \Lambda$, $\lambda \neq \lambda'$.

If

$$\liminf_{|\lambda|\to\infty, \lambda\in\Lambda} \frac{\log|f'(\lambda)|}{\log f^\#(|\lambda|)} > 0,$$

where $f^\#(r) = \prod_{\lambda\in\Lambda}(1 + r/|\lambda|)$ then the measure

$$\mu = \sum_{\lambda\in\Lambda} \frac{1}{|f'(\lambda)|^2}\delta_\lambda$$

is indeterminate and Nevanlinna extremal.

Remark 1.3. The sequences Λ in the present paper are all positive in which case $f^\#(r) = M(f, r)$. Furthermore, $\sum_{\lambda\in\Lambda} 1/|\lambda| < \infty$ and $|\lambda - \lambda'| \geq \text{Const}$.

In Borichev and Sodin's paper other results depending on proximate orders could also be used (Theorem B and Theorem D). In order to keep our application of their results relatively simple we have postponed the discussion of proximate orders to the end of the present paper. Let us remark that the starting point for their investigations was an attempt in [3] to characterize the Nevanlinna extremal measures.

2. Asymptotic results for a class of entire functions

In this section we suppose that the zeros of an entire function are of the form $\{A(n)\}_{n\geq 1}$, where A is an increasing C^1-function defined on $[0,\infty)$ such that $A(0)>0$. We consider the following conditions:

$$\frac{A(t)}{t} \quad \text{increases eventually,} \tag{2}$$

$$\frac{A(t)}{t}\to\infty \quad \text{as } t\to\infty, \tag{3}$$

$$\frac{A'(t)t}{A(t)}\to\infty \quad \text{as } t\to\infty, \tag{4}$$

$$\int_0^\infty \frac{\mathrm{d}t}{A(t)}<\infty. \tag{5}$$

Notice that by partial integration and (3),

$$\int_0^\infty \frac{A'(t)t}{A(t)^2}\mathrm{d}t=\int_0^\infty \frac{\mathrm{d}t}{A(t)}<\infty.$$

Lemma 2.1. *Suppose that the positive C^1-function A satisfies (3)–(5). Then A has the following properties*

$$A(r)\int_r^\infty \frac{\mathrm{d}t}{A(t)}=o(r),$$

$$\log A(r)=o\left(\int_0^r \frac{A'(t)t}{A(t)}\mathrm{d}t\right),$$

$$r=o\left(\int_0^r \frac{A'(t)t}{A(t)}\mathrm{d}t\right)$$

as r tends to infinity.

Proof. Let ϵ be given. Choose r_0 such that

$$\frac{A(r)}{A'(r)r}\leq\epsilon$$

for all $r\geq r_0$. This gives first of all for $r\geq r_0$

$$\begin{aligned}\int_r^\infty \frac{\mathrm{d}t}{A(t)} &= \int_r^\infty \frac{A(t)A'(t)t}{A(t)^2A'(t)t}\mathrm{d}t\\ &\leq \epsilon\int_r^\infty \frac{A'(t)t}{A(t)^2}\mathrm{d}t\\ &= \epsilon\left(\frac{r}{A(r)}+\int_r^\infty\frac{\mathrm{d}t}{A(t)}\right).\end{aligned}$$

Hence

$$(1-\epsilon)\int_r^\infty \frac{\mathrm{d}t}{A(t)}\leq\epsilon\frac{r}{A(r)},$$

or, equivalently,

$$A(r)\int_r^\infty \frac{\mathrm{d}t}{A(t)}\leq\frac{\epsilon}{1-\epsilon}r.$$

The first assertion in the lemma is verified.

To show that

$$\lim_{r\to\infty}\frac{\log A(r)}{\int_0^r \frac{A'(t)t}{A(t)}\mathrm{d}t}=0$$

we use L'Hospital's rule:

$$\lim_{r\to\infty}\frac{A'(r)/A(r)}{A'(r)r/A(r)}=0.$$

The third assertion is also verified using L'Hospital's rule. □

Proposition 2.2. *Let* $f(z) = \prod_{k=1}^{\infty}(1 - z/A(k))$*, where the function A has the properties* (3)–(5)*. Then*

$$\log M(f, A(r)) = \int_0^r \frac{tA'(t)}{A(t)} \mathrm{d}t + o\left(\int_0^r \frac{tA'(t)}{A(t)} \mathrm{d}t\right).$$

Proof. We have

$$\log M(f, A(r)) = \sum_{k=1}^{\infty} \log\left(1 + \frac{A(r)}{A(k)}\right).$$

Since $k \mapsto \log(1 + A(r)/A(k))$ decreases we have

$$\int_1^{\infty} \log\left(1 + \frac{A(r)}{A(t)}\right) \mathrm{d}t \le \log M(f, A(r)) \le \int_0^{\infty} \log\left(1 + \frac{A(r)}{A(t)}\right) \mathrm{d}t.$$

Furthermore, using that $A(t) \ge A(0)$ for all t,

$$\begin{aligned}\int_0^1 \log\left(1 + \frac{A(r)}{A(t)}\right) \mathrm{d}t &= \log A(r) + \int_0^1 \log\left(\frac{1}{A(r)} + \frac{1}{A(t)}\right) \mathrm{d}t \\ &\le \log A(r) + \log \frac{2}{A(0)}\end{aligned}$$

so

$$\log M(f, A(r)) = \int_0^{\infty} \log\left(1 + \frac{A(r)}{A(t)}\right) \mathrm{d}t + O(\log A(r)).$$

We proceed to investigate the integral in this relation. First of all

$$\begin{aligned}\int_0^r \log\left(1 + \frac{A(r)}{A(t)}\right) \mathrm{d}t &= \int_0^r \log \frac{A(r)}{A(t)} \mathrm{d}t + \int_0^r \log\left(1 + \frac{A(t)}{A(r)}\right) \mathrm{d}t \\ &\le \int_0^r \log \frac{A(r)}{A(t)} \mathrm{d}t + (\log 2)r.\end{aligned}$$

Here, integration by parts shows that

$$\int_0^r \log \frac{A(r)}{A(t)} \mathrm{d}t = \int_0^r \frac{A'(t)t}{A(t)} \mathrm{d}t.$$

Furthermore,

$$\int_r^{\infty} \log\left(1 + \frac{A(r)}{A(t)}\right) \mathrm{d}t \le A(r) \int_r^{\infty} \frac{\mathrm{d}t}{A(t)},$$

which by Lemma 2.1 is $o(r)$.

We conclude that

$$\begin{aligned}\log M(f, A(r)) &= \int_0^r \frac{A'(t)t}{A(t)} \mathrm{d}t + O(r) + O(\log A(r)) \\ &= \int_0^r \frac{A'(t)t}{A(t)} \mathrm{d}t + o\left(\int_0^r \frac{A'(t)t}{A(t)} \mathrm{d}t\right),\end{aligned}$$

again using Lemma 2.1. □

Remark 2.3. For an entire function f (with $f(0) \neq 0$) of genus 0 and zero counting function $n(t)$ one has

$$N(r) \le \log M(f, r) \le N(r) + Q(r),$$

where

$$N(r) = \int_0^r \frac{n(t)}{t} \mathrm{d}t, \quad \text{and} \quad Q(r) = r \int_r^{\infty} \frac{n(t)}{t^2} \mathrm{d}t.$$

In the proposition above we have computed $N(A(r))$ and verified that $Q(r) = o(N(r))$.

Proposition 2.4. *Let $f(z)=\prod_{k=1}^{\infty}(1-z/A(k))$, where the function A has the properties (2)–(5). Then*

$$\log|f'(A(n))|=\int_0^n \frac{tA'(t)}{A(t)}\mathrm{d}t+o\left(\int_0^n \frac{tA'(t)}{A(t)}\mathrm{d}t\right).$$

Proof. It readily follows that

$$\log|f'(A(n))|=\sum_{k=1}^{n-1}\log\left(\frac{A(n)}{A(k)}-1\right)+\sum_{k=n+1}^{\infty}\log\left(1-\frac{A(n)}{A(k)}\right)-\log A(n).$$

Since $t\mapsto \log(A(n)/A(t)-1)$ decreases for $t\in[0,n]$ we have

$$\int_1^n \log\left(\frac{A(n)}{A(t)}-1\right)\mathrm{d}t\le\sum_{k=1}^{n-1}\log\left(\frac{A(n)}{A(k)}-1\right)\le\int_0^{n-1}\log\left(\frac{A(n)}{A(t)}-1\right)\mathrm{d}t.$$

Now,

$$\int_1^n \log\left(\frac{A(n)}{A(t)}-1\right)\mathrm{d}t=\int_1^n\log\frac{A(n)}{A(t)}\mathrm{d}t+\int_1^n\log\left(1-\frac{A(t)}{A(n)}\right)\mathrm{d}t.$$

For a given positive number ϵ we choose n_0 such that $A(t)/(A'(t)t)\le\epsilon$ for all $t\ge n_0$. Then $1/A'(A^{-1}(s))\le\epsilon A^{-1}(s)/s$ for $s\ge A(n_0)$. Since the integrands in the formulas below are negative we get

$$\begin{aligned}\int_{n_0}^n \log\left(1-\frac{A(t)}{A(n)}\right)\mathrm{d}t&=\int_{A(n_0)}^{A(n)}\log\left(1-\frac{s}{A(n)}\right)\frac{\mathrm{d}s}{A'(A^{-1}(s))}\\&\ge\epsilon\int_{A(n_0)}^{A(n)}\log\left(1-\frac{s}{A(n)}\right)\frac{A^{-1}(s)}{s}\mathrm{d}s\\&\ge\epsilon n\int_{A(n_0)}^{A(n)}\log\left(1-\frac{s}{A(n)}\right)\frac{\mathrm{d}s}{s}\\&\ge\epsilon n\int_0^1\log(1-s)\frac{\mathrm{d}s}{s}=-\epsilon n\frac{\pi^2}{6}.\end{aligned}$$

Hence

$$\begin{aligned}\int_1^n \log\left(\frac{A(n)}{A(t)}-1\right)\mathrm{d}t&\ge\int_1^n\log\frac{A(n)}{A(t)}\mathrm{d}t+\int_1^{n_0}\log\left(1-\frac{A(t)}{A(n)}\right)\mathrm{d}t-\epsilon n\frac{\pi^2}{6}\\&\ge\int_0^n\log\frac{A(n)}{A(t)}\mathrm{d}t-\int_0^1\log\frac{A(n)}{A(t)}\mathrm{d}t+(n_0-1)\log\left(1-\frac{A(n_0)}{A(n)}\right)-\epsilon n\frac{\pi^2}{6}\\&=\int_0^n\log\frac{A(n)}{A(t)}\mathrm{d}t+O(\log A(n))+o(n).\end{aligned}$$

Furthermore,

$$\begin{aligned}\int_0^{n-1}\log\left(\frac{A(n)}{A(t)}-1\right)\mathrm{d}t&\le\int_0^n\log\frac{A(n)}{A(t)}\mathrm{d}t+\int_0^{n-1}\log\left(1-\frac{A(t)}{A(n)}\right)\mathrm{d}t\\&\le\int_0^n\log\frac{A(n)}{A(t)}\mathrm{d}t,\end{aligned}$$

and we conclude that

$$\sum_{k=1}^{n-1}\log\left(\frac{A(n)}{A(k)}-1\right)=\int_0^n\log\frac{A(n)}{A(t)}\mathrm{d}t+O(\log A(n))+o(n).$$

Next we consider the sum

$$\sum_{k=n+1}^{\infty}\log\left(1-\frac{A(n)}{A(k)}\right).$$

The summands are negative and increasing (for $k\ge n+1$) and therefore

$$\int_n^{\infty}\log\left(1-\frac{A(n)}{A(t)}\right)\mathrm{d}t\le\sum_{k=n+1}^{\infty}\log\left(1-\frac{A(n)}{A(k)}\right)\le 0.$$

To estimate the integral in this relation we define $K(n) = A^{-1}(2A(n))$ and split it into two parts:

$$\int_n^\infty \log\left(1-\frac{A(n)}{A(t)}\right)\mathrm{d}t = \int_n^{K(n)} \log\left(1-\frac{A(n)}{A(t)}\right)\mathrm{d}t + \int_{K(n)}^\infty \log\left(1-\frac{A(n)}{A(t)}\right)\mathrm{d}t.$$

Here,

$$\begin{aligned}\int_n^{K(n)} \log\left(1-\frac{A(n)}{A(t)}\right)\mathrm{d}t &= \int_{A(n)}^{2A(n)} \log\left(1-\frac{A(n)}{s}\right)\frac{\mathrm{d}s}{A'(A^{-1}(s))}\\ &\geq \epsilon\int_{A(n)}^{2A(n)} \log\left(1-\frac{A(n)}{s}\right)\frac{A^{-1}(s)}{s}\mathrm{d}s\\ &\geq \epsilon K(n)\int_{A(n)}^{2A(n)} \log\left(1-\frac{A(n)}{s}\right)\frac{\mathrm{d}s}{s}\\ &= \epsilon K(n)\int_1^2 \log\left(1-\frac{1}{s}\right)\frac{\mathrm{d}s}{s},\end{aligned}$$

and since $K(n) \leq 2n$ (this is obtained using that $A(t)/t$ increases) it follows that indeed

$$\int_n^{K(n)} \log\left(1-\frac{A(n)}{A(t)}\right)\mathrm{d}t = o(n).$$

Furthermore, since $\log(1-x) \geq -(2\log 2)x$ for $x \in [0, 1/2]$, it follows that

$$\begin{aligned}\int_{K(n)}^\infty \log\left(1-\frac{A(n)}{A(t)}\right)\mathrm{d}t &\geq -(2\log 2)A(n)\int_{K(n)}^\infty \frac{\mathrm{d}t}{A(t)}\\ &\geq -(2\log 2)A(n)\int_n^\infty \frac{\mathrm{d}t}{A(t)},\end{aligned}$$

and hence by the lemma above this yields altogether

$$\log|f'(A(n))| = \int_0^n \log\frac{A(n)}{A(t)}\mathrm{d}t + O(\log A(n)) + o(n). \quad \square$$

Theorem 2.5. *Let $f(z) = \prod_{k=1}^\infty (1 - z/A(k))$, where the function A has the properties (2)–(5). Then $f \in \mathcal{H}$ and*

$$\lim_{n\to\infty} \frac{\log|f'(A(n))|}{\log M(f, A(n))} = 1.$$

Proof. It follows from Proposition 2.4 and Lemma 2.1 that $f \in \mathcal{H}$. That the limit above is equal to 1 follows from Propositions 2.2 and 2.4. $\square$

Remark 2.6. The values of the function A on any finite interval do not play any role in the result above.

Proof of Theorem 1.1. In each of the cases (a)–(e) in the theorem the function $A(t)$, for which $\{A(n)\} = \{\lambda_n\}$, is eventually positive and increasing, and the relations (2)–(4) hold. The relation (5) also holds on e.g. the interval $[2, \infty)$. Combining Theorem 2.5, 1.2 and the remark above the discrete measures listed in (a)–(e) are all indeterminate and Nevanlinna extremal.

It remains to compute the logarithmic order and type of each of the canonical products. This follows easily from Proposition 2.2. For $A(t) = a^{t^b}$ we find

$$\log M(f, A(r)) \sim \int_0^r \frac{A'(t)t}{A(t)}\mathrm{d}t = \log a\frac{b}{b+1}r^{b+1}$$

(where f is the corresponding canonical product), so that (with $R = A(r)$)

$$\log M(f, R) \sim \frac{b}{b+1}(\log a)^{-1/b}(\log R)^{1+1/b}.$$

From this relation it follows that the logarithmic order is $1 + 1/b$ and that the logarithmic type is $(b/(b+1))(\log a)^{-1/b}$.

Similarly, if $A(t) = \mathrm{e}^{(\log t)^2}$ then $\log M(f, A(r)) \sim 2r\log r$ so that

$$\log M(f, R) \sim 2\sqrt{\log R}\,\mathrm{e}^{\sqrt{\log R}}.$$

Therefore the logarithmic order is infinite.

For $A(t) = e^{e^t}$ it follows that $\log M(f, A(r)) \sim re^r$ so that $\log M(f, R) \sim \log R \log\log R$. This results in logarithmic order equal to 1 and infinite logarithmic type.

For $A(t) = e^{(t\log t)^b}$ (resp. $A(t) = e^{(t/\log t)^b}$) it follows that $\log M(f, A(r)) \sim b/(b+1)(r\log r)^{1+1/b}/\log r$ (resp. $\sim b/(b+1)(r/\log r)^{1+1/b}\log r$). This results in logarithmic order equal to $1+1/b$ and zero (resp. infinite) logarithmic type. □

Remark 2.7. A function ρ defined on $(0, \infty)$ is called a proximate order if

$$\lim_{r\to\infty} \rho(r) \geq 0 \quad \text{and} \quad \lim_{r\to\infty} r\rho'(r)\log r = 0.$$

A proximate order ρ is a proximate order for an entire function f if

$$\tau = \limsup_{r\to\infty} \frac{\log M(f, r)}{r^{\rho(r)}} \in (0, \infty),$$

and in this case the number τ is called the type of f relative to the proximate order ρ. The growth of $\log M(f, r)$ is thus compared with the growth of the function $V(r) = r^{\rho(r)}$. See [6]. In [2, Theorem D] there is a result about proximate orders:

Let ρ be a proximate order and suppose that $\lim_{r\to\infty} \rho(r) \leq 1/2$, that V is increasing and that $V(r) = o(r^{1/2})$.

Suppose furthermore that $\Lambda = \{\lambda_n\} \subset (0, \infty)$ and that $\lambda_n/\Phi(n)$ is an increasing sequence, where Φ is the inverse function to V. Then Λ is the zero set of some Hamburger class function f and the measure

$$\mu = \sum_{n=1}^{\infty} \frac{1}{|f'(\lambda_n)|^2}\delta_{\lambda_n}$$

is an indeterminate Nevanlinna extremal measure.

In order to keep our results relatively simple we have chosen to find the asymptotic relation of both the functions f and f' (in Propositions 2.2 and 2.4). However, let us briefly indicate how the results in the present paper can also be obtained by the use of [2, Theorem D]. Proposition 2.4 yields a proximate order for the functions in question, namely ρ defined as

$$r^{\rho(r)} = V(r) = \int_0^r \frac{A'(t)t}{A(t)}dt.$$

To show that $A(n)/\Phi(n)$ increases one can make the change of variable $n = V(r)$ and show that $A(V(r))/r$ increases. Now,

$$\frac{A(V(r))}{r} = \frac{A(V(r))}{V(r)}\frac{V(r)}{r},$$

where the first fraction increases (since $V(r)$ and $A(x)/x$ increase). The second fraction increases if $V'(r)r - V(r) \geq 0$ and this is the same as

$$\int_0^r \frac{A'(t)t}{A(t)}dt \leq r\frac{A'(r)r}{A(r)}.$$

This is certainly the case if the function $A'(t)t/A(t)$ increases.

3. Ordinary growth

Even though the focus in the present paper is an investigation of moment problems of ordinary order equal to zero we briefly describe the situation of positive ordinary order. We limit ourselves to the construction of the support of a Nevanlinna extremal measure of any prescribed order $\rho \in (0, 1/2)$ and any prescribed type $\tau \in (0, \infty)$.

Proposition 3.1. *Let $\beta > 0$ and $\alpha > 2$. The sequence $\lambda_n = \beta n^\alpha$, $n \geq 1$ is the support of a Nevanlinna extremal measure and the corresponding Stieltjes moment problem is of order $1/\alpha$ and type $\beta(\pi/\alpha)/\sin(\pi/\alpha)$.*

This result is merely a reformulation of the remarks in [2, Appendix 2].

If one considers the corresponding symmetric Hamburger moment problem it follows that any positive order in (0, 1) and any positive and finite type can appear as the common order and type in an indeterminate moment problem.

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