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The two largest distances in finite planar sets

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Abstract

We determine all homogenous linear inequalities satisfied by the numbers of occurrences of the two largest distances among n points in the plane.

1. Introduction

Let S be a set of n points in the Euclidean plane. Let N_1 AND n_2 denote the number of times the largest and second largest distances occur among points in S. It is well known [1,2] that $n_1 \leqslant n$ and it was proved in [4] that $n_2 \leqslant 3n/2$. Are there any further relations among these numbers? To formalize this question, we consider the pairs $(n_1/n, n_2/n)$, for every finite set S, and the convex hull C of this set in the plane. Determining this convex hull is tantamount to finding all homogeneous linear inequalities involving n, n_1 and n_2.

In an earlier paper [3] the analogously defined set was described for the case when the points in S form a convex n-gon. We also gave a complete description for the case when the points in S form a convex n-gon in the hyperbolic plane.

In this paper we give a complete set of inequalities determining C, for arbitrary set S in the Euclidean plane.

2. Inequalities

Let S be a set of n arbitrary points in R^2. We denote the largest distance between two points by d_1, the second largest by d_2 and by n_1 resp. n_2 the number of distances equal to d_1 resp. d_2. We give a complete system of homogenous, linear inequalities for n_1, n_2, n in the sense that any homogenous, linear inequality for n_1, n_2, n is a consequence of these.

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Theorem. *For any set of n points in* R^2,

(i) $n_2 \leqslant 3/2n$,

(ii) $n_2 \leqslant n + 2n_1$,

(iii) $n_1 + n_2 \leqslant 2n$.

Let us consider, for arbitrary point systems S in the Euclidean plane, the point $(n_1/n, n_2/n)$ corresponding to S. Let C be the set of these points. For any point in C the inequalities (i), (ii) and (iii) are valid. Furthermore, since for any set S, $n_1 \leqslant n$ [1, 2] and obviously $n_i > 0$. Thus, for the closure of C the following inequalities are valid:

$$
\begin{aligned}
x_1 &\geqslant 0,\\
x_2 &\geqslant 0,\\
x_1 &\leqslant 1,\\
x_2 &\leqslant \tfrac{3}{2},\\
x_1 + x_2 &\leqslant 2,\\
-2x_1 + x_2 &\leqslant 1.
\end{aligned}
$$

We shall show that every vertex of this domain is realized (at least asymptotically) by some point set S. This implies that every linear inequality valid for C is implied by these (see Fig. 1).

Before giving the proof of the theorem we give some definitions and recall some results from [4] (called propositions below).

We call segments of length d_1 *red*, and segments of length d_2 *blue*. Let us consider the convex hull of S. We call a point *outer* if it is on the convex hull, *inner* if it is inside the convex hull. We call a blue edge *inner blue edge* if it has inner endpoint.

Proposition 1. *Both endpoints of a red edge are outer points. At least one endpoint of a blue edge is an outer point.*

A path of length 3 consisting of red and blue edges in which the two end-edges do not intersect is called an 'N'. An 'N' with a red end-edge and a blue middle edge is called *forbidden*.

Proposition 2. *No forbidden 'N' exists.*

Definition. The number of blue and red edges incident with a point u will be called the *blue degree* and *red degree*, respectively, and will be denoted by $d_b(u)$ and $d_r(u)$. For an outer point u, the number of outer (blue) edges starting at u is the *outer degree* of u, the number of inner (blue) edges starting at u is the *inner degree* of u.

Proposition 3. *For every inner point u,* $d_b(u) \leqslant 3$. *If* $d_b(u) = 3$ *then at least one of the neighbors of u has blue degree* 1.

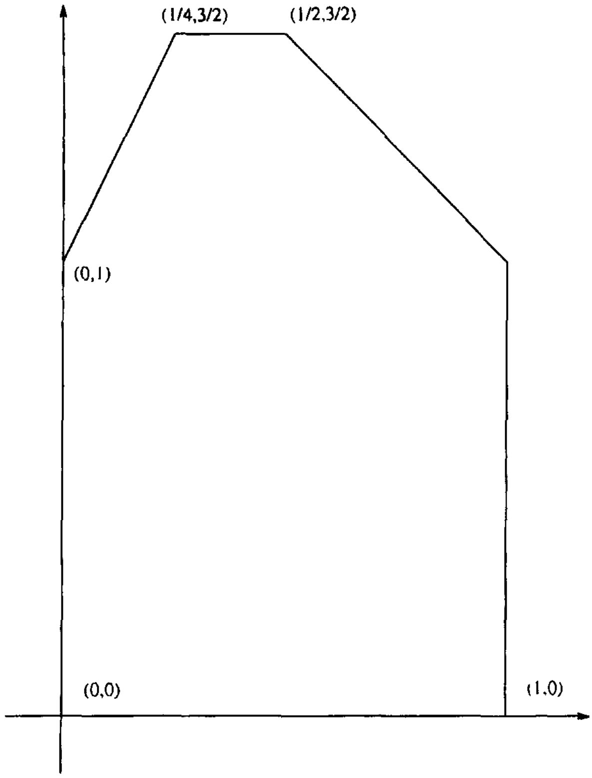

Fig. 1.

Proof of the Theorem. If there is a point v of blue degree at most 1 then the inequalities (i), (ii) and (iii) follow by induction. So in the sequel we suppose

$$d_b(v) > 1 \tag{1}$$

for every point in S. By Proposition 3, this implies that for every inner point v, $d_b(v) = 2$. □

Definition. Let u be an outer point and $v_1, \ldots, v_t$ all outer points of S such that the edges (u, v_i) are blue. A point v_i is a *middle neighbor* of u if on both sides of the line uv_i, u has neighbors at distance d_2. The edge uv_i is called a *middle edge* incident with u.

Proposition 4. *Let u be an outer vertex of outer degree larger than 2, and v a middle neighbor of u. Then no inner blue edge can start in v, and at most one outer blue edge can start from v on each side of the line uv.*

Proposition 5. *The inner degree of every outer point is at most 2.*

Proposition 6. *If the outer degree of u equals 3, then the inner degree of u is at most 1.*

Proposition 7. *If the outer degree of u equals* 4, *then no inner edge can start at u.*

Proposition 8. *No outer point of S has outer degree higher than* 4.

Definition. Let $G(S, = 2)$ be the following graph: the vertices of the graph are the points in S, and vertices u and v are connected by an edge iff their distance is exactly d_2. Let $G(S, \leqslant 2)$ be the graph on the same set of vertices in which two vertices are connected by an edge iff their distance is at least d_2.

Now we summarize the previous propositions in graph terms.

Let us consider the graph $G(S, = 2)$. Let us define the following directed graph G^*. The vertices are the outer points of S, the edges for every outer point v are the middle edges from v directed away from v (an edge may be directed in both ways). Propositions 4 and 8 imply that in G^* every vertex has degree at most 2. So G^* is a point-disjoint union of circuits, paths and isolated points. It also follows from Proposition 4 and Lemma 1 that if a point has degree 2 in G^* then either both edges are directed in, or both are directed out or form a pair of antiparallel edges. Thus, the edges along each circuit or path alternate in orientation. The doubly directed edges of G^* arise of connecting points in S of outer degree 3, then form separate components of the graph, and we consider these as (degenerate) circuits of length 2. Hence, the components of G^* are the following:

— isolated vertices, which correspond to vertices in $G(S, = 2)$ of outer and inner degree $\leqslant 2$.

— circuits of length $\geqslant 4$, whose vertices have in $G(S, = 2)$ outer degree 4 and 2 alternatingly (where the points of degree 2 are the middle neighbors of the points of degree 4).

— circuits of length 2, which have points of outer degree 3 and are the middle neighbors of each other. We have seen in Propositions 5 and 7 that the inner degree of any point on a circuit is 0.

— paths whose edges alternate in direction. The outer degree of each point is its outdegree plus 2. The inner degree of all points except the endpoints is 0.

We can partition the vertices of G^* using the description above:

m_1: the number of isolated vertices of G^*,

m_2: the number of vertices belonging to the circuits of G^*,

m_3: the number of vertices belonging to the paths of G^*,

k: the number of paths in G^*,

p_1: the number of outer points,

q_1: the number of outer blue edges.

Let us denote by p_2 the number of inner points in S and by q_2 the number of inner blue edges in $G(S, = 2)$.

Then we have

$$p_1 = m_1 + m_2 + m_3$$

and

$$q_1 \leqslant \tfrac{1}{2}(2m_1 + 3m_2 + 3m_3 - k)$$

because on every circuit in G^* the average degree in $G(S, =2)$ equals 3, on the other hand on every path it is < 3, because on every path at least one edge is missing. We get also for the number of inner blue edges

$$q_2 \leqslant 2m_1 + 2k.$$

Since we may suppose that every inner point is of blue degree exactly 2, from this we get

$$p_2 = \tfrac{1}{2}q_2 \leqslant m_1 + k.$$

From these we get for the number of edges of $G(S, =2)$, first proved in [3]

$$\begin{aligned} n_2 &= q_2 + q_1 = 2p_2 + q_1 = \tfrac{3}{2}p_2 + q_1 + \tfrac{1}{2}p_2 \\ &\leqslant \tfrac{3}{2}p_2 + m_1 + \tfrac{3}{2}m_2 + \tfrac{3}{2}m_3 - \tfrac{1}{2}k + \tfrac{1}{2}m_1 + \tfrac{1}{2}k \\ &= \tfrac{3}{2}p_2 + \tfrac{3}{2}(m_1 + m_2 + m_3) = \tfrac{3}{2}(p_2 + p_1) = \tfrac{3}{2}n. \end{aligned}$$

Now we give the proof of (ii). We may assume the following inequality on the blue and red degrees of the graph $G(S, \leqslant 2)$:

$$d_b(v) > 1 + 2d_r(v) \tag{2}$$

holds in every vertex of the graph. Suppose the contrary is true for a certain v, then omitting v we may proceed by induction. So in the sequel we suppose that (2) is true in every vertex.

Lemma 9. *Suppose an outer vertex u is connected to the outer vertices v_1, v_2 by blue edges in such a way that the edges uv_i are middle edges of the vertices v_i, then a red edge must start at both v_1 and v_2.*

Proof. Suppose rv_1 (resp. tv_2) is the blue edge starting in v_1 (resp. in v_2) where r (resp. t) is on the other side of the line uv_1 (resp. uv_2), than v_2 (resp.v_1). Then in the quadrangle rv_1v_2u (resp. uv_1v_2t), the diagonal rv_2 must be red (resp. tv_1). □

Lemma 10. *If we have an all-blue N on the vertices r, u, t, v then at least one red edge must start at each of u and t.*

Proof. Obviously $d(r, v) = d_1$, since ru and tv are avoiding blue edges. If a blue edge different from ru starts at r, and it is on the other side of the line r, v than t, then this forces a red edge starting at t. So all blue edges other than ru, must go to the same side of the line r, v where t is. On the other hand, at least 2 blue edges must start at r because of inequality (2). Let the other endpoints of these blue edges be r_i.

The same argument is true for v, so the blue edges starting at v are vv_i. But then in these vertices r_i and v_i no other blue edge may start, because it would give a forbidden N-configuration. Thus, inequality (1) is violated. □

Lemma 11. *Every nonisolated point of G^* is incident with a red edge.*

Proof. Obviously, any nonisolated point u of G^* is covered by an all-blue N. Furthermore, u is an inner point of the N, so by Lemma 10, a red edge must start in u. □

The isolated points of G^* arise from the vertices of $G(S, =2)$ of degree 2, and the number of outer blue edges starting at these vertices is at most 2. Furthermore, if an inner blue edge starts at an isolated vertex then, obviously, at least one red edge starts as well. Let m_1' be the number of those isolated vertices of G^* incident with inner blue edges. Then obviously $m_1' \leqslant m_1$.

By Lemma 11 and the above observations we get the following lower bound for the number of red edges:

$$m_1' + m_2 + m_3 \leqslant 2n_1 . \tag{3}$$

Using what we know for n_2 from the above description, we get

$$2n_2 = 2(q_1 + q_2) \leqslant 2m_1 + 3m_2 + 3m_3 - k + 2m_1' + 2k + 2p_2 . \tag{4}$$

Combining the inequalities (3) and (4) with the trivial inequality $k \leqslant m_3$, we get (ii).

Proof of the inequality (iii). Suppose first that there exists an inner vertex u with $d_b(u) = 3$. Then every other point of S is contained in the circle K with radius d_2 about u. Let the vertices v_1, v_2, v_3 be the neighbours of u, then we have one of the following cases [4]:

(a) the vertices v_1, v_2, v_3 form a regular triangle of side d_1;

(b) the vertices v_1, v_2, v_3 form a triangle where two sides are of length d_1 and the third is at most of length d_2.

(a) Draw the circles of radius d_1 around the v_i's (see Fig. 2(a)). Then no point of S can be outside of the intersection C of these three circles. On the other hand, any point which is not on the boundary of C would give a distance between d_1 and d_2 with one of the v_i's. Draw the circles of radius d_2 around the v_i's and let the intersection of these circles with the boundary of C be the points $a_1, a_2, b_1, b_2, c_1, c_2$. Then only one of the arcs may contain points of S at all, say the arc a_1a_2, because otherwise we would get a distance between d_1 and d_2. This means that we have $n - 4$ points on the arc a_1a_2, these points are of distance d_1 from v_1 but no other blue or red edge starts at any of them. In this case we have at most 5 blue edges and $3 + n - 4$ red edges, so (iii) holds.

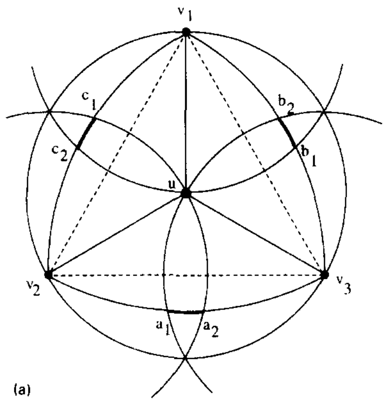

Fig. 2(a).

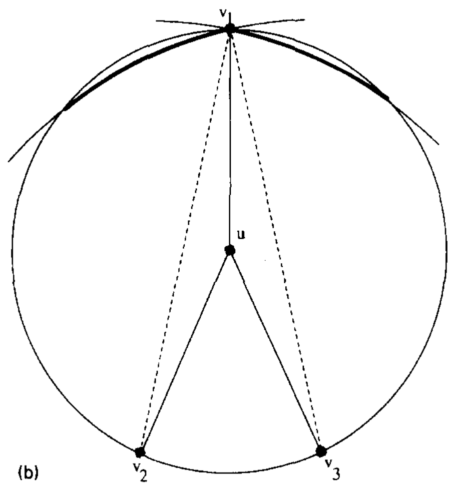

Fig. 2(b).

(b) Suppose $d(v_1,v_2) = d(v_1,v_3) = d_1$ and $d(v_2,v_3) \leqslant d_2$ (see Fig. 2(b)). Observe that the triangle $v_1v_2v_3$ does not contain any inner point, since the distance of such a point from at least one of v_1, v_2 and v_3 would be strictly between d_1 and d_2.

Let w be any outer point different from v_1, v_2 and v_3. We claim that w lies inside the segment v_2uv_3 of K. Assume not, so that w lies in the segment v_3uv_1 of K, say. Then w must be on the same side of the line v_2u as v_3, else its distance from v_3 would lie strictly between d_1 and d_2. Hence the triangle v_1v_2w contains the triangle v_1v_2u, and hence its circumference is larger. Hence at least one of the segments v_1w and v_2w must be red. But all points in the circle K at a distance d_1 from v_1 lie in the angle v_2uv_3, so $d(v_2,w) = d_1$. But then an elementary argument shows that the distance of w and v_3 is strictly between d_1 and d_2, a contradiction.

Thus we know that every point of S other than v_1 lies in the segment v_2uv_3. Hence it follows that no red segment other than wv_1 starts from w, and no blue segment starts at w. If we omit w, we may proceed by induction.

This leaves us with the trivial case when $S = \{u, v_1, v_2, v_3\}$.

So we may suppose that at most 2 blue edges may start at every inner point. Omitting the inner points we can proceed by induction. Thus we may suppose there are no inner points. For such 'convex' sets, however, even the stronger inequality

$$n_1 + 2n_2 \leqslant 3n$$

holds [3]. This completes the proof of the Theorem. □

3. Construction

Let $n = 2m$, $n \geqslant \delta$ and let the points $v_1, \ldots, v_m$ be the vertices of a regular m-gon in this order. The v_i are the outer points of S. In this configuration d_2 occurs m times.

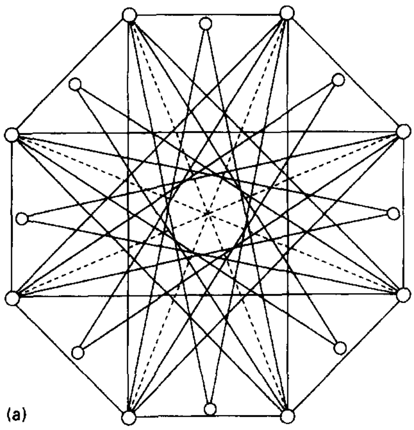

Fig. 3(a).

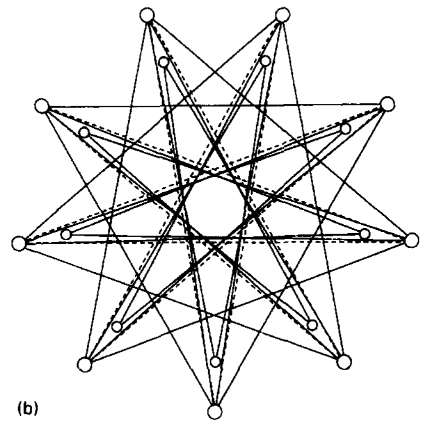

Fig. 3(b).

We construct $t_1, \ldots, t_m$ in the following way. Let the t_i be the point inside the m-gon determined by the equation

$$d(v_i, t_i) = d(t_i, v_{i+1}) = d_2$$

(where the indices are understood modulo m). Obviously, adding the points t_i does not change d_1 and d_2. In this configuration $n = 2m$, $n_2 = 3n/2$, and $n_1 = n/4$ if m is even, while $n_1 = n/2$ if m is odd (see Figs. 3(a) and 3(b)).

Acknowledgements

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References

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