Bounds on the Spectral Radius of Graphs with \( e \) Edges

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ABSTRACT

We give upper bounds for the spectral radius of a graph with \( e \) edges provided that there is no complete graph with \( e \) edges. Our bounds are sharp for (i) the complete graphs with one, two, or three edges removed; (ii) the complete graph with one added vertex and edge.

Let \( G \) be a undirected graph with \( e \) edges and without loops. Then \( G \) is represented by its adjacency 0-1 symmetric matrix \( A \) with zero trace. Denote by \( \rho(G) = \rho(A) \) the spectral radius of \( G \). It is of interest to find \( \rho(e) \), the maximum spectral radius of all \( G \) having \( e \) edges. Clearly, \( \rho(e) \) is achieved for a connected graph. Moreover \( \rho(e) \) is an increasing function of \( e \). For

\[
e = \frac{m(m + 1)}{2}
\]

(that is, there exists a complete graph on \( e \) edges), it was shown by Brualdi and Hoffman that \( \rho(e) = m \), the spectral radius of the corresponding complete graph. Furthermore, the complete graph is the only maximal graph. Brualdi and Hoffman used a variation of Schwarz's rearrangement theorem to prove the above inequality. They showed that in order to compute \( \rho(e) \) it is

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enough to consider the following graphs:

(i) $G$ has a complete subgraph on $\mu$ vertices, say $\{1, \ldots, \mu\}$.

(ii) All other vertices are connected only to the vertices of the complete subgraph $\{1, \ldots, \mu\}$. Furthermore, it is possible to arrange these vertices $\{\mu + 1, \ldots, n\}$ so that any vertex $i$, $i > \mu + 1$, is connected to a subset of neighbors of the vertex $i - 1$.

The following observation is crucial to our arguments. If $\nu$ is the dimension of the maximal complete subgraph of a graph $G$ satisfying (i)-(ii), then the degree of each vertex that does not lie in the maximal complete subgraph is less than $\nu$. In that case $G$ also satisfies (i)-(ii) with $\mu = \nu - 1$.

In [2] we studied at length the maximal eigenvalues of 0-1 matrices—non-symmetric, and symmetric with 0 diagonal—using Schwarz’s and Brualdi and Hoffman’s rearrangement theorems. In the case of undirected graphs with no loops we showed the following:

For $e = m(m + 1)/2 - 1$ (i.e., there exists a complete graph with one edge removed on $e$ edges), $\rho(e)$ is the spectral radius of this graph [2, Theorem 11, p. 651]. For

$$e = \frac{m(m - 1)}{2} + s,$$

where $s$ is fixed and $m$ big enough, we showed that $\rho(e)$ is achieved for $G$ as described in (i)-(ii) with $\mu = m$, where $G$ is a graph on $m + 1$ vertices [2, Theorem 12, p. 671]. More precisely, $G$ is a complete graph with deleted edges, where all the deleted edges are connected to a common vertex.

We also showed that the spectral radius of a graph of the form (i)-(ii) is bounded by

$$R(\mu, e) = \frac{\mu - 1 + (4e - \mu^2 + 1)^{1/2}}{2}.$$ [2, (7.4)]. Hence for a fixed $e$ of the form (2) with $0 \leq s \leq m$, the expression (3) achieves its maximum for $\mu = m$. The inequality (7.6) states

$$\rho(e) \leq \frac{m - 1 + [(m - 1)^2 + 4s]^{1/2}}{2}.$$ (4)
In a recent paper Stanley [3] gave the following simple bound:

\[ \rho(e) \leq \frac{-1 + (1 + 8e)^{1/2}}{2}. \]  

(5)

This inequality is sharp for \( e = m(m + 1)/2 \). In all other cases (5) is better than (4). The paper of Stanley is elegant, short, and straightforward. The purpose of this paper is to improve Stanley's inequality using the machinery developed by us in [2].

**Theorem 1.** Let \( G \) be an undirected graph with \( e \) edges and without loops. Assume that \( e \) is of the form of (2) with \( 0 < s < m \). Then the spectral radius \( \rho(G) \) of \( G \) satisfies

\[ \rho(G) \leq \frac{m - 2 + (m^2 + 4s)^{1/2}}{2}. \]  

(6)

This inequality is sharp if \( G \) is a complete graph with one edge removed.

**Proof.** We first observe that \( R(\mu, e) \) is a strictly increasing function in \( \mu \) on the interval \([0, (2e + \frac{r}{2})^{1/2}]\) for a fixed \( e \). So

\[ R(\mu, e) \leq R(m - 1, e) \quad \text{for} \quad \mu \leq m - 1. \]

To complete the proof we need to show that \( R(m - 1, e) \) majorizes the spectral radius of the graph \( G \) given by (i)–(ii) with \( \mu = m \). Since \( s < m \), we can assume that vertex \( m + 1 \) is not connected to all the vertices \( 1, \ldots, m \). To be specific, we may suppose that vertex \( m + 1 \) is not connected to vertex \( m \). That is, our graph also satisfies conditions (i)–(ii) with \( \mu = m - 1 \), and the theorem follows.

A straightforward calculation shows that the bound (6) is sharper than (5). We now improve the bound (6) in case \( 0 < s < m - 1 \). To do that we need deeper results established in [2]. Assume that \( G \) is of the form (i)–(ii). Then \( \rho(G) \) is dominated by the unique positive root of the cubic equation:

\[ \frac{\mu}{r + 1} + \frac{t}{(r + 1)[r(r + 1) - s]} - 1 = 0, \]

\[ s = e - \frac{\mu(\mu - 1)}{2}, \quad t = \|Bu\|^2. \]  

(7)
Here $B$ is a 0-1 matrix having $\mu$ columns with $s$ ones, and $u$ is a column vector all whose $\mu$ coordinates are equal to 1 [2, (7.14), p. 67]. Note that the positive root $r(\mu, s, t)$ is an increasing function of all its arguments for positive values of $\mu$, $s$, and $t$. The value $t$ is maximal when $B$ has the minimal number of nonzero rows and the maximal number of rows all filled with ones. In particular

$$t \leq s^2.$$  

Moreover, if $e$ is of the form (2), $0 \leq s \leq m$, then for $\mu = m$ the unique positive root of (7) with $t = s^2$ is equal to $\rho(G)$, where $G$ is the unique graph on $m + 1$ vertices of the form (i)-(ii) with $\mu = m$. We denote by $r(m, e)$ the spectral radius of this $G$. Note that $r(m, e)$ for $e = m(m + 1)/2 - 1$ is the spectral radius of a complete graph on $m + 1$ vertices with one edge removed.

**Theorem 2.** Let $G$ be a graph with $e$ vertices. Assume that $e$ is of the form (2) with $0 < s < m$. Then

$$\rho(G) \leq r(m, e) \quad \text{for} \quad e = \frac{m(m - 1)}{2} + s, \quad s = 1, m - 3, m - 2, m - 1.$$  

For $s = m - 3, m - 2, m - 1$ the equality holds if and only if $G$ is a complete graph with one, two, or three edges removed from exactly one vertex. For $s = 1$ the equality sign holds only for a complete graph with an added vertex and an edge. In all other cases for $m > 6$ we have the inequality

$$\rho(G) \leq \frac{(m - 3) + (m^2 + 2m + 4s - 3)^{1/2}}{2},$$

$$e = \frac{m(m - 1)}{2} + s, \quad 1 < s < m - 3, \quad m > 5.$$  

In general the following inequality holds:

$$\rho(G) \leq \max(R(m - 2, e), r(m, e)).$$
Proof of Theorem 2. The case \( s = m - 1 \) was established in Theorem 1. Assume that \( 0 < s < m - 1 \), and let \( G \) be of the form (i)–(ii). For \( \mu = m - 2 \) we deduce that \( \rho(G) \leq R(\mu, e) \leq R(m - 2, e) \) as in the proof of Theorem 1. For \( \mu = m \) we have the inequality \( \rho(G) \leq r(m, e) \). Assume that \( \mu = m - 1 \). If the degree of some vertex \( k > m - 1 \) is \( m - 1 \), then actually we are in the case \( \mu = m \). If the degrees of all vertices \( k > m - 1 \) are less than \( m - 1 \), we are back in the case \( \mu = m - 2 \). This establishes the inequality (11).

We now prove that

\[
R(m - 2, e) < r(m, e) \quad \text{for} \quad s = 1, m - 3, m - 2. \tag{12}
\]

Let \( R = R(m - 2, e) \). Then \( R \) satisfies the quadratic equation

\[
R^2 = (m - 3)R + 2m - 3 + s. \tag{13}
\]

We now claim that

\[
\frac{m - s}{R + 1} + \frac{sR}{(R + 1)[(R + 1)R - s]} - 1 = \frac{m - s}{R + 1} + \frac{sR}{(R + 1)R - s} - 1
\]

is positive for \( s = 1, m - 3, m - 2 \). Using (13), we have the following expressions for the left-hand side of (14):

\[
\frac{m - s}{R + 1} + \frac{sR}{(m - 2)R + 2m - 3} - 1 = \frac{m - s}{R + 1} + \frac{(2 - m + s)R - 2m + 3}{(m - 2)R + 2m - 3}
\]

\[
= \frac{-R + 2m - 3 + s(2 - m + s)}{(R + 1)[(m - 2)R + 2m - 3]}.
\]

For \( s = 1, m - 3 \) the numerator of the above ratio is \(-R + m\). As \( R < m \), the expression (14) is positive. For \( s = m - 2 \) the numerator of the above expression is \(-R + 2m - 3\), which is positive for \( m \geq 3 \). As \( r(m, e) \) is a unique positive root of a strictly decreasing function for \( r(r + 1) > s \) given in (7), it follows that \( R(m - 2, e) < r(m, e) \) for \( s = 1, m - 3, m - 2 \).

Next note that

\[
2m - 3 + s(2 - m + s) \leq 5 \quad \text{for} \quad 2 \leq s \leq m - 4. \tag{15}
\]

A straightforward calculation shows that \( R(m - 2, e) > 5 \) for \( e \geq 17 \), i.e.,
m ≥ 6 and 2 ≤ s ≤ m. In that case the expression (14) is negative. Hence \( r(m, e) < R(m - 2, e) \) in this case, and (10) follows from (11). The proof of the theorem is concluded.

The above results verify the conjecture stated in [1], that \( \rho(e) \) is always achieved for a complete graph with deleted edges having a common vertex. To prove this conjecture or to improve the inequality (10) one has to use the bound given by the cubic equation (7) for different values of \( p \) as in [2].

REFERENCES


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