Coherence for Categories with Group Structure: An Alternative Approach

MIGUEL L. LAPLAZA

Department of Mathematics, University of Puerto Rico, Mayagüez, Puerto Rico 00708

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INTRODUCTION

A category with group structure is a monoidal category with a contravariant functor *, natural isomorphisms $I \to A^* \otimes A$, $I \to A \otimes A^*$ and coherence axioms for the commutativity of some diagrams. The structure is abelian if the category is symmetric monoidal and additional coherence axioms are satisfied.

In a recent paper K.-H. Ulbrich [3] has studied the coherence of these categories and proved that any arrow composed of elementary arrows of the structure depends only upon the formal expression of its domain and codomain: any diagram commutes if the vertices and edges are such expressions and composites, respectively. This result is similar to that of S. Mac Lane for monoidal categories in [2].

We present here an alternative way for the study of the coherence of these categories. This paper was written after we read [3] and became convinced that the interest and applicability of the results could justify a more conceptual treatment of the topic which could make it more accessible. For the case of the categories with group structure we use essentially a "diamond lemma" argument after some necessary transformations: a similar argument was used in [2]. For the symmetric case the coherence result is a corollary of [1]: we provide only the convenient background. This approach gives some insight into the relation between categories with group structure and compact categories which can be explored more deeply when, and if, the applications require it. We also detail some elementary results and consequences which are related to our treatment.

In the last part of the paper we show that a group structure on a monoidal category can be given only by the assignment for any object $A$ of an object...
$A^*$ and an isomorphism $j: I \to A^* \otimes A$, without any axiom, naturality or functoriality required. The structure defined from these data turns out to be abelian only for the symmetric monoidal categories whose symmetries $c: A \otimes B \to B \otimes A$ are identities whenever $A = B$.

As general background for the paper we have assumed some familiarity with the theory of monoidal categories. For a good understanding of our arguments a working knowledge of coherence for monoidal categories is sufficient and for this the best reference that we can give is [2].

1. BASIC CONCEPTS ON CATEGORIES WITH GROUP STRUCTURE

We will recall some basic terminology for monoidal categories. A monoidal category is a category $\mathcal{M}$ with the structure provided by a functor $\otimes: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, an object $I$ and natural isomorphisms

$$a_{A,B,C}: (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad r_A: A \otimes I \to A, \quad l_A: I \otimes A \to A,$$

making commutative some types of diagrams.

A symmetric monoidal category is a monoidal category with an additional natural isomorphism $c_{A,B}: A \otimes B \to B \otimes A$, called symmetry and additional commutativity for diagrams.

We will now fix some terminology. Let $f$ be an arrow of a monoidal category: an expanded instance of $f$ is any combination by $\otimes$ of $f$ with identities. Examples of expanded instances of $f$ are $f, 1 \otimes f, f \otimes 1, 1 \otimes (f \otimes 1), \ldots$. Any expanded instance of an arrow of type $a, l, r$, identities or their inverses is called a monoidal arrow. If we are dealing with a symmetric monoidal category the expanded instance of $c$ will also be considered to be monoidal.

We will make the convention of representing with the same symbol an arrow and its inverse whenever such abuse of formal language is harmless. In the use of parenthesis we will use the same criterion.

A category with a group structure or simply a gs-category is a monoidal category $\mathcal{G}$ with a functor $*: \mathcal{G}^{op} \to \mathcal{G}$ and natural isomorphisms $j_A: A^* \otimes A \to I$, $i_A: A \otimes A^* \to I$, making commutative the diagrams

$$(A \otimes A^*) \otimes A \xrightarrow{\alpha} A \otimes (A^* \otimes A)$$

$$\begin{array}{ccc}
I \otimes A & \xrightarrow{l} & A \\
\downarrow & & \downarrow \quad (1) \otimes j \\
I & \xleftarrow{r} & I.
\end{array}$$

The above commutativities are the coherence axiom (1) of the gs-structure.
A category with an abelian group structure or simply an ags-category is a symmetric monoidal category with a group structure for which the diagrams

\[
\begin{array}{ccc}
A \otimes A^* & \xrightarrow{f} & I \\
\downarrow c & & \downarrow i \\
A^* \otimes A & \xrightarrow{g} & A \otimes A^*
\end{array}
\]

are commutative. This is coherence axiom (2).

The relation \( c_{A,A} = 1_A \otimes A: A \otimes A \to A \otimes A \) is imposed in [3] for the coherence of an ags-category, but it is a consequence of our axioms; we will see it as a corollary. One can also find a direct proof of it by means of a diagram-chase argument. The results of the second part of Satz 2 of [3] suggest other possible axiomatizations of ags-categories.

The following proposition is an immediate consequence of the definitions.

1.1. PROPOSITION. For any object \( A \) of a gs-category \( \mathcal{G} \), the functors \( - \otimes A, A \otimes -: \mathcal{G} \to \mathcal{G} \) are equivalences with "quasi-inverses" \( - \otimes A^* \) and \( A^* \otimes - \), respectively.

1.2. COROLLARY. For any object \( A \) of a gs-category \( \mathcal{G} \) the set of arrows \( \mathcal{G}(A,A) \) is a commutative monoid for the composition of arrows.

Proof. It is easy to check that the canonical map given by

\[
\mathcal{G}(I,I) \to \mathcal{G}(I \otimes A, I \otimes A) \to \mathcal{G}(A,A)
\]

commutes with the composition of arrows and it is well-known that \( \mathcal{G}(I,I) \) is commutative for composition because \( \mathcal{G} \) is monoidal. This map is a bijection because both arrows above are so: the left one by the preceding proposition and the other by its definition.

1.3. PROPOSITION. In any gs-category \( \mathcal{G} \) the diagrams

\[
\begin{array}{ccc}
A^* \otimes A \otimes A^* & \xrightarrow{a} & A^* \otimes (A \otimes A^*) \\
\downarrow j \otimes 1 & & \downarrow 1 \otimes l \\
I \otimes A^* \xrightarrow{1 \otimes l} A & \xrightarrow{r} & A^* \otimes I
\end{array}
\]

are commutative.

Proof. Consider the canonical natural bijections described by the composites.
p: \( \mathcal{F}(A \otimes B, C) \to \mathcal{F}(A^* \otimes A \otimes B, A^* \otimes C) \to \mathcal{F}(B, A^* \otimes C) \),

\( q: \mathcal{F}(B, A^* \otimes C) \to \mathcal{F}(A \otimes R, A \otimes A^* \otimes C) \to \mathcal{F}(A \otimes B, C) \).

By using Yoneda one can check that \( qp: \mathcal{F}(A \otimes B, C) \to \mathcal{F}(A \otimes B, C) \) is the identity and because \( p \) and \( q \) are bijections the composite

\[ pq: \mathcal{F}(B, A^* \otimes C) \to \mathcal{F}(A \otimes B, C) \to \mathcal{F}(B, A^* \otimes C) \]

is also the identity. If one computes the formal expression of \( pq(r: A^* \to A^* \otimes I) \) one can see easily that the statement of the proposition is equivalent to \( pq(r_{A^*}) = r_{A^*} \), and we are done.

1.4. **Proposition.** In any gs-category \( \mathcal{F} \) the action of the functor \( * \) can be described as the composite,

\[ \mathcal{F}(A, B) \to \mathcal{F}(A^* \otimes A \otimes B^*, A^* \otimes B \otimes B^*) \to \mathcal{F}(B^*, A^*). \]

**Proof.** By Yoneda it is enough to check that the composite takes identities onto identities and this is equivalent to the statement of Proposition 1.3. From this proposition one can see that for any arrow \( f: A \to B, f \) can be expressed as the composite represented by \( B^* \to I \otimes B^* \to A^* \otimes A \otimes B^* \to A^* \otimes B \otimes B^* \to A^*. \)

1.5. **Proposition.** In a gs-category \( \mathcal{F} \) there exists a natural isomorphism \( u_A: A \to A^{**} \) making commutative the diagrams

\[
\begin{array}{ccc}
A \otimes A^* & \xrightarrow{i} & A^* \otimes A \\
\downarrow{\jmath} & & \downarrow{1 \otimes u} \\
A^{**} \otimes A^* & & A^* \otimes A^{**}.
\end{array}
\]

**Proof.** By means of Yoneda, Proposition 1.3 and the coherence axiom (1) one can show the existence of the commutative diagrams

\[
\begin{array}{ccc}
\mathcal{F}(A, B) & \to & \mathcal{F}(I, A^* \otimes B) \\
\downarrow & & \downarrow \\
\mathcal{F}(I, B \otimes A^*) & \to & \mathcal{F}(A^*, A^* \otimes B \otimes A^*), \\
\mathcal{F}(A^{**}, B) & \to & \mathcal{F}(I, A^* \otimes B) \\
\downarrow & & \downarrow \\
\mathcal{F}(I, B \otimes A^*) & \to & \mathcal{F}(A^*, A^* \otimes B \otimes A^*)
\end{array}
\]
whose edges are canonical bijections: therefore they are pullbacks hence there is a bijection \( \mathcal{F}(A, B) \to \mathcal{F}(A**, B) \) for which the diagram

\[
\begin{array}{ccc}
\mathcal{F}(I, A^* \otimes B) \\
\uparrow \\
\mathcal{F}(A**, B) \longrightarrow \mathcal{F}(A, B) \\
\downarrow \\
\mathcal{F}(I, B \otimes A^*)
\end{array}
\]

is commutative. All the bijections are natural hence by Yoneda the horizontal one is of type \( \mathcal{F}(u_A, 1): \mathcal{F}(A**, B) \to \mathcal{F}(A, B) \) with \( u_A: A \to A** \) an isomorphism. The commutativity of the two partial subdiagrams is equivalent by Yoneda to the last part of the proposition. One can use this proposition to obtain a concrete description of \( u_A \) such as the composite

\[
A \to I \otimes A \to A** \otimes A^* \otimes A \to A** \otimes I \to A**.
\]

1.6. COROLLARY. (i) The arrows \( i_A \) can be expressed as composites of expanded instances of \( j \) and monoidal arrows. (ii) The functor \( * \) is an equivalence and any arrow \( f** \) can be expressed as a composite of monoidal arrows and expanded instances of \( f \) and \( j \).

Proof. Part (i) is a consequence of (1.5). Part (ii) is a consequence of (i) and 1.4. Notice that in the construction of an expanded instance the functor \( * \) is not involved.

1.7. PROPOSITION. In any gs-category the diagrams

\[
\begin{array}{ccc}
A \otimes A^* & \xrightarrow{i} & A^* \otimes A \\
\downarrow u \otimes u & & \downarrow u \otimes u \\
A** \otimes A*** & \xrightarrow{i} & A*** \otimes A**
\end{array}
\]

are commutative.

Proof. By Proposition 1.5 we have

\[
i_A \cdot (u_A \otimes u_A) = i_A \cdot (1 \otimes u_A) \cdot (u_A \otimes 1) = j_A \cdot (u_A \otimes 1) = i_A.
\]

The proof for the other diagrams is similar.
We will later need a natural arrow \( b_{A,B} : (A \otimes B)^* \to B^* \otimes A^* \) making commutative the diagram

\[
\begin{array}{c}
(A \otimes B)^* \otimes A \otimes B \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
B^* \otimes I \otimes B \\
\quad \downarrow \\
I \\
\quad \downarrow \\
B^* \otimes B.
\end{array}
\]

An immediate consequence of Proposition 1.1 is the existence and unicity of such an arrow.

2. The Coherence of GS-Categories

Let \( \mathcal{G} \) be a gs-category. The object \( I \) and the functors \( *, \otimes \) define operations in \( \text{Ob} \mathcal{G} \) making it a \( \{I, *, \otimes\} \)-algebra. We will say that \( \mathcal{G} \) is free for a set of objects \( S \) if \( \text{Ob} \mathcal{G} \) is the free \( \{I, *, \otimes\} \)-algebra over \( S \): this means essentially that any object of \( \mathcal{G} \) can be expressed uniquely as a well-formed combination of elements of \( S \), parenthesis and symbols \( I, *, \otimes \).

When \( \mathcal{G} \) is a gs-category free for \( S \) we can define a map \( F : \text{Ob} \mathcal{G} \to \text{Ob} \mathcal{G} \) and for each object \( A \) an isomorphism \( f_A : A \to FA \). They are determined by the following rules:

1. \( FX = X \) and \( f_X \) is the identity for \( X \) in \( S \) or \( X = I \).
2. \( F \) is a morphism for \( \otimes \), \( F(A \otimes B) = FA \otimes FB \), and \( f_{A \otimes B} = f_A \otimes f_B \).
3. The action of \( F \) on an object of type \( W^* \) is described separately for the cases \( W \) in \( S \), \( W = I \), \( W = A \otimes B \) and \( W = A^* \) by taking \( F(X^*) = X \) for \( X \) in \( S \), \( F(I^*) = I \), \( F(A \otimes B)^* = F(B^*) \otimes F(A^*) \) and \( F(A^{**}) = FA \). In this situation \( f_{X^*} \) is the identity and the remaining arrows \( f \) are given by the composites

\[
\begin{align*}
    f_{I^*} &= jr : I^* \otimes I \to I, \\
    f_{(A \otimes B)^*} &= (f_{B^*} \otimes f_{A^*}) b_{A,B} : (A \otimes B)^* \to B^* \otimes A^* \to FB^* \otimes FA^*, \\
    f_{A^{**}} &= f_A u_A : A^{**} \to A \to FA,
\end{align*}
\]

with the definitions of \( u \) and \( b \) given in Proposition 1.5 and after Proposition 1.7, respectively: we also use our convention on the identification of symbols for an arrow and its inverse.
We can extend $F$ to a functor $F: \mathcal{S} \to \mathcal{S}$ if we define $F(g: A \to B)$ by the commutativity of the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f_A} & FA \\
\downarrow g & & \downarrow Fg \\
B & \xrightarrow{f_B} & FB.
\end{array}
$$

Clearly $F$ is a functor and one can easily check the following properties:

(i) $F$ is monoidal, that is, $FI = I$, $Fa = a$, $Fl = l$, $Fr = r$, $F(g \otimes h) = Fg \otimes Fh$.

(ii) For arrows $u$ and $b$ as before $Fu$ and $Fb$ are identities.

(iii) $FA = A$ if and only if $A$ is a $\otimes$-product of $I$ and elements of type $X, X^*$ with $X$ in $S$.

(iv) If $FA = A$ then $f_A$ is an identity.

(v) The functor $F$ is an equivalence and $FF = F$.

2.1. **Lemma.** For $F$ defined as before $Fj_A$ can be expressed as a composite of monoidal arrows and expanded instances of $i_X, j_X$ with $X$ in $S$.

**Proof.** With the definition of $F$ for the arrows $Fj$ is determined by the commutativity of

$$
\begin{array}{ccc}
A^* \otimes A & \xrightarrow{f_A \otimes f_A} & FA^* \otimes FA \\
\downarrow j_A & & \downarrow Fj_A \\
I & & I.
\end{array}
$$

It follows immediately from the definitions that for $X$ in $S$, $Fj_X = j_X$. If we take into account Proposition 1.4 we can also get $Fj_X^* = i_X$ for $X$ in $S$, and by also considering the coherence axiom (i) we can easily obtain the relations $Fj_I = Fj_I = l_I = r_I$. For the general case we will use induction on the number of symbols in the expression of $A$. When $A = U \otimes V$ by the definition of $b$ we have the relation

$$
j_{U \otimes V} = j_V(1 \otimes l)(1 \otimes j_U \otimes 1)(b_{U,V} \otimes 1 \otimes 1)
$$

and by properties (i) and (ii) of $F$, $Fj_{U \otimes V} = (1 \otimes Fj_U \otimes 1)$ and we can apply the induction hypothesis. If $A = (U \otimes V)^*$ and we use the naturality of $\gamma$ we obtain

$$
Fj_{(U \otimes V)^*} = j_{V^\vee \otimes U^\vee}(b^* \otimes b):
$$

$$(U \otimes V)^* \otimes (U \otimes V)^* \to (V^* \otimes U^*)^* \otimes (V^* \otimes U^*) \to I,$$
so that $F_{j(U \otimes V)} = F_{jV \otimes U}$, and after the consideration of the previous case we can apply the induction hypothesis. For the case $A = V^*$ we apply Proposition 1.5 to get $f_{jV} = (u_{V^*} \otimes 1) i_{V^*} = (u_V \otimes 1)(1 \otimes u_V) f_V$ hence $F_{jV^*} = F_{jV}$ and we are within the induction hypothesis.

Let $G_{\text{scat}}$ be the category with the gs-categories for objects and with arrows the functors preserving strictly the gs-structure. The forgetful functor, $\text{Ob}: G_{\text{scat}} \to \text{Sets}$, which takes a category onto the set of its objects, has a left adjoint, $G_s: \text{Sets} \to G_{\text{scat}}$. For any set $S$ the category $G_s(S)$ will be called the free gs-category over $S$ and we are going to describe it in terms of "generators and relations." The description is long, straightforward and rather deceptive because it hides the structural simplicity of $G_s(S)$. One cannot avoid this presentation because it appears in the usual concrete situations which require the application of coherence theory.

The objects of $G_s(S)$ are the elements of the free $(I, *, \otimes)$-algebra over $S$, that is, they are the words of a formal language composed according to the following axioms: (a) The symbol $I$ and each element of $S$ are objects of $G_s(S)$. (b) If $X$ and $Y$ are objects of $G_s(S)$ then $X \otimes Y$ is an object of $G_s(S)$. (c) If $X$ is an object of $G_s(S)$ then $X^*$ is an object of $G_s(S)$.

We now describe the edges of a graph $\mathcal{R}$ with the objects of $G_s(S)$ for vertices. For objects $X, Y, Z$ there are to be edges

$$a_{XYZ}: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z), \quad l_x: I \otimes X \to X, \quad r_x: X \otimes I \to X,$$

$$i_x: X \otimes X^* \to I, \quad j_x: X^* \otimes X \to I.$$

Moreover for each of the above edges we have the reverse edge represented by its addition of $-$ upon its name. The description of $\mathcal{R}$ is completed by the rule that for an edge $t: X \to Y$ there are to be edges $Z \otimes t: Z \otimes X \to Z \otimes Y, \ t \otimes Z: X \otimes Z \to Y \otimes Z, \ t^*: Y^* \to X^*$, and the remark that these edges are supposed to be words in a formal language and are distinct if they have different names. We describe now a relation $\sim$ in the category $\mathcal{L}$, generated by the graph $\mathcal{R}$. The relations that define $\sim$ are as follows. First we have the relations

$$(X \otimes p)(X \otimes q) \sim X \otimes (pq), \quad (p \otimes X)(q \otimes X) \sim (pq) \otimes X,$$

$$(t \otimes W)(X \otimes s) \sim (Y \otimes s)(t \otimes Z),$$

$$1_{X \otimes Y} \sim X \otimes 1_Y \sim 1_X \otimes Y, \quad (1_X)^* \sim 1_X, \quad (pq)^* \sim q^*p^*$$

that we need for $\otimes$ and $*$ to be functors. Then we can describe the relations that assert the naturality of $a$, $a$, $l$, $l$, $r$, $r$, $i$, $i$, $j$, $j$ and that each of them of type $e$ is the inverse of $e$. Next we have the coherence axioms for $a$, $l$, $r$, $i$ and $j$. We come now to all the expansions by $\otimes$ and $*$ of these relations: for each relation $a \sim b$ we include the relations $a \otimes X \sim b \otimes X$, $X \otimes a \sim X \otimes b$, ...
a^* \sim b^*$, and all obtained from them by a finite number of applications of this process. We end by defining $G_\mathcal{S}(S)$ as the quotient of the category $\mathcal{L}$ module the relation $\sim$, that is, module the category congruence generated by $\sim$.

It follows easily from this description that $G_\mathcal{S}(S)$ is a gs-category. Its basic property is that for any gs-category $\mathcal{S}$ and any map $g: S \to \text{Ob } \mathcal{S}$ there is exactly one functor $G: G_\mathcal{S}(S) \to \mathcal{S}$ which is an extension of $g$ and strictly preserves the gs-structure. Moreover, by Corollary 1.6 any arrow of $G_\mathcal{S}(S)$ is a composite of monoidal arrows and expanded instances of $j_\delta$: notice that in such expanded instances we do not use the action of $\ast$.

The objects of $G_\mathcal{S}(S)$ are free for $S$ and therefore we have an idempotent equivalence $F: G_\mathcal{S}(S) \to G_\mathcal{S}(S)$ defined in the beginning of this section.

2.2. Proposition. If $g: A \to B$ is an arrow of $G_\mathcal{S}(S)$ and $FA = A$, $FB = B$, then $g$ can be expressed as a composite of monoidal arrows and expanded instances of $i_x, j_x$ with $X$ in $S$.

Proof. From property (iv) of $F$ it follows easily that $Fg = g$. Moreover, as we have remarked before, $g$ is a composite of monoidal arrows and expanded instances of $j$, and by property (i) of $F$ we are reduced to the case $g = j_\delta$ and for it we can use Lemma 2.1.

We now need a convenient criterion for the existence of an arrow of type $A \to I$. Let $\mathcal{S}$ be the set of finite sequences of elements of type $X, X^*$ with $X$ in $S$. A chain is a formal expression of type $a_1 \to a_2 \to \cdots a_\tau$, where each $a_i$ is in $\mathcal{S}$ and can be obtained from $a_{i-1}$ by the deletion of an occurrence of type $XX^*$ or $X^*X$. A simple argument shows that if we have one-link chains of type $a \to \beta, a \to \gamma$ and $\beta \neq \gamma$, then there exists a diagram of one-link chains of type

$$
\begin{array}{c}
\alpha \\
\downarrow \\
\gamma \\
\downarrow \\
\beta
\end{array}
$$

A reduction is a chain $a_1 \to a_2 \to \cdots a_\tau$ with no occurrences of $XX^*$ or $X^*X$ in $a_\tau$, that is, a reduction is a chain that cannot be extended. An easy consequence of the preceding fact is that the existence of a chain of type $\alpha \to \beta$ makes sure that two reductions with origin in $\alpha$ and $\beta$ have the same end and therefore the end of a reduction is uniquely determined by its origin. We will denote by $\mathcal{S}'$ the set of $\emptyset$ and all the elements of $\mathcal{S}$ which are the origin of a reduction with the end $\emptyset$. It is immediate the equivalence of the following relations: $a\beta \in \mathcal{S}'$, $aXX^*\beta \in \mathcal{S}'$, $aX^*X\beta \in \mathcal{S}'$.

Let $A$ be an object of $G_\mathcal{S}(S)$ such that $FA = A$. By property (iii) of $F$ if we
omit in \( A \) the occurrences of \( \otimes, I \) and parenthesis we obtain an element of \( \mathcal{S}' \) denoted by \( s(A) \).

2.3. Lemma. If \( A \) is an object of \( \mathcal{G}_S(S) \) and \( FA = A \), then \( \mathcal{G}_S(S)(A, I) \neq \emptyset \) if and only if \( s(A) \) is in \( \mathcal{S}' \).

Proof. Suppose that \( g: A \rightarrow I \) is an arrow of \( \mathcal{G}_S(S) \). By Proposition 2.2 \( g \) can be expressed as \( g: A = A_1 \rightarrow A_2 \rightarrow \cdots A_r = I \), where each \( A_i \rightarrow A_{i+1} \) is either a monoidal arrow or an expanded instance of \( j_x \) or \( i_x \) with \( X \) in \( S \) and \( FA_i = A_i, FA_{i+1} = A_{i+1} \). Therefore either \( s(A_i) = s(A_{i+1}) \) or one of \( s(A_i) \rightarrow s(A_{i+1}), s(A_{i+1}) \rightarrow s(A_i) \) is a chain of elements of \( \mathcal{S} \) and it follows immediately that \( s(A) = s(A_i) \) is in \( \mathcal{S}' \). Suppose now that \( s(A) \) is in \( \mathcal{S}' \): after trivial considerations we can assume the existence of an occurrence in \( A \) of type \( X \otimes X^* \) or \( X^* \otimes X \) with \( X \) in \( S \), so that we can find an expanded instance of \( j_x, i_x \) \( g: A \rightarrow B \), such that the number of symbols is lower in \( B \) than in \( A \): a trivial induction shows that \( \mathcal{G}_S(S)(A, I) \neq \emptyset \).

An arrow of \( \mathcal{G}_S(S) \) is simple if it is an identity or a composite of expanded instances of \( a \) or expanded instances of \( r, l, i_x, j_x \) with \( X \) in \( S \) and such that the number of symbols in the codomain of these instances is not larger than in the domain. Therefore, the arrow \( i_x: X \otimes X^* \rightarrow I \) is simple, but the arrow with the same name \( i_x: I \rightarrow X \otimes X^* \) is not simple.

2.4. Lemma. Let \( A \) be an object of \( \mathcal{G}_S(S) \) such that \( FA = A \) and \( \mathcal{G}_S(S)(A, I) \neq \emptyset \). Then there exists a simple arrow \( g: A \rightarrow I \).

Proof. Taking into account Lemma 2.3 a proof is almost a repetition of the second part of the proof of that lemma.

2.5. Lemma. If \( A \) is an object of \( \mathcal{G}_S(S) \), \( FA = A \) and \( g, g': A \rightarrow I \) are simple arrows, then \( g = g' \).

Proof. The basic result of coherence for monoidal categories (2) makes sure that between two objects of \( \mathcal{G}_S(S) \) there exists at most one arrow which can be expressed by composing monoidal arrows, that is, any composite of monoidal arrows is determined uniquely by its domain and codomain. In the proof of this lemma we will omit parenthesis and composites of monoidal arrows: they can be recovered easily.

Suppose now that \( h: A \rightarrow B \) is simple and an expanded instance of \( r \) or \( l \). By induction on the number of symbols in \( A \) one can prove easily that any simple arrow \( g: A \rightarrow I \) can be decomposed as \( g = ph: A \rightarrow B \rightarrow I \), where \( p \) is also simple.

Suppose now that \( k: A \rightarrow C \) is simple and an expanded instance of \( i \) or \( j \). By the same induction on \( A \) we can prove that any simple arrow \( g: A \rightarrow I \) can be decomposed as \( g = qk: A \rightarrow C \rightarrow I \), where \( q \) is also simple. The proof
depends essentially upon the naturality of $\otimes$, the coherence axiom (i) and Proposition 1.3: the last two imply the commutativity of the diagrams

$$
\begin{align*}
X \otimes X^* \otimes X &\longrightarrow X \otimes I \\
I \otimes X &\longrightarrow X,
\end{align*}
\begin{align*}
X^* \otimes X \otimes X^* &\longrightarrow X^* \otimes I \\
I \otimes X^* &\longrightarrow X^*.
\end{align*}
$$

After this we can prove the lemma by induction on the number of symbols occurring in $A$. If $A = I$, then $g = g' = 1_I$. If $A \neq I$ and $I$ has an occurrence in $A$ denote by $h: A \to B$ the expanded instance of $I$ or $r$ that erases this occurrence: both $g$ and $g'$ factor through $h$ and we can use the induction hypothesis on $B$. If no occurrence of $I$ is in $A$, by Lemma 2.3 there is in $A$ an occurrence of type $X \otimes X^*$ or $X^* \otimes X$; denote by $k: A \to C$ the expanded instance of $i_X$ or $j_X$ acting on this occurrence: both $g$ and $g'$ factor through $k$ and we can apply the induction hypothesis to $C$.

2.6. Lemma. Let $A$ be an object of $G_\delta(S)$ such that $FA = A$. Then there is at most one arrow in $G_\delta(S)(A, I)$.

Proof. By Proposition 2.2 any arrow $g: A \to I$ can be decomposed as $g: A = A_1 \to A_2 \to \cdots A_r$ where each $A_i \to A_{i+1}$ or its inverse is simple. By Lemmas 2.4 and 2.5 for each $A_i$ we have a unique simple arrow $A_i \to I$ and the diagrams

$$
\begin{tikzcd}
A_i & A_{i+1} \\
& I
\end{tikzcd}
$$

are commutative. From this it follows that the arrow $g$ coincides with the simple arrow from $A$ onto $I$.

2.7. Coherence Theorem. If $A$ and $B$ are objects of $G_\delta(S)$, there exists at most one arrow from $A$ to $B$.

Proof. We have a sequence of canonical bijections.

$$
G_\delta(S)(A, B) \to G_\delta(S)(A \otimes B^*, B \otimes B^*) \to G_\delta(S)(A \otimes B^*, I),
$$

and we are reduced to the case $B = I$. Moreover $F$ is full and faithful hence we have a bijection $G_\delta(S)(A, I) \to G_\delta(S)(FA, FI)$, and because $FFA = FA$, $FI = I$ we are reduced to the case $A = FA$ and this has been done in Lemma 2.5.
After this theorem we can describe $G_s(S)$ completely. Its objects are the free $\{I, *, \otimes\}$-algebra over $S$. Between two objects $A$ and $B$ of $G_s(S)$ there is at most one arrow and this is so when $sF(A \otimes B^*)$ is in $\mathcal{S}'$, a fact which can be decided effectively by eliminating in $sF(A \otimes B^*)$ all occurrences of type $XX^*$ and $X^*X$: one can get the empty set if and only if $sF(A \otimes B^*)$ is in $\mathcal{S}'$.

We will reformulate the coherence of gs-categories in more elementary terms. We define the *shapes* of objects of a gs-category $\mathcal{C}$ as the words of a formal language whose alphabet are the objects of $\mathcal{C}$, symbols $\ast, \otimes$ and parenthesis. The rules of formation of words are the following: (1) Any object of $\mathcal{C}$ is a shape. (2) If $A$ is a shape so is $(A)^\ast$. (3) If $A$ and $B$ are shapes then $(A) \otimes (B)$ is a shape. Each shape has for "translation" the object of $\mathcal{C}$ obtained by "reading" its symbols in the gs-structure of $\mathcal{C}$.

In an analogous way we describe the formal canonical arrows (or simply fc-arrows) of $\mathcal{C}$. They are words of a formal language and each fc-arrow has a shape for domain and another for codomain. The construction, domain and codomain of the fc-arrows are described by the following rules:

1. For shapes $A, B, C$ the words
   
   \[
   l_A : A \to A, \quad a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad r_A : A \otimes I \to A,
   \]
   
   \[
   l_A : I \otimes A \to A, \quad j_A : A^\ast \otimes A \to I, \quad i_A : A \otimes A^\ast \to I
   \]

   are fc-arrows.

2. If $f : A \to B$ is an fc-arrow so are $(f)^{-1} : A \to B$ and $(f)^\ast : (B)^\ast \to (A)^\ast$.

3. If $f : A \to B$, $f' : A' \to B'$ are fc-arrows, then $(f) \otimes (f') : (A) \otimes (A') \to (B) \otimes (B')$ is an fc-arrow.

We can also translate fc-arrows as arrows of $\mathcal{C}$: the translations preserve domains and codomains.

Our Theorem 2.7 is equivalent to the statement that any two formal canonical arrows with the same domain and codomain have identical translations. Another equivalent version is the following: any diagram of shapes and formal canonical arrows can be translated as a diagram of objects and arrows and such a translation is always a commutative diagram.

3. The Coherence of AGS-Categories

We are going to recall some results on compact categories. They are immediate consequences of more general results of (1) and will be used in our study of the coherence of ags-categories.
A compact category is a symmetric monoidal category \( \mathcal{P} \) with the additional structure of a map, *: Ob \( \mathcal{P} \to \text{Ob} \mathcal{P} \), and for each object \( A \) a pair of arrows, \( d_A: I \to A \otimes A^* \), \( e_A: A^* \otimes A \to I \), such that the composites

\[
r(1 \otimes e) a(d \otimes 1): A \to I \otimes A \to (A \otimes A^*) \otimes A \to A \otimes (A^* \otimes A) \to A \otimes I \to A,
\]

\[
l(e \otimes 1) a(1 \otimes d) r: A^* \to A^* \otimes I \to A^* \otimes (A \otimes A^*) \to (A^* \otimes A) \otimes A^* \to I \otimes A^* \to A
\]

are identities.

If \( \mathcal{A} \) is an ags-category and we take \( d_A = i_A: I \to A \otimes A^* \), \( e_A = j_A: A^* \otimes A \to I \), the coherence theorem 2.7 makes sure that they provide a compact structure for \( \mathcal{A} \).

The next example of a compact category will be used later. Let \( S \) be a set and denote by \( S^* \) the set of elements of \( S \) signed with *, that is, \( S^* = \{ X^* | X \in S \} \). For \( \sigma = (U_1, U_2 \cdots U_r) \) a non-empty sequence of elements of \( S \cup S^* \) we define \( \sigma^* = (Y_1, \ldots, Y_r) \), where \( Y_i = X^* \) if \( U_i = X \) or \( Y_i = X \) if \( U_i = X^* \) with \( X \) in \( S \): for \( \sigma \) the empty sequence we define \( \sigma^* = \emptyset \). Let now \( M \) be the free commutative monoid over \( S \), that is,

\[ M = \{ n_1 X_1 \cdots n_r X_r | X_i \in S, n_i \text{ is a natural number} \}. \]

We will define now a compact category \( \text{Seq}(S) \). Its objects are the finite sequences of elements of \( S \cup S^* \) and the arrows from \( A \) onto \( B \) are pairs \((\theta, p)\), where \( \theta \) is a fixed-point-free involution of the concatenation of \( A^* \) and \( B \), and \( p \) is an element of the monoid \( M \). We can also describe \( \theta \) as a graph of undirected edges between the occurrences of elements of \( S \) in \( A^* \) and \( B \) such that the ends of each edge are different occurrences of the same element of \( S \) with different sign and each element of \( A^* \) or \( B \) is exactly at the end of one edge. For \((\theta, p): A \to B \) and \((\theta', p'): B \to C \) arrows of \( \text{Seq}(S) \) their composite is the arrow \((\theta'', p''): A \to C \), where \( \theta'' \) is the set of all edges whose ends are in \( A \) and in \( C \) obtained by linking alternatively edges of \( \theta \) and \( \theta' \), and \( p'' = p + p' + \sum A_i \), where \( A_i \) is an element of \( S \) and the sigma is extended to all the elements of \( S \) included in a loop made by linking together alternatively edges of \( \theta \) and \( \theta' \); each element appears in \( \sum \) as many times as is included in different loops. The tensor product is defined by concatenation for the objects, by juxtaposition for the first component of the arrows and by addition for the second component of the arrows. The remaining part of the compact structure of \( \text{Seq}(S) \) is defined canonically.

For any set \( S \) we described in Section 2 the free gs-category over \( S \) denoted \( \text{Gs}(S) \). In an analogous way we can construct the free ags-category \( \text{Ags}(S) \) and the free compact category \( \text{Com}(S) \). They both have a similar
universal property: if $\mathcal{A}$ is an ags-category and $\mathcal{C}$ is a compact category, any maps $g': S \to \text{Ob} \mathcal{A}$, $g'': S \to \text{Ob} \mathcal{C}$, can be extended uniquely to functors $G': \text{Ags}(S) \to \mathcal{A}$, $G'': \text{Com}(S) \to \mathcal{C}$, which strictly preserve the structure under consideration. We have to remark that both $\text{ObAgs}(S)$ and $\text{ObCom}(S)$ are the free $\{I, *, \otimes\}$-algebra over $S$. Moreover any arrow of $\text{Ags}(S)$ is a composite of monoidal arrows and expanded instances of $i$ and $j$ without the action of $*$: this can be deduced easily from the construction or the universal property of $\text{Ags}(S)$ and Corollary 1.6.

As we have remarked before the ags-category $\text{Ags}(S)$ is compact and the natural injection $S \to \text{ObAgs}(S)$ extends to a functor $G: \text{Com}(S) \to \text{Ags}(S)$ which is the identity for the objects and preserves the compact structure.

3.1. Lemma. The functor $G: \text{Com}(S) \to \text{Ags}(S)$ is full and the arrows $G(e_Acd_A : I \to A \otimes A^* \to A^* \otimes A \to I)$ are identities.

Proof. The preservation of the compact structure implies

\[
G(d_A : I \to A \otimes A^*) = (i_A : I \to A \otimes A^*),
\]

\[
G(e_A : A^* \otimes A \to I) = (j_A : A^* \otimes A \to I),
\]

and by the coherence axiom (ii),

\[
G(cd_A : I \to A \otimes A^* \to A^* \otimes A) = (ci_A : I \to A \otimes A^* \to A^* \otimes A)
= (j_A : I \to A^* \otimes A),
\]

\[
G(e_A c : A \otimes A^* \to A \otimes X A \to I) = (j_A c : A \otimes A^* \to A^* \otimes A \to I)
= (i_A : A \otimes A \to I),
\]

and therefore all monoidal arrows and expanded instances of $i$ and $j$ are in the image of $G$, which is full. The last statement of the lemma follows from coherence axiom (ii).

We have also a canonical map $h: S \to \text{ObSeq}(S)$ which can be extended to a functor $H: \text{Com}(S) \to \text{Seq}(S)$ which preserves the compact structure. The coherence results of (1) imply that this functor $H$ is full and faithful. One can easily check that $H(I) = \emptyset$, and for $X$ in $S$, $H(e_xcd_x : I \to X \otimes X^* \to X^* \otimes X \to I) = (\emptyset, X)$. Any arrow of $\text{Seq}(S)(\emptyset, \emptyset)$ is of type $(\emptyset, \sum X_i)$ and the preceding relation shows that it is the image by $H$ of an arrow of type $t = m((e_x cd_x) \otimes (e_x cd_x) \otimes \cdots) m'$, where $m: I \to I \otimes I \otimes \cdots I$ and $m' I \otimes I \otimes \cdots I \to I$ are monoidal arrows. The functor $H$ is faithful hence any arrow of $\text{Com}(S)(I, I)$ is of the type of $t$.

3.2. Coherence Theorem. For any objects $A$ and $B$ of $\text{Ags}(S)$ there is at most one arrow from $A$ onto $B$. 
Proof. By Lemma 3.1 the functor \( G: \text{Com}(S) \to \text{Ags}(S) \) is full. We have seen that any arrow \( t: I \to I \) in \( \text{Com}(S) \) is of type \( t = m((e_X,cd_X) \otimes (e_X',cd_X')) m' \). By Lemma 3.1 \( G(t) = G(m) G(m') \), where \( G(m) G(m') \) is a monoidal arrow from \( I \) onto \( I \) hence \( G(t) = 1 \), according to the coherence for monoidal categories [2]. Therefore \( \text{Ags}(I, I) \) has only the identity \( 1_I \). By Proposition 1.1 we have bijections,

\[
\text{Ags}(S)(I, I) \to \text{Ags}(A \otimes I, A \otimes I) \to \text{Ags}(S)(A, A),
\]

and therefore in \( \text{Ags}(S)(A, A) \) is only the identity. Suppose now that \( g, h: A \to B \) are arrows: all the arrows in \( \text{Ags}(S) \) are isomorphism and therefore, \( 1_A = g^{-1}g = g^{-1}h: A \to A \) and \( g = h \).

3.3. Corollary. For any object \( A \) of an ags-category \( \mathcal{A} \), \( c_{A,A} = 1_A \otimes_A: A \otimes A \to A \otimes A \).

Proof. We can construct a category \( \text{Ags}(S) \) and a map \( q: S \to \text{Ob} \mathcal{A} \) such that for \( X \) in \( S \) \( q(X) = A \). For the extension of \( q \), \( Q: \text{Ags}(S) \to \mathcal{A} \), \( Q(c_{X,X}) = c_{A,A} \), and after the Coherence Theorem \( c_{X,X} \) is an identity and so is \( c_{A,A} \).

3.4. Corollary. If \( p, t: A \to B \) are arrows of an ags-category, then \( p \otimes t = t \otimes p \).

Proof. An immediate consequence of Corollary 3.3 and the naturality of \( c \).

We can now describe the category \( \text{Ags}(S) \). Its objects are the free \( \{I, *, \otimes\} \)-algebra over \( S \) and between two of its objects there exists at most one arrow. We will describe a simple method for deciding the existence of such an arrow. Let \( G(S) \) be the free abelian group over \( S: G(S) \) is a discrete ags-category with the group operation \( + \) for \( 0 \), the opposite \( - \) for \( * \) and \( 0 \) for \( I \). The natural injection \( t: S \to G(S) \) can be extended to a functor \( T: \text{Ags}(S) \to G(S) \) that can be described by the rules: (1) \( T(I) = 0 \) and for \( X \) in \( S \), \( T(X) = X \). (2) \( T(A \otimes B) = TA + TB \). (3) \( T(A*) = TA \). (4) For any arrow \( g \) of \( \text{Ags}(S) \), \( T(g) \) is an identity. It is clear that if \( \text{Ags}(S)(A, B) \neq \emptyset \), then \( T(A) = T(B) \). Suppose now that \( T(A) = T(B) \); from this it follows immediately that \( T(FA) = T(FB) \) with \( F \) the functor defined in the beginning of Section 3 and this means that if we disregard the occurrences of \( I \) and the order of such occurrences \( FA \) and \( FB \) have the same occurrences of elements of \( S \) and therefore \( \text{Ags}(FA, FB) \neq \emptyset \) and \( \text{Ags}(A, B) \neq \emptyset \). Therefore we have proved that \( \text{Ags}(A, B) \neq \emptyset \) if and only if \( T(A) = T(B) \). In more sophisticated terms, \( T: \text{Ags}(S) \to G(S) \) is an equivalence.

It is possible to explain the coherence of ags-categories in terms of shapes
and formal canonical arrows: this requires only obvious modifications in the similar exposition for gs-categories included in the end of Section 2.

4. Another Definition of Group Structure on a Category

Let $\mathcal{M}$ be a monoidal category with a map $\ast: \text{Ob}\mathcal{M} \to \text{Ob}\mathcal{M}$ and for each object $A$ an isomorphism $j_A: I \to A^* \otimes A$. No claims of naturality can be made on $j_A$ because $\ast$ is not a functor. We are going to see that this data can be extended to a group-structure on $\mathcal{M}$ in a unique way.

4.1. Proposition. For any object $A$ of $\mathcal{M}$, $A \otimes A^*$ and $I$ are isomorphic.

Proof: We have a chain of isomorphisms, $A \otimes A^* =: I \otimes A \otimes A^* \cong A^{**} \otimes A^* \otimes A \otimes A^* \cong A^{**} \otimes A^* \cong I$.

4.2. Corollary. The functors $A \otimes -$ and $- \otimes A$ are equivalences.

4.3. Proposition. The map $\ast: \text{Ob}\mathcal{M} \to \text{Ob}\mathcal{M}$ can be extended uniquely to a functor $\ast: \mathcal{M}^{\text{op}} \to \mathcal{M}$, in such a way that the arrows $j_A: I \to A^* \otimes A$ are natural.

Proof. The naturality of $j_A: I \to A^* \otimes A$ is equivalent to the commutativity of

$$
\begin{array}{ccc}
I & \xrightarrow{j_A} & A^* \otimes A \\
\downarrow j_B & & \downarrow 1 \otimes f \\
B^* \otimes B & \xrightarrow{f \otimes 1} & A^* \otimes B
\end{array}
$$

for any arrow $f: A \to B$, and by Corollary 4.2 we can use this commutativity for the definition of $f^\ast$. One can check easily that this definition provides a functor.

4.4. Lemma. There exists a natural isomorphism $u_A: A \to A^{**}$.

Proof. We define $u_A$ as the composite

$$r(1 \otimes j) a(j \otimes 1): A \to I \otimes A \to A^{**} \otimes A^* \otimes A \to A^{**} \otimes I \to A^{**}.$$ 

One can check easily the naturality of the definition.
4.5. **Lemma.** There is a natural isomorphism $i_A : I \to A \otimes A^*$ making commutative the diagram

\[
\begin{array}{ccc}
(A \otimes A^*) \otimes A & \xrightarrow{a} & A \otimes (A^* \otimes A) \\
\downarrow i_A \otimes 1 & & \downarrow 1 \otimes i_A \\
I \otimes A & \xleftarrow{1} & A \otimes I.
\end{array}
\]

**Proof.** We define $i_A$ as the composite, $i_A = (u_A \otimes 1)j_A : I \to A^{**} \otimes A^* \to A \otimes A^*$. From the definition of $u$ given in 4.4 there follows the commutativity of

\[
\begin{array}{ccc}
(A^{**} \otimes A^*) \otimes A & \xrightarrow{a} & A^{**} \otimes (A^* \otimes A) \\
\downarrow j \otimes 1 & & \downarrow 1 \otimes j \\
I \otimes A & \xleftarrow{1} & A^{**} \otimes I \\
\downarrow i & & \downarrow r \\
A & \xrightarrow{u} & A^{**}
\end{array}
\]

and the naturality of $a$ implies the commutativity of

\[
\begin{array}{ccc}
(A \otimes A^*) \otimes A & \xrightarrow{a} & A \otimes (A^* \otimes A) \\
\downarrow (u \otimes 1) \otimes 1 & & \downarrow u \otimes 1 \\
(A^{**} \otimes A^*) \otimes A & \xrightarrow{a} & A^{**} \otimes (A^* \otimes A).
\end{array}
\]

For the proof, glue together the two diagrams and use the definition of $i_A$, the naturality of $r$ and the functoriality of $\otimes$.

4.6. **Corollary.** The data given on $\mathcal{M}$ can be extended uniquely to a group-structure.

**Proof.** By Proposition 4.5 the definition of $i_A$ provides a group-structure and it is determined uniquely by $j_A$ according to coherence axiom (1) and Proposition 1.1. Moreover the naturality of $j$ imposes a unique way of extending $*$ to a functor as we have pointed out in 4.3.

Let $\mathcal{E}$ be a monoidal category and denote by $M$ the set of all objects $A$ for which there exist objects $A'$, $A''$ and isomorphism $A' \otimes A \to I$, $A \otimes A'' \to I$. The chains of isomorphisms,

\[
A \otimes A' \approx A \otimes A' \otimes I \approx A \otimes A' \otimes A \otimes A'' \approx A \otimes I \otimes A'' \approx A \otimes A'' \approx I,
\]

\[
A'' \otimes A \approx I \otimes A'' \otimes A \approx A' \otimes A \otimes A'' \otimes A \approx A' \otimes I \otimes A \approx A' \otimes A \approx I,
\]
show that the objects $A'$ and $A''$ are also in $M$ hence for each object $X$ in $M$ one can find another object $Y$ in $M$ and an isomorphism $X \otimes Y \to I$. It is easy to check that $M$ includes $I$ and is closed for $\otimes$. Let $\mathcal{M}$ be the full subcategory of $\mathcal{F}$ with $M$ for objects. After Corollary 4.6 $\mathcal{M}$ has a well-defined $gs$-structure if we fix for each object $A$ of $\mathcal{M}$ an object $A^*$ and an isomorphism $A^* \otimes A \to I$.

4.7. PROPOSITION. If the category $\mathcal{M}$ is symmetric monoidal then the group-structure just defined is abelian if and only if for any object $A$, $c_{A,A} = 1_{A \otimes A} : A \otimes A \to A \otimes A$.

Proof. By Corollary 3.3 the condition is necessary. Suppose now that the arrows $c_{A,A}$ are identities. We redefine the arrows $i_A$ by $i_A = j_A c : A \otimes A^* \to A^* \otimes A \to I$, so that the coherence condition (2) is fulfilled. The coherence of symmetric monoidal categories imposes the commutativity of the diagram

$$
\begin{array}{ccc}
(A \otimes A^*) \otimes A & \xrightarrow{c \otimes 1} & (A^* \otimes A) \otimes A \\
& a \downarrow & \downarrow 1 \otimes c_{A,A} \\
A \otimes (A^* \otimes A) & \xrightarrow{c} & (A^* \otimes A) \otimes A
\end{array}
$$

so that the diagram

$$
\begin{array}{ccc}
(A \otimes A^*) \otimes A & \xrightarrow{c \otimes 1} & (A^* \otimes A) \otimes A \\
& a \downarrow & \downarrow c \\
A \otimes (A^* \otimes A)
\end{array}
$$

is commutative. From this there follows easily the commutativity of the diagram

$$
\begin{array}{ccc}
(A \otimes A^*) \otimes A & \xrightarrow{c \otimes 1} & (A^* \otimes A) \otimes A \\
& a \downarrow & \downarrow c \\
A \otimes (A^* \otimes A) & \xrightarrow{1 \otimes c} & A \otimes I
\end{array}
$$

hence the coherence axiom (1) holds and we have an $ags$-structure. Proposition 1.1 and coherence axiom (1) imply that the arrow $i_A$ is uniquely determined by $j_A$ and therefore the new definition of $i_A$ coincides with the former one.

This proposition shows that the construction detailed after Corollary 4.6 does not necessarily produce an $ags$-category when applied to the symmetric case. We need some preliminary results to describe a convenient
construction. It is well known that in a symmetric monoidal category \( \mathcal{A} \),
\[ c_{I,I} = 1_{I \otimes I} : I \otimes I \to I \otimes I \]
and it follows from the naturality of \( c \) that \( P \simeq I \)
implies \( c_{P,P} = 1_{P \otimes P} : P \otimes P \to P \otimes P \). The coherence of the symmetric
monoidal categories implies the commutativity of a diagram that without
parenthesis and associativities looks like

\[
\begin{array}{ccc}
D \otimes D \otimes E \otimes E & \xrightarrow{c_{D,D} \otimes 1 \otimes 1} & D \otimes D \otimes E \otimes E \\
1 \otimes c_{D,E} \otimes 1 & \downarrow & 1 \otimes c_{D,E} \otimes 1 \\
D \otimes E \otimes D \otimes E & \xrightarrow{c_{D \otimes E, D \otimes E}} & D \otimes E \otimes D \otimes E.
\end{array}
\]

Now if \( c_{D \otimes E, D \otimes E} \) is an identity the relations \( c_{D,D} = 1 \) and \( c_{E,E} = 1 \) are
equivalent in a gs-category, and if \( c_{E,E} \) and \( c_{D,D} \) both are identities so is
\( c_{D \otimes E, D \otimes E} \). We define \( M \) as the set of all the objects \( A \) of \( \mathcal{A} \) such that
\( c_{A,A} = 1_{A \otimes A} \) and that there exist objects \( A', A'' \) and isomorphisms \( A' \otimes A \to I \)
and \( A \otimes A'' \to I \). Taking into account the construction after Corollary 4.6
and the preceding considerations it is easy to show that \( A' \) and \( A'' \) are in \( M \),
so that we can find an object \( A^* \) in \( M \) and an isomorphism \( A^* \otimes A \to I \).
Also \( M \) includes \( I \) and is closed for \( \otimes \): notice that the construction takes
place in the inside of the gs-category defined after Corollary 4.6. Now by
Proposition 4.7 and Corollary 4.6 the full subcategory \( \mathcal{M} \) with objects \( M \)
has a well-defined abelian group-structure if we fix for each object \( A \) an
object \( A^* \) and an isomorphism \( A^* \otimes A \to I \).

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