Available online at www.sciencedirect.com
SciVerse ScienceDirect
Systems Engineering Procedia 4 (2012) 424 – 430
Procedia
Systems Engineering

Percolation criticality for complex networks

Xiaojing ZHENG[a,b,]*, Xusong XU[a]

[1]Wuhan University,Wuhan, 430072, China
2Harbin University of Commerce, Harbin, 150086, China

Abstract

A random Multi-Local-World complex networks directed model is constructed to mine the evolution laws of topological properties for this complex engineering system according to the rules about what the behaviors operate in complex system is gotten by calculating the avalanche-size distribution. It concludes that: It is the robustness and vulnerability that behaviors show at the same moment in complex system, when facing random attacking and intention attacking respectively.

Keywords: random Multi-Local-Worlds complex networks; percolation; criticality; scale-free; robustness and vulnerability

1. Introduction

Recently there has been much interest in studying new random complex networks models that capture certain observed common features of many large-scale, real-world networks. Although the recent activity in this area perhaps started with the "small-world" model of Watts and Strogatz, the main focus now seems to be on so called "scale-free" random graphs, whose degree distributions follow power laws.

Two objectivities to scale-free models have been proposed. The first is to explain how and what distribution the parameters of the random complex networks, degree sequence, strength sequence, clustering coefficient sequence, and so on, satisfy in terms of simple underlying principles. The second objectivity is to master the properties that we consider here are among the most basic properties of real-world networks, namely robustness, i.e., resistance to random damage, and vulnerability, i.e., vulnerability to malicious attack. These were considered experimentally in Refs. [1-5], where, attackness, means some vertices and the corresponding links or arrows in a graph would be deleted. Writing n for the total number of vertices of the graph, we ask when the damaged graph contains a giant component, i.e., a component whose order is $\Theta(n)$ as $n \to \infty$. In particular, we measure robustness or vulnerability by asking what fraction p_c of vertices must remain in order to have a giant component, when vertices are deleted at random or so as to cause the most damage. This kind of measure is usual in random complex networks, and corresponds to the critical probability in percolation. This case is introduced in Refs. [6-15], and the first largest component and the

* Corresponding author. Tel.: +00-86-15004647157.
Supported by National Natural Science Foundation of China(No. 71040001)
E-mail address: tkwiloi 75@126.com.

 doi:10.1016/j.sepro.2012.01.006

second largest component, and the corresponding condition, are obtained. Whether there exist a criticality for a random complex networks or not is decided by the system's properties. Furthermore, if the system is a scale-free one and a dense one, there does exist a critical probability for malicious attack and the probability is very small; if the system is a scale-free one and a spare one, it is totally different case. The results is adjusted and complemented in Refs. [16-19].

What remains of a graph when leaves are iteratively removed until none remains? The answer depends on what is meant by leaves. In the most standard definition, a leaf is a vertex with exactly one neighbor, and leaf removal deletes this vertex and the adjacent edge. In the context of large random graphs where the connectivity α (the average number of neighbors of a vertex) is kept vertex and the number of vertices $N \to \infty$, the answer is well known and interesting. When $\alpha < 1$, the remnant after leaf removal is made of $O(N)$ isolated vertices, plus a sub-graph of size $o(N)$ without leaves. When $\alpha > 1$, the remnant still contains $O(N)$ isolated points, but the rest is a sub-graph of size $O(N)$, which is dominated by a single connected component usually called the backbone. The a priori surprising, but rather general, fact that backbone percolation and standard percolation occur at the same point, namely at $\alpha = 1$, has a very simple explanation for random graphs. Indeed, a large random graph of average connectivity $\alpha < 1$ consists of a forest (union of finite trees) plus a finite number of finite connected components with one closed loop. Obviously, each tree shrinks to a single isolated point after leaf removal. However, at $\alpha = 1$ the percolation transition occurs and when $\alpha > 1$, a random graph consists of a forest plus a finite number of components with one closed loop, plus a "giant" connected component containing a finite fraction of the vertices and an extensive number of loops. No loop is destroyed by leaf removal so that the giant component leads to a macroscopic connected remnant after leaf removal. The percolation transition was discovered and studied by many scientists. This has initiated a lot of work on the random graph model, and many fine details concerning the structure of the percolation transition have been computed. References [3, 6, 8-9, 12, 16, 18-19] give the corresponding results for this field.

The matching problem had already led mathematicians to a thorough study of leaf removal. In fact, parts of our analytical results have already been obtained in this context. However, we have obtained them independently by a direct enumeration technique which turned out to be quite similar to a counting lemma for bicolored trees.

As we well known, the true world should consist of different "Locals", in this case, every agent can just know the limited information from "Locals", not from the "World". In this sense, the results mentioned above should be reconsidered to fit for the reality. So, a random Multi-Local-Worlds complex networks directed model is introduced in this paper, and the corresponding analytic percolation criticality will be obtained after analyzed.

The rest of this paper is organized as follow. In section 2, we describe the random Multi-Local-Worlds complex networks directed model. The result of percolation criticality for the system described in the model is given in section 3. To obtain this result, the evolution law for this system should be specified in section 4, in this section, two theorems are introduced to explain the evolution law. The last section, we specify the theorems about percolation criticality for the system described in the model.

2. The model

Provided that a complex system can be regarded as the integration of many local-worlds, which is large enough that means there are several agents in this local-world, and which is small enough that means there are several this kinds of local-worlds in the system. Furthermore, in a short time-scale, interaction between agents in a same local can be regarded as cooperation, which makes the local-world synchronize or coordinate because their benefits are identification and be coordinated; interaction between agents in different locals can be regarded as non-cooperation because their benefits are discrepant. In a long time-scale, these interactions make the system converge to his attractor, which makes the system phase transition make sense, in this sense, the configuration of the system will be transferred due to several agents in a local-world perhaps enter another one, or some agents enter or quit this system, this game configuration changing makes the analysis more complex.

Considering the hypothesis mentioned above, we can not consider the dual problems simultaneously, so, we omit the interaction in detail and the flow dynamics of the system from a relative large-scale, system behaviours' synchronization should be considered. In this sense, the interaction between agents becomes very easy: yes and no, the interaction in this case can be described to 0 and 1 as for yes and no, respectively.

The following model is created according to the rules mentioned above. On modeling, a Boolean Multi-Local-World networks directed is considered in this paper. That is to say, to simply this question, we just consider all inhomogeneous interaction, here 1 is chosen.

We consider a directed Boolean network, $G(t)=\{a_{ij}\}$, where $a_{ij}=\begin{cases}1\\0\end{cases}$ denotes the directed flowing from i to j, which means in the complex adaptive system the working flow from agent i to agent j if they interact. The model of random Multi-Local-Worlds complex networks directed should be specified as follow.

We consider a graph which grows by adding single edges at discrete time steps. At each such a vertex may or may not also be added. For simplicity we allow multiple edges and loops. In what follows, to choose a vertex v of $G(t)$ according to $d_{out}+\delta_{out}$ means to choose v so that $P(v=v_i)$ is proportional to $d_{out}(v_i)+\delta_{out}$, i.e., so that $P(v=v_i)=(d_{out}(v_i)+\delta_{out})/(t+\delta_{out}n(t))$. To choose v according to $d_{in}(v_i)+\delta_{in}$ means choose v so that $P(v=v_i)=(d_{in}(v_i)+\delta_{in})/(t+\delta_{in}n(t))$, where all degrees are measured in $G(t)$. To choose a vertex v of $G(t)$ according to $(d_{out}(v_i)+\delta_{out})'$ means to choose v so that $P(v=v_i)$ is proportional to $(d_{out}(v_i)+\delta_{out})'$, i.e., so that $P(v=v_i)=1-(d_{out}(v_i)+\delta_{out})/(t-2+\delta_{out}n(t))$. To choose v according to $(d_{in}(v_i)+\delta_{in})'$ means choose v so that $P(v=v_i)=1-(d_{in}(v_i)+\delta_{in})/(t-2+\delta_{in}n(t))$, where all degrees are measured in $G(t)$.

For $t\ge t_0$ we form $G(t+1)$ from $G(t)$ according to the following rules:

A. With probability β_1, add a new local-world that has some vertices and edges.

B. With probability β_2, we add a new vertex v together with an edge from v to an existing vertex w in a local-world chosen arbitrarily, where w is chosen according to $d_{in}+\delta_{in}$.

C. With probability β_3, add an new vertex w and an edge from an existing vertex v to w in a local-world chosen arbitrarily, where v is chosen according to $d_{out}+\delta_{out}$.

D. With probability β_4, add an edge from an existing vertex v to an existing vertex w in the same local-world, where v and w are chosen independently, v according to $d_{out}+\delta_{out}$,and w according to $d_{in}+\delta_{in}$.

E. With probability β_5, delete an edge in a single local-world from v to w, where v and w is chosen according to $(d_{out}+\delta_{out})'$, $(d_{in}+\delta_{in})'$.

F. With probability β_6, add an new edge from an existing vertex w belonging to an local-world to anther existing vertex w belonging to anther local-world chosen arbitrarily, w and v are chosen respectively according to $d_{in}+\delta_{in}$ and $d_{out}+\delta_{out}$.

Our model allows loop and multiple edges; there seems no reason to exclude them. However, there will not be very much, so excluding them would not significantly affect our conclusions.

Setting $c_1=(\beta_1+\beta_2+\beta_4+\beta_5+\beta_6)/(1+\delta_{in}(\beta_1+\beta_2-\beta_5))$, and $c_2=(\beta_1+\beta_2+\beta_3+\beta_5+\beta_6)/(1+\delta_{in}(\beta_1+\beta_2-\beta_5))$

And writing $x_i(t)$ for the number of vertices of $G(t)$ with in-degree i, and $y_i(t)$ for the number of vertices of $G(t)$ with out-degree i, the result for complex networks evolution law is given in following sections.

3. Main Results

How the complex networks operate and how a certain behaviour emerges in this networks must be the important and valuable thing to study. Many conclusions point out that there exist a threshold for percolating, if the number of agents' excess to this threshold, which are called percolation threshold. Above the thresholds, the corresponding behaviours shall be emergence suddenly, or, these behaviours can not be noted because they are so small that shall be ignored. So, it is important to finds the percolation thresholds of these behaviours.

There are two methods to consider these behaviours in this complex system, which is random control and intention control. As for random control, i.e., we need not discover the most serious place to induce these behaviours happen and control blindly and randomly to some places. On the contrary, the intention control means we can find the most serious place to induce these behaviours happen and control intentionally to the correspond place. We name the former random attack and the latter intention attack respectively. Where, random attack means destroy the agents randomly, and intention one means destroy them from large to small due to their in-degree and out-degree.

Set $K_1 = max\{x_i(t)\}$, $m_1 = min\{x_i(t)\}$, $K_2 = max\{y_i(t)\}$, $m_2 = min\{y_i(t)\}$ denote the maximum and minimum value for in-degree and out-degree of vertices respectively, some analytic conclusions are drawn here.

Theorem 1. *The network of behaviours in complex system shall be vanished by deleted some vertices randomly with probability* α *.(Alternatively, a fraction* $\beta = 1-\alpha$ *of nodes is retained) . The percolation threshold* α *or* β *is*

$$\alpha = 1-\beta = \frac{<k>}{<jk>} = \frac{2-X_{in}}{2(1-A)BC_{in}(K_1^{2-X_{in}} - m_1^{2-X_{in}}) + (2-X_{in})AB(K_1^2 - m_1^2)\delta_{j,j(k)}}$$

Where $j(k) = k^{\frac{X_{out}-1}{X_{in}-1}}, \delta_{x,y} = \begin{cases} 0, x \neq y \\ 1, x = y \end{cases}$.

Theorem 2. *The network of behaviours in complex system shall be vanished by deleted some vertices that in-degree cutoff* $K_1 (m_1 \ll K_1 < K_1)$ *and out-degree* $K_2 (m_2 \ll K_2 < K_2)$ *intentionally with probability* α *. (Alternatively, a fraction* $\beta = 1-\alpha$ *of nodes is retained) . The percolation threshold* α *or* β *is*

$$\alpha = \sum_{j=K_1,k=K_2}^{K_1,K_2} \frac{jkP(j,k)}{<j_0k_0>} \sim \left(\frac{K_1}{m_1}\right)^{2-X_{in}} \left(\frac{K_2}{m_2}\right)^{2-X_{out}} = \alpha^{(2-X_{in})(2-X_{out})/(1-X_{in})(1-X_{out})},\ \beta = 1-\alpha$$

Where, α *is the percolation threshold of random attack.*

Firstly, the evolution rule for of behaviours in complex system is described, then, based on this rule, the percolation critical value can be gotten.

4. The Evolution Law of Random Multi-Local-Worlds Complex Networks Directed

The evolution process of these behaviours in this complex system satisfies power-law distribution, as follow.

4.1. Evolution Law

The system evolution process of these behaviours in this complex system satisfies power-law distribution, as follow.

Theorem 3. *Let* $i \geq 0$ *be fixed. There are constants* p_i *and* q_i *such that* $x_i(t) = p_i t + o(t)$ *and* $y_i(t) = q_i t + o(t)$ *hold with* 1. *Furthermore, as* $i \to \infty$ *we have*

$p_i \sim C_{in} i^{-X_{in}}$,

where $X_{in} = 1 + 1/c_1$ *and* C_{in} *is a positive constant. As* $i \to \infty$ *we have*

$q_i \sim C_{out} i^{-X_{out}}$,

where $X_{out} = 1 + 1/c_2$ *and* C_{out} *is a positive constant.*

4.2. Remarks

(1) Remark 1: The Theorems

We can review these theorem proved above, and can know that the in-degree and out-degree are changed by corresponding operators p_i and q_i respectively and the operators distribution satisfies different power-law distribution, which is the properties of the networks' parameters. So, the networks is a scale-free one. Furthermore, it is $X_{in}=1+1/c_1$ and $X_{out}=1+1/c_2$ that denote the parameter of the in-degree and out-degree. Similar to the in-out-degree of the networks, $X'_{in}=1+1/c_1+c_2/c_1(\delta_{out}+1_{\{\gamma\delta_{out}=0\}})$ $X'_{out}=1+1/c_2+c_1/c_2(\delta_{in}+1_{\{\gamma\delta_{in}=0\}})$ denote the parameter.

Because c_i is decided by β_i, $d_{out}(v_i)+\delta_{out}$ and $d_{in}(v_i)+\delta_{in}$ that represent the property of the these behaviours in different categories complex system, the diversity and complexity of the these behaviours in this complex system can be gotten, which makes managing these behaviours in this complex system difficult. So, all β_i, $d_{out}(v_i)+\delta_{out}$ and $d_{in}(v_i)+\delta_{in}$ should be gotten according to the certain situation before concluding what to be done.

(2) Remark 2: The Behaviour Itself in The Complex System

Theorem 3 specify the evolution law about these behaviours in this complex system, a system of Multi-Local-World networks directed, but the certain law relies on the complex system structure and the function of the complex system, which need to analyze according to every certain complex system. We can analyze the behaviours for several kinds complex system exampled above, however, it is not the objectivity of this paper, so, this analyzing is left to others.

It is power-law distribution that Theorem 3 and Theorem 4 want to give. Behaviours can be produced from every agent randomly, and can be transferred, as an arbitrarily probability, to the next agents who interact with. Furthermore, the behaviours can not transfer to the next agent once the first agent interacts with, similar to arbitrary agent in the process. If and only if the interaction time super to a certain threshold, the behaviours can be imitated by the next agent and the transferring process happening, which is a stochastic process for every graph of behaviours from time 1 to arbitrary time *t*, to *t*+1, and to the infinite.

(3) The Exponents of Power-Law

Though we always to say that behaviours in complex have the property of scale-free, there are many categories complex systems and many categories behaviours in every kind complex system. So, it is extraordinary complex to identify them. It is well known that the exponents of all parameters of the networks do decide the properties of behaviours, such as inducing and transferring. If and only if the details of them are known clear, the corresponding controlling methods can be gotten.

Generally, the larger the exponent, the heavier diversity the behaviours have, the easier behaviours transferring if the agents that have high in-degree or out-degree produce behaviours.

(4) Robustness and Vulnerability

It is known that the Multi-Local-World network of behaviours in complex system satisfied scale-free law from the analyzing above, so, there must be the property of robustness and vulnerability to the system. That is to say, if the behaviours in complex system randomly, the complex system will be stability; otherwise, if the behaviours are induced intentionally, the system will be collapsed soon. Furthermore, it is the critical point of the behaviours about agents in the system that we need really to know. Following sections will give a reasonable method to obtain the critical point by analytical method.

5. Percolation Criticality for Random Multi-Local-Worlds Complex Networks Directed

From the Theorem 3, the power law of the distribution for in- and out-degree of the network is given, so we can calculate the avalanche-size distribution as the following formula.

From analyzing above, we obtained the distribution about the in-, out- and in-out-degree of the network, and the network of complex system is a scale-free network, which the degree satisfies the power law distribution. According to the law, the network has the properties of robustness and vulnerability when random attacking and intentional attacking happens. It is the percolation of the network that is important to the system, and the percolation threshold can be given by Theorem 1 and Theorem 2, the main process is prove them are right.

6. Conclusions

Many systems can be regarded as a Multi-Local-Worlds directed networks. In this case, system is separated into several sub-systems and every sub-system can be separated into some sub-sub-system. The interaction frequency between agents in the same sub-system is larger than the frequency between agents in different sub-systems. Every agent can know the local information of its sub-system and the average and superficial knowledge about the system, but can not know the information coming from the other sub-system. When interacting, every agent operate according to some certain rules which perhaps consists of several behaviours that can be simplified to a single one under certain conditions stricken, so, the complexity about this system is very interesting, and the behaviour emerged suddenly in this system, as called percolation, is more important to us. We can say, a certain behaviour can be produced and transferred in the system because of his/her legal/illegal system operating in the process of interaction, which must give some positive/negative effect to the other. This behaviour will be learned by the other persons interacted with this agent if and only if this kind of behaviour does good to the accepted, which must be happened in the sub-system, and to another sub-system when certain conditions are satisfied and this behaviour can be transferred from one person to another and from one sub-system to anther sub-system.

References

1. M. Bauer and O. Golinelli, "Core percolation in random graphs: a critical phenomena analysis", The European Physical Journal B(Springer), Vol. 24, pp. 339~352, 2001
2. Shankar Bhamidi, "First passage percolation on locally treelike networks. I. Dense random graphs", Journal of Mathematical Physics(AIP EurekAlert), Vol. 49, pp. 125218-1~125218-27, 2008
3. Tonći Antunović, Ivan Veselić, "Sharpness of the Phase Transition and Exponential Decay of the Subcritical Cluster Size for Percolation on Quasi-Transitive Graphs", J Stat Phys(Springer), Vol. 130, pp. 983~1009, 2008
4. Robert Kozma, Marko Puljic, Paul Balister, Béla Bollobàs, Walter J. Freeman, "Phase transitions in the neuropercolation model of neural populations with mixed local and non-local interactions", Biological Cybernetics(Springer), Vol. 92, pp. 367–379, 2005
5. M. Franceschetti, L. Booth, M. Cook, R. Meester and J. Bruck, "Continuum percolation with unreliable and spread-out connections", Journal of Statistical Physics(Springer), Vol. 118 pp. 721–734, 2005
6. Béla Bollobàs and Oliver Riordan, "Robustness and Vulnerability of Scale-Free Random Graphs", Internet Mathematics(A K Peters, Ltd), Vol. 1, No. 1, pp. 1~35, 2008
7. Béla Bollobàs and Oliver Riordan, "The Phase Transition and Connectedness in Uniformly Grown Random Graphs", LNCS(Springer), Vol. 3243, pp. 1-18, 2004.
8. Béla Bollobàs, S. Janson, and O. Riordan, "The phase transition in the uniformly grown random graph has infinite order", Random Structures Algorithms(Wiley), Vol. 26, pp. 1-36, 2005
9. Béla Bollobàs, S. Janson, and O. Riordan, "The phase transition in inhomogeneous random graphs", Random Structures Algorithms(Wiley), Vol. 31, pp. 3-122, 2007
10. Béla Bollobàs, S. Janson, and O. Riordan, "Line-of-sight percolation", Combinatorics Probability and Computing(Cambridge), Vol. 18, pp. 83–106, 2009
11. Béla Bollobàs, O. Riordan, "Slow emergence of the giant component in the growing m-out graph", Random Structures Algorithms(Wiley), Vol. 27, pp. 1-24,2005
12. Béla Bollobàs, S. Janson, and O. Riordan, "Spread-out percolation in $\mathbb{R}^d$", Random Structures Algorithms(Wiley), Vol. 31, pp. 239-246, 2007
13. N. Fountoulakis, "Percolation on sparse random graphs with given degree sequence", Internet Math.(Springer), Vol. 4, No. 4, pp. 329-356, 2007
14. A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, "*k*-core (bootstrap) percolation on complex networks: critical phenomena and nonlocal effects", Phys. Rev. E.(American Physical Society),Vol. 73, No. 10, 056101, 2006
15. Béla Bollobàs, O. Riordan, Percolation. Cambridge University Press(Cambridge), English, 2006
16. J. Wang, Y.-H. Liu1, Y. Jiao, and H.-Y. Hu, "Cascading dynamics in congested complex networks", The European Physical Journal B(Springer), Vol. 67, pp. 95-100,2009
17. Daniela Bertacchi, Fabio Zucca, "Characterization of Critical Values of Branching Random Walks on Weighted Graphs through Infinite-

Type Branching Processes", J Stat Phys(Springer), Vol. 134, pp. 53-65, 2009
18. Tatyana S. Turova, "Phase Transitions in Dynamical Random Graphs", Journal of Statistical Physics(Springer), Vol. 123, No. 5, pp. 1007-1032, 2006
19. Balázs Ráth, Bálint Tóth, "Triangle Percolation in Mean Field Random Graphs—with PDE", J Stat Phys (Springer), Vol. 131, pp. 385-391, 2008
20. DAVID SMYTH AND ALFRED HU BLER, "A Conductivity-Dependent Phase Transition from Closed-Loop to Open-Loop Dendritic Networks", Complexity(Morgan Kauffman), Vol. 9, No. 1, pp. 56-60, 2006