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Real-Valued and 2-Rational Group Characters

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The work presented here is largely concerned with the interaction between the real-valued irreducible characters of a finite group and the 2-regular elements (elements of odd order) in that group. This interaction can be studied in some detail because we can exploit two types of pairing: that of an irreducible character which is not real-valued with its complex conjugate, and that of a nonreal conjugacy class with the corresponding class of inverses. Similar ideas can be used to investigate the influence of 2-regular elements on irreducible characters of a group that are 2-rational, and we have included some simple results concerned with the number of 2-rational characters possessed by a group. Our methods show certain similarities with those that have been employed in the theory of 2-blocks, and so we have given some block-theoretic interpretations to our results.

The paper is divided into seven sections. Section 1 briefly describes how the Brauer–Witt theorem can be used to reduce certain problems about real-valued characters to the same problems stated in terms of real-valued characters of two classes of subgroup of G. Section 2 investigates the real-valued characters of one of the classes of subgroup described in the Brauer–Witt theorem. Such subgroups have the form AU, where A is the cyclic subgroup generated by a non-identity real element h of odd order and U is a Sylow 2-subgroup of $C^*(h)$. Attention is drawn to what we have called weakly real 2-regular elements, that is, elements of odd order that are real but are not inverted by an involution. Section 3 deals with estimates of the number of irreducible 2-rational characters. In Section 4, we give some sufficient conditions for the existence of real-valued irreducible characters with Schur index 2 over the real numbers.

In Section 5, we apply some of our results from Sections 3 and 4 to the study of 2-blocks of characters. In particular, some properties of real 2-blocks are given. We investigate the influence of normal subgroups on real-valued characters in Section 6. The final section is devoted to real-valued characters of 2-nilpotent groups. We give a solution to Brauer's problem concerning the number of real-valued irreducible characters of Frobenius–Schur invariant 1 for

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a 2-nilpotent group whose Sylow 2-subgroup is Abelian. The answer is given in terms of the number of strongly real and weakly real classes in the group.

The notation used is standard and conforms to that used in an earlier paper of the author [6]. All groups considered in the paper are taken to be finite.

1. The Brauer–Witt Theorem

Let χ be an irreducible character of the group G. The Frobenius–Schur invariant, $\epsilon(\chi)$, of χ is defined by $\epsilon(\chi) = (1/|G|) \sum \chi(g^2)$. We have $\epsilon(\chi) = 0$ if and only if χ is not real-valued, $\epsilon(\chi) = 1$ if and only if χ is the character of a real representation of G, and $\epsilon(\chi) = -1$ if and only if χ is real-valued but is not the character of a real representation. The Frobenius–Schur invariant (hereafter shortened to F–S invariant) of a real-valued irreducible character can be studied by means of the Brauer–Witt theorem [5, 15.12, p. 84]. We state here a version of this theorem, whose proof follows easily from maximality arguments of a kind used by Benard in [9, Sect. 3].

THEOREM 1.1. *Let χ be a real-valued irreducible character of the group G. There is a subgroup H of G and a real-valued irreducible character θ of H for which (θ, χ_H) is odd. H can be taken either to be a Sylow 2-subgroup of G or to have the form AU, where A is a cyclic subgroup of odd order generated by a real non-identity element h and U is a Sylow 2-subgroup of $C^*(h)$. In the second case it can be assumed that A is not in the kernel of θ. We have $\epsilon(\theta) = \epsilon(\chi)$.*

2. Evaluation of Some F–S Invariants

We begin our discussion in a general setting and then specialize our arguments to groups of the form AU described in Theorem 1.1. Let S be a subgroup of index 2 in the group T. For each irreducible character θ of S, we define an invariant, $\eta(\theta)$, of θ (the invariant depends on T). Like the F–S invariant, $\eta(\theta)$ takes only the values 0, 1 or -1. The numbers $\eta(\theta)$ also give information about the number of involutions in $T - S$. We will describe only the circumstances in which $\eta(\theta) = -1$ since this is what we will need in our applications.

LEMMA 2.1. *Let S be a subgroup of index 2 in the group T. If θ is an irreducible character of S, define $\eta(\theta) = (1/|S|) \sum_{T-S} \theta(t^2)$. Then $\eta(\theta) = 0, 1$ or -1. We have $\eta(\theta) = -1$ if and only if one of the following holds: θ^T is irreducible and $\epsilon(\theta^T) = -1$; $\epsilon(\theta) = 1$ and θ can be extended to a character of T that is not real-valued; $\epsilon(\theta) = -1$ and θ can be extended to a real-valued character of T. The sum $\sum \eta(\theta)\, \theta(1)$, extending over all irreducible characters of S, equals the number of involutions in $T - S$.*

Proof. Let t be an element of $T - S$ and let θ_1 be the conjugate character θ^t of θ. Let us suppose first that $\theta_1 \neq \theta$. It follows that θ^T is an irreducible character of T. Since we clearly have $\epsilon(\theta) = \epsilon(\theta_1)$ and $\eta(\theta) = \eta(\theta_1)$, evaluation of $\epsilon(\theta^T)$ shows that $\epsilon(\theta^T) = \epsilon(\theta) + \eta(\theta)$.

First suppose that $\epsilon(\theta^T) = 1$. If $\epsilon(\theta) = 0$, $\eta(\theta) = 1$. If $\epsilon(\theta) \neq 0$, we must have $\epsilon(\theta) = 1$ and so $\eta(\theta) = 0$. If $\epsilon(\theta^T) = 0$, $\epsilon(\theta) = 0$ also and then $\eta(\theta) = 0$. If $\epsilon(\theta^T) = -1$, and $\epsilon(\theta) = 0$, $\eta(\theta) = -1$. If $\epsilon(\theta^T) = -1$ and $\epsilon(\theta) \neq 0$, we must have $\epsilon(\theta) = -1$ and then $\eta(\theta) = 0$. This describes all the possibilities when θ^T is irreducible.

If $\theta_1 = \theta$, θ can be extended to a character ϕ of T. This time we have $2\epsilon(\phi) = \epsilon(\theta) + \eta(\theta)$. If $\epsilon(\phi) = 1$, $\epsilon(\theta) = 1$ also and thus $\eta(\theta) = 1$. If $\epsilon(\phi) = -1$, $\epsilon(\theta) = -1$ also and thus $\eta(\theta) = -1$. Finally we consider the case $\epsilon(\phi) = 0$. If $\epsilon(\theta) = 0$, $\eta(\theta) = 0$. If $\epsilon(\theta) = 1$, then $\eta(\theta) = -1$ and if $\epsilon(\theta) = -1$, $\eta(\theta) = 1$. This describes all possibilities when $\theta_1 = \theta$.

Examination of the proof above shows that there are three ways in which $\eta(\theta) = -1$ can arise and these are exactly those described in the statement of the lemma. The formula for the number of involutions in $T - S$ is a consequence of the familiar Frobenius–Schur involution formula. This completes the proof.

We now restrict our attention to groups of the form $H = AU$, where A is generated by a nonidentity real 2-regular element h and U is a Sylow 2-subgroup of $C^*(h)$. Let V be the subgroup of index 2 in U that centralizes h. It is easy to see that the irreducible characters of H that are nontrivial on A have the form $(\lambda\theta)^H$, where λ is a nontrivial irreducible character of A and θ is an irreducible character of V. The F–S invariant of these characters is decided by the values of $\eta(\theta)$, as the next lemma indicates.

LEMMA 2.2. *Let $\chi = (\lambda\theta)^H$ be an irreducible character of H, where λ is a nontrivial irreducible character of A. Let $\eta(\theta) = (1/|V|)\sum_{U-V}\theta(t^2)$. Then $\epsilon(\chi) = \eta(\theta)$.*

Thus H has real-valued irreducible characters χ with $\epsilon(\chi) = -1$ whenever V has characters θ with $\eta(\theta) = -1$. The next lemma is a sufficient (but not necessary) condition for the existence of characters with $\eta(\theta) = -1$.

LEMMA 2.3. *Suppose that U does not split over V. Then V has at least one irreducible character θ with $\eta(\theta) = -1$.*

Proof. By assumption, there are no involutions in $U - V$. Thus by Lemma 2.1, $\sum \eta(\theta)\,\theta(1) = 0$, the sum extending over all irreducible characters of V. Since the identity character of V clearly contributes 1 to this sum, there must be some θ with $\eta(\theta) = -1$, as required.

Now U does not split over V precisely when h is not strongly real. We will say that an element that is real but not strongly real is *weakly real*. Weakly real elements of odd order often have an influence on the F-S invariants of the characters of an arbitrary group as the next result and Theorem 4.1 show.

THEOREM 2.4. *Suppose that G is a group whose Sylow 2-subgroup is either dihedral or semidihedral. Then if G has an irreducible character* χ *with* $\epsilon(\chi) = -1$, *G has a weakly real element of odd order.*

Proof. Since neither a dihedral nor a semidihedral group has an irreducible character with F–S invariant -1, it follows from Theorem 1.1 that there is a subgroup H of G and an irreducible character ϕ of H with $\epsilon(\phi) = -1$. $H = AU$, where A is generated by a real nonidentity element h of odd order and U is a Sylow 2-subgroup of $C^*(h)$.

Let us first assume that the Sylow 2-subgroup of G is dihedral. It follows that U is cyclic or dihedral. Since H has a real-valued character ϕ with $\epsilon(\phi) = -1$, Lemmas 2.1 and 2.2 imply that U has an irreducible character which is either not real-valued or has F–S invariant -1. Neither of these cases can occur in a dihedral group and so U must be cyclic of order at least 4. This means that h is weakly real.

We turn to the case where the Sylow 2-subgroup is semidihedral. U must either be cyclic of order at least 4, quaternion, dihedral, or semidihedral. The first two possibilities imply that h is weakly real, as both types of group contain a unique involution. The third possibility cannot arise, by the argument of the previous paragraph. Thus we can assume that U is semidihedral. The faithful irreducible characters of U are not real-valued, whereas those that are not faithful have F–S invariant 1. Let V be subgroup of index 2 in U that centralizes h. Reference to Lemmas 2.1 and 2.2 indicates that ϕ has the form $(\lambda\theta)^H$, where θ is a character of V with $\epsilon(\theta) = 1$ and θ can be extended to a character θ_1 of U that is not real-valued. It follows that θ_1 must be faithful. V is either cyclic, quaternion, or dihedral. Since θ must also be a faithful real-valued character of V, we can rule out the possibility that V is cyclic. Similarly, as $\epsilon(\theta) = 1$, while all faithful irreducible characters of a quaternion 2-group have F–S invariant -1, V cannot be quaternion. Finally if V is dihedral, it contains all involutions in U and so h is weakly real (this case can actually arise). This completes the proof.

Let us consider the converse of 2.4. Suppose that G has a dihedral Sylow 2-subgroup. It is easy to see that if h is a weakly real element of odd order, $C^*(h)$ has a cyclic Sylow 2-subgroup. It follows from [6, Theorem 2] that G has an irreducible character χ with $\epsilon(\chi) = -1$. Thus we have the following result:

THEOREM 2.5. *Let G be a group whose Sylow 2-subgroup is dihedral. Then G has real-valued irreducible characters* χ *satisfying* $\epsilon(\chi) = -1$ *if and only if G has weakly real elements of odd order.*

The converse of Theorem 2.4 for a group whose Sylow 2-subgroup is semi-dihedral will be investigated in the Section 4.

3. Estimates of the Number of 2-Rational Characters

Let G be a group of order $2^a b$, $(b, 2) = 1$, and let $Q_{|G|}$, Q_b be the fields obtained by adjoining primitive $|G|$th and bth roots of unity to the rational field. Let S be the ring of algebraic integers in Q_b and let P be a maximal ideal in S containing the prime 2. We will write $x \equiv 0$ if the element x of S belongs to P. A character χ of G is said to be 2-rational if the values taken by χ lie in the field Q_b. Calculations of the exact number of irreducible 2-rational characters possessed by a group tend to be complicated by the fact that the Galois group of $Q_{|G|}$ over Q_b is not cyclic, so that Brauer's permutation lemma for group characters cannot be applied. However, the Galois group is a 2-group and this enables us to give estimates of the number of irreducible 2-rational characters in terms of the number of 2-regular classes in the group. Our proof uses a method due to Brauer [1, p. 412].

THEOREM 3.1. *Let G be a group which has t 2-regular classes. Let $h_1, \ldots, h_t$ be representatives of these classes. There exist t irreducible 2-rational characters $\chi_1, \ldots, \chi_t$ of G which satisfy* $\det \chi_i(h_j) \not\equiv 0$, $1 \leq i, j \leq t$.

Proof. G has at least one irreducible 2-rational character, the trivial character. If our statement above is not true, we can find elements $u_1, \ldots, u_t$ of S, with some $u_k \not\equiv 0$, such that for any irreducible 2-rational character χ of G, $\sum_{i=1}^{t} u_i \chi(h_i) \equiv 0$. Thus if θ is any S-linear combination of 2-rational irreducible characters, $\sum_{i=1}^{t} u_i \theta(h_i) \equiv 0$.

Let A be the cyclic subgroup generated by $h = h_k$, with $|A| = m$. Let V be a Sylow 2-subgroup of $C(h)$ and let $H = AV$. Let $\lambda_1, \ldots, \lambda_m$ be the irreducible characters of A. Each λ_i can be extended to a linear character μ_i of H by putting $\mu_i(h^j v) = \lambda_i(h^j)$ for v in V. Let θ_i be the induced character μ_i^G. We observe that θ_i is 2-rational. Now put $\theta = \sum_{i=1}^{m} \bar{\lambda}_i(h)\theta_i$; θ is an S-linear sum of 2-rational characters and is thus 2-rational. The definition of induced characters and the orthogonality relations for group characters in A show that $\theta(h_j) = 0$ if $j \neq k$, $\theta(h) \not\equiv 0$.

Let W be the Galois group of $Q_{|G|}$ over Q_b. If $\theta = \sum a_i \chi_i$, where the a_i are in S and the χ_i are irreducible characters of G, we have $\theta = \theta^\sigma = \sum a_i \chi_i^\sigma$ for σ in W. The action of W on the irreducible characters of G partitions the characters into orbits; an orbit consisting of a single character if and only if the character is 2-rational. Moreover, the number of characters in an orbit is a power of 2. Suppose that W has r orbits on the irreducible characters of G and that ϕ_j denotes the sum of the characters in the jth orbit. We have $\theta = \sum_{j=1}^{r} a_j \phi_j$. Since the elements h_i are 2-regular, we have $\chi^\sigma(h_i) = \chi(h_i)$ for any irreducible character χ and σ in W. Thus $\phi_j(h_i) = 0$ unless the jth orbit contains a single character. It follows that $\theta = \theta' = \sum a_j\chi_j$, where the sum extends only over irreducible characters that are 2-rational. We have $\theta'(h_k) \not\equiv 0$, $\theta'(h_i) = 0$, $i \neq k$, and so

$$\sum_{i=1}^{t} u_i\theta'(h_i) = u_k\theta'(h_k) \not\equiv 0.$$

Since θ' is an S-linear combination of irreducible 2-rational characters, this contradicts our earlier supposition. It follows that 2-rational irreducible characters with the required property exist.

It is possible to use this argument to obtain similar information on the number of irreducible 2-rational characters that are real-valued. Suitable modification of the argument used in [6, Theorem 1] leads to the following result.

THEOREM 3.2. *Let G be a group of even order which possesses s real classes of 2-regular elements. Then G possesses at least $s + 1$ irreducible 2-rational real-valued characters with* F–S *invariant* 1.

If we take into account 2-regular elements whose centralizers have even order, the estimates in Theorem 3.1 can be improved.

THEOREM 3.3. *Let G be a group of even order which has t 2-regular classes. Let $h_1, \dots, h_t$ be representatives of these classes. Suppose that the notation is chosen so that $C(h_i)$ has even order, $1 \leq i \leq r$. Then G has at least $t + r$ irreducible 2-rational characters.*

Proof. Let $\chi_1, \dots, \chi_n$ be the irreducible characters of G, with $\chi_1, \dots, \chi_m$ the irreducible 2-rational characters. By the orthogonality relations for group characters we have

$$\sum_k \chi_k(h_i)\,\bar{\chi}_k(h_j) = 0, \qquad i \neq j,$$

$$\sum_k \chi_k(h_i)\,\bar{\chi}_k(h_i) = |C(h_i)|.$$

We again let W denote the Galois group of $Q_{|G|}$ over Q_b. We have $\chi_k^\sigma(h_i)\,\bar{\chi}_k^\sigma(h_j) = \chi_k(h_i)\,\bar{\chi}_k(h_j)$ for σ in W. Thus if we partition the irreducible characters into W-orbits, we obtain

$$\sum_{k=1}^{m} \chi_k(h_i)\,\bar{\chi}_k(h_j) = 0, \qquad i \neq j,$$

$$\sum_{k=1}^{m} \chi_k(h_i)\,\bar{\chi}_k(h_i) \equiv 0, \qquad 1 \leq i \leq r,$$

$$\sum_{k=1}^{m} \chi_k(h_i)\,\bar{\chi}_k(h_i) \not\equiv 0, \qquad i \geqslant r+1,$$

where the sums now run over 2-rational characters.

Let B be the $t \times m$ matrix $\chi_i(h_j)$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant t$. B has rank t modulo P by Theorem 3.1. If C is the conjugate transpose matrix of B, C also has rank t modulo P. Our modified orthogonality relations show that BC has rank $t - r$ modulo P. General results from the theory of matrices imply that $t + t - m \leqslant t - r$ and so $m \geqslant t + r$, as required.

Again, we can take into account those 2-regular classes that are real. An analogous proof yields:

THEOREM 3.4. *Let G be a group of even order which has v 2-regular real classes. Suppose that w of these classes have centralizers of even order. Then G has at least $v + w$ real-valued irreducible 2-rational characters.*

This number exceeds the estimate of Theorem 3.2 by $w - 1$. However, we cannot expect to say much about the Schur indices of these additional real-valued characters. The results of this section admit interpretations in block theory, which will be explained in Section 5.

4. SUFFICIENT CONDITIONS FOR CHARACTERS WITH INVARIANT -1

We continue with the notation of the previous section. In [6, Theorem 2] we showed that if h is a weakly real element of odd order whose centralizer has an Abelian Sylow 2-subgroup, then G has an irreducible real-valued character χ satisfying $\epsilon(\chi) = -1$ and $\chi(h) \neq 0$. We wish to give a simple generalization of this result to a larger class of 2-groups. We will show later that this generalization is the best that we can hope for without sacrificing the condition that $\chi(h) \neq 0$.

THEOREM 4.1. *Let G be a group that has t weakly real classes of odd order. Let $h_1, \ldots, h_t$ be representatives of these classes. Let U_i be a Sylow 2-subgroup of $C^*(h_i)$ and V_i a subgroup of index 2 in U_i that centralizes h_i. Suppose that U_i/V_i' does not split over V_i/V_i', $1 \leqslant i \leqslant t$. Then G has t real-valued irreducible characters $\chi_1, \ldots, \chi_t$ with $\epsilon(\chi_i) = -1$, $1 \leqslant i \leqslant t$, and* $\det \chi_i(h_j) \not\equiv 0$, $1 \leqslant i, j \leqslant t$.

Proof. Let h denote any of the elements h_i and let $U = U_i$, $V = V_i$. Put $H = AU$, where A is the cyclic group generated by h. Our arguments will apply to H/V', so we may as well suppose that V is Abelian. Thus we are assuming that U does not split over V. By Theorems 2.3 and 2.2 we know that for each nontrivial irreducible character λ of A, there is a real-valued irreducible character θ of H with $\epsilon(\theta) = -1$ and $(\theta_A, \lambda) \neq 0$. It is clear that θ has degree 2 and so $\theta_A = \lambda + \bar{\lambda}$. We use the characters θ to construct real-valued class functions ϕ satisfying $\phi(h) \neq 0$. The rest of the proof follows that in [6].

The next result shows how strongly real elements behave with respect to irreducible characters of F–S invariant -1.

LEMMA 4.2. *Let h be a strongly real element of the group G. Then if χ is a real-valued irreducible character of G with $\epsilon(\chi) = -1$ and F is a representation of G with character χ, each distinct eigenvalue of $F(h)$ occurs with even multiplicity. In particular, $\chi(h) = 2c$, for some algebraic integer c.*

Proof. Let u be an involution that inverts h. Let H be the dihedral subgroup of G generated by h and u. Any irreducible character θ of H satisfies $\epsilon(\theta) = 1$. It follows that θ must occur with even multiplicity in χ_H. This proves our contention.

Even if h is a weakly real element of odd order, it is usually the case that $\chi(h) = 2c$, for some algebraic integer c, unless we are in the situation described in Theorem 4.1.

THEOREM 4.3. *Let χ be a real-valued irreducible character of the group G that satisfies $\epsilon(\chi) = -1$. Suppose that there exists a real element h of odd order for which $\chi(h) \neq 2c$ for any algebraic integer c. Then h is weakly real and if U is a Sylow 2-subgroup of $C^*(h)$ and V is a subgroup of index 2 in U that centralizes h, U/V' does not split over V/V'.*

Proof. That h is weakly real follows from Lemma 4.2. Let $H = AU$, where A is the cyclic subgroup generated by h. Since $\chi(h) \neq 2c$, there must be a real-valued irreducible character ϕ of H with (χ_H, ϕ) an odd integer and $\phi(h) \neq 2d$. Now h cannot be in the kernel of ϕ, so $\phi = (\lambda\theta)^H$, where λ is a nontrivial irreducible character of H and θ an irreducible character of V. We observe that $\phi(h) = \theta(1)(\lambda(h) + \bar{\lambda}(h))$; since $\phi(h) \neq 2d$, θ must be linear. Thus θ can be considered to be a character of V/V'. Furthermore, V' is contained in the kernel of ϕ and so ϕ can be considered to be a character of H/V'. H/V' has a normal Abelian subgroup AV/V' of index 2. Since $\epsilon(\phi) = -1$, we see from [5, Proof of 11.7, p. 64] that H/V' cannot be a split extension of AV/V', as required.

If we take into account 2-singular elements, the method of induced characters that was applied in 4.1 can still be used to show the existence of characters of F–S invariant -1 in circumstances not covered by Theorem 4.1.

THEOREM 4.4. *Let h be a weakly real 2-regular element of the group G. Suppose that $C^*(h)$ has a generalized quaternion or semidihedral Sylow 2-subgroup of order at least 32. Then G has real-valued irreducible characters of F–S invariant -1.*

Proof. We begin by assuming that $C^*(h)$ has a generalized quaternion Sylow 2-subgroup. Let U be a Sylow 2-subgroup of $C^*(h)$ and let V be a Sylow

2-subgroup of $C(h)$ contained in U. If V is cyclic, Theorem 4.1 shows that our theorem is true. Thus we can suppose that V is also generalized quaternion.

Let A be the cyclic subgroup generated by h, with $|A| = 2n + 1$, and let $H = AU$. Let w be a primitive $(2n + 1)$th root of unity and let λ_i be the irreducible character of A defined by $\lambda_i(h) = w^i$, $1 \leqslant i \leqslant n$. Let θ be any faithful irreducible character of V. Then since V is generalized quaternion, θ is real-valued, satisfies $\epsilon(\theta) = -1$, and extends to a real-valued character of U. It follows that $\sigma_i = (\lambda_i\theta)^H$ is a real-valued irreducible character of H with $\epsilon(\sigma_i) = -1$. Let D be the $n \times n$ matrix with entries $\mu_j(h^i)$, $1 \leqslant i, j \leqslant n$, where $\mu_j = \lambda_j + \bar{\lambda}_j$ is a real-valued character of A. By [6, p. 265], there is an $n \times n$ matrix E, whose entries are algebraic integers, that satisfies $DE = dI_n$; here d is an algebraic integer with $d \neq 0$. Let $\rho_i = \sigma_i^G$ and let ρ be the class function on G defined by

$$\rho = \sum_{i=1}^{n} e_i\rho_i ,$$

where $e_1 ,..., e_n$ are the entries of the first column of E.

Let $|V| = 2^{m+1}$, $m \geqslant 3$. V contains an element v of order 2^m and we can assume that $\theta(v) = \alpha + \alpha^{-1}$, where α is a 2^mth root of unity. It follows that

$$\sigma_i(vh^r) = (\alpha + \alpha^{-1})\, \mu_i(h^r).$$

We wish to calculate $\rho(vh)$. We have

$$\rho(vh) = (1/|H|) \sum_{x\in G} \sum_{i} e_i\sigma_i(xvhx^{-1}),$$

where $\sigma_i(xvhx^{-1})$ is set equal to 0 if $xvhx^{-1}$ is not in H. If $xvhx^{-1}$ belongs to H it belongs to AV, as its order is not a power of 2. Now v generates the unique subgroup of order 2^m in V, so $xvhx^{-1} = v^rh^s$ for suitable integers r and s. Then $\sigma_i(xvhx^{-1}) = (\alpha^r + \alpha^{-r})\, \mu_i(h^s)$ and the choice of $e_1 ,..., e_n$ shows that

$$\sum e_i(\alpha^r + \alpha^{-r})\, \mu_i(h^s) = 0,$$

unless $s = 1$ or -1. If $xvhx^{-1} = v^rh$ or v^rh^{-1}, then x is in $C^*(h)$. Since $C^*(h)$ has a quaternion Sylow 2-subgroup, v is conjugate only to v and v^{-1} in $C^*(h)$. Thus if M is the subgroup $\{x \in G; xvhx^{-1} = vh, vh^{-1}, v^{-1}h, \text{ or } v^{-1}h^{-1}\}$, we have

$$\rho(vh) = d\, |M : N|\, (\alpha + \alpha^{-1}).$$

However, M is a subgroup of $C^*(h)$ and so the 2-part of $|M|$ equals the 2-part of $|N|$. Thus $\rho(vh) = \beta(\alpha + \alpha^{-1})$, where β is some algebraic integer with $\beta \neq 0$. Similarly we have $\rho(v^rh) = \beta(\alpha^r + \alpha^{-r})$ for each odd integer r.

We turn to showing that some $\rho_i = \sigma_i^G$ contains a real-valued irreducible character χ of G with odd multiplicity. We must have $\epsilon(\chi) = \epsilon(\sigma_i) = -1$ for such a character. Certainly if χ is an irreducible character of G with $\epsilon(\chi) = 1$, χ occurs with even multiplicity in each ρ_i. Similarly if χ is an irreducible character that is not real-valued, χ and $\bar{\chi}$ both occur with the same multiplicity in ρ_i, since ρ_i is real-valued. For such characters that are not real-valued $\chi(v^r h) = \bar{\chi}(v^r h)$. Thus if we suppose that ρ_i contains no real-valued irreducible character with odd multiplicity, we have $\rho_i(v^r h) = 2c_{ir}$, for some algebraic integer c_{ir}. It follows that $\rho(v^r h) = 2b_r$, for some algebraic integer b_r, if none of the ρ_i contains a real-valued irreducible character with odd multiplicity. We have then

$$\rho(v^r h) = \beta(\alpha^r + \alpha^{-r}) = 2b_r .$$

But $(\alpha + \alpha^{-1})(\alpha^3 + \alpha^{-3}) \cdots (\alpha^s + \alpha^{-s}) = -2$, where $s = 2^{m-1} - 1$. It follows that $-2\beta^{2^{m-2}} = 2^{2^{m-2}} b_1 b_3 \cdots b_s$. This implies that $\beta^{2^{m-2}} \equiv 0$, which is not the case, as $\beta \not\equiv 0$. We have reached a contradiction and so it follows that G has real-valued irreducible characters of F–S invariant -1.

We turn to the case where U is semidihedral. Let V be a Sylow 2-subgroup of $C(h)$ contained in U. Since h is weakly real, V must be dihedral. Let θ be any faithful irreducible character of V; θ extends to a character of U that is not real-valued. It follows from Lemma 2.2 that if λ is a nontrivial irreducible character of A, then $(\lambda\theta)^H$ is a real-valued character of H with F-S invariant -1. We can now proceed exactly as in the first case. This completes the proof.

Notes. It is not hard to see from the proof of Theorem 4.4 that there is a real-valued constituent of ρ that has F–S invariant -1 but is not 2-rational. Thus by taking algebraic conjugates, we obtain more irreducible characters with invariant -1 (the number of conjugates is $|V|/8$). The proof developed here breaks down when the subgroup U has order 16. However, it seems probable that calculations of suitable inner products of the form (ρ_H, χ), where χ is an irreducible character of H, can be used to show that not all real-valued irreducible constituents of ρ can occur with multiplicity equal to twice an algebraic integer. Thus the same conclusion should hold in general. It should also be noted that a similar proof can be used to show that if G is any group with a generalized quaternion Sylow 2-subgroup of order 16 or more, then G has real-valued irreducible characters with F–S invariant -1. The proof can be modified to yield the same conclusion if the Sylow 2-subgroup is quaternion of order 8.

If we combine Theorems 2.4 and 4.4, we obtain the next result. It probably holds without any restriction on the order of the Sylow 2-subgroup.

THEOREM 4.5. *Let G be a group whose Sylow 2-subgroup is semidihedral of order at least 32. Then G has real-valued irreducible characters with F–S invariant -1 if and only if G has weakly real elements of odd order.*

5. Real 2-Blocks

The theory of real 2-blocks, as described by Brauer in [2, Sect. 8], provides a setting for some of the results in the previous sections, as we will now explain. Certain of our findings apply to arbitrary 2-blocks, so we have included these as well. To begin with, we recall some of the essential features of block theory.

Let p be a prime and let B be a p-block of characters of G. If χ is an irreducible character in B, there is a corresponding character ω defined on the center of the complex group algebra by $\omega(\hat{K}) = |K| \chi(g)/\chi(1)$, where $\hat{K}$ is the sum of the elements in the conjugacy class K of G and g is a representative of K.

In future, we will write $\omega(g)$ for $\omega(\hat{K})$. The $\omega(g)$ are algebraic integers and thus we can consider their values modulo some maximal ideal in a ring of p-local integers. Two irreducible characters χ_1 and χ_2 are in B if and only if their associated central characters ω_1, ω_2 satisfy $\omega_1(g) \equiv \omega_2(g) \equiv \omega(g)$ for all g in G. If p^a is the p-part of $|G|$ and p^{a-d} is the highest power of p that divides the degree of each irreducible character in B, d is said to be the defect of B. There is also a set of conjugacy classes of G, called the defect classes of B. If K is a defect class of B and g is a representative of K, we have $\omega(g)\, \chi(g^{-1}) \not\equiv 0$, where χ is an irreducible character in B whose degree is not divisible by p^{a-d-1}. Such a character χ is said to have height 0 in B. If K is any conjugacy class of G and g is in K, a Sylow p-subgroups D of $C(g)$ is called a defect group of K. Defect groups are defined up to conjugacy in G. It can be proved that if K_1, K_2 are defect classes of B, with defect groups D_1 and D_2, then D_1 and D_2 are conjugate in G and $|D_1| = p^d$. Such p-subgroups are called the defect groups of B.

If B contains the irreducible characters χ_i there is a corresponding block B^*, called the contragredient of B, which consists of the complex conjugates of the χ_i. The block B is said to be real if $B = B^*$. We observe that B is real if and only if $\omega(g) \equiv \omega(g^{-1})$ for all g in G. It is not hard to see that if B contains a real-valued irreducible character, then B is a real block. We will show that the converse statement is true if B is a real 2-block. This fact is one feature of what appears to be a reasonable theory of real 2-blocks.

THEOREM 5.1. *Let B be a real 2-block of the group G. Then B contains a real-valued irreducible 2-rational character of height 0. This character can be taken to have Schur index 1 over the rational field.*

Proof. Let χ be an irreducible character of height 0 in B and let d be the defect of B. We define a class function θ on G by

$$\theta(g) = 2^d \chi(g), \qquad \text{if } g \text{ has odd order,}$$

$$\theta(g) = 0, \qquad \text{otherwise.}$$

It is shown in [3, 62.1] that θ is a generalized character of G whose irreducible constituents are precisely the irreducible characters in B. Moreover, an irreducible character occurs in θ with odd (possibly negative) multiplicity if and only if it has height 0 in B.

Let m be the odd part of $\chi(1)$ and let U be a Sylow 2-subgroup of G. We have $\theta(h) = 0$ if h is a nonidentity element of U and $\theta(1) = m \mid U \mid$. It follows that $\theta_U = m\rho$, where ρ is the regular character of U. Thus if 1_U is the identity character of U, $(\theta_U, 1_U) = m$. Let $\theta = \sum a_i\chi_i$ be the decomposition of θ into irreducible characters of G belonging to B and let b_i be the multiplicity of 1_U in $(\chi_i)_U$. We have $\sum_i a_ib_i = m$. Since B is a real 2-block, θ contains the complex conjugate of each of its irreducible constituents. Let χ_j be an irreducible constituent of θ and let χ_k be its complex conjugate. We have $b_j = b_k$ and $a_j = a_k$ since χ_j and χ_k both have the same height in B. Thus by pairing each nonreal irreducible constituent of θ with its complex conjugate, we obtain $\sum_i a_ib_i \equiv \sum_j a_jb_j \equiv 1$, where the second sum extends over real-valued irreducible constituents of θ of height 0. There must be a real-valued irreducible character χ_j of height 0 in B with the corresponding b_j odd. Moreover, we can choose χ_j to be 2-rational, since 2-conjugates of characters in B also belong to B and their restrictions to U contain 1_U with the same multiplicity. Now χ_j occurs in $1_U{}^G$ with odd multiplicity. It follows from general results that the Schur index of χ_j over the rationals is coprime to 2. However, as χ_j is real-valued, the Brauer–Speiser theorem [4] implies that the rational Schur index is 1 or 2. Thus χ_j has Schur index 1 over the rationals and the proof is finished.

Calculation of the number of irreducible characters in a p-block has proved to be difficult. For the prime 2, some simple divisibility criteria are available.

THEOREM 5.2. *Let B be a 2-block of defect at least 2. Then the number of irreducible characters in B of height 0 is divisible by 4. If B has defect at least 3 and the number of irreducible characters of height 0 in B is not divisible by 8, B has an odd number of irreducible 2-rational characters of height 1.*

Proof. Let d be the defect of B and let $\chi_1, \ldots, \chi_t$ be the irreducible characters in B. Let U be a Sylow 2-subgroup of G. The block orthogonality relations [7, 15.23, p. 273] show that

$$\sum_{i=1}^{t} \chi_i(1)\,\chi_i(u) = 0$$

if u is a nonidentity element of U. It follows that $\sum \chi_i(1)\,\chi_i$ is a multiple of the regular character ρ of U. As 2^{a-d} divides each $\chi_i(1)$, we have $\sum \chi_i(1)\chi_i = c2^{a-d}\rho$, where c is some integer. Thus $\sum \chi_i(1)^2 = c2^{2a-d}$.

Suppose now that the notation has been chosen so that $\chi_1, \ldots, \chi_s$ are the irreducible characters of height 0 in B, and $\chi_{s+1}, \ldots, \chi_r$ those of height 1. Let $\chi_i(1) = 2^{a-d}b_i$, where b_i is odd, $1 \leqslant i \leqslant s$, and $\chi_j(1) = 2^{a-d+1}c_j$, where c_j is odd, $s+1 \leqslant j \leqslant r$. Substituting these values into the equality $\sum \chi_i(1)^2 = c2^{2a-d}$, we obtain

$$2^{2a-2d} \sum_{i=1}^{s} b_i^2 + 2^{2a-2d+2} \sum_{j=s+1}^{r} c_j^2 + f2^{2a-2d+4} = c2^{2a-d},$$

where we have taken characters of height greater than 1 into account in the term $f2^{2a-2d+4}$. As $d \geqslant 2$, $2^{2a-2d+2}$ divides 2^{2a-d}. It follows that 4 divides $\sum b_i^2$. Now if w is an odd integer, $w^2 \equiv 1 \pmod 8$. Thus the sum of the squares of odd integers is divisible by 4 only if the number of summands is divisible by 4. This means that the number of characters of height 0 in B is divisible by 4, as required.

If the number of characters of height 0 in B is not divisible by 8, the term $\sum b_i^2$ is not divisible by 8. It follows that if $d \geqslant 3$, the term $\sum c_j^2$ is odd. This means that there is an odd number of irreducible characters of height 1 in B, and hence an odd number of irreducible 2-rational characters of height 1 in B.

A simple application of the above ideas gives the next result.

THEOREM 5.3. *Let B be a real 2-block of positive defect. Then the number of real-valued irreducible 2-rational characters in B is even and greater than* 0.

A general result of Brauer shows that if p is an odd prime, then the number of irreducible p-rational characters in a p-block is at least as large as the number of irreducible Brauer characters in the block. A similar result holds for the prime 2, but the proof is different.

THEOREM 5.4. *Let B be a 2-block of positive defect. If B contains u irreducible Brauer characters, it contains at least $u+1$ irreducible 2-rational ordinary characters.*

Proof. Let $h_1, \ldots, h_t$ be representatives of all the 2-regular classes in G. By Theorem 3.1, there are t irreducible 2-rational characters $\chi_1, \ldots, \chi_t$ in G with $\det \chi_i(h_j) \not\equiv 0$, $1 \leqslant i, j \leqslant t$. We can suppose that the notation has been chosen so that $\chi_1, \ldots, \chi_v$ are in B but the remaining characters are not in B. We observe that the $(t-v) \times t$ matrix $\chi_k(h_j)$, $v+1 \leqslant k \leqslant t$, $1 \leqslant j \leqslant t$, has rank $t-v$ modulo P. It follows that there must be $t-v$ irreducible Brauer characters $\phi_{v+1}, \ldots, \phi_t$ which are modular constituents of $\chi_{v+1}, \ldots, \chi_t$ such that $\phi_k(h_j)$, $v+1 \leqslant k \leqslant t$, $1 \leqslant j \leqslant t$, also has rank $t-v$ modulo P. These Brauer characters belong to blocks different from B since the $t-v$ ordinary characters do not belong to B. Thus G has at least $t-v+u$ irreducible Brauer characters. Since the number of irreducible Brauer characters is t, we have $u \leqslant v$. By considering the distribution of the remaining irreducible Brauer characters among the blocks of G, we find that we actually have $u = v$.

We now assume that the h_i are numbered so that $\det \chi_i(h_j) \not\equiv 0$, $1 \leqslant i, j \leqslant u$. The argument of Theorem 5.1 applied to an arbitrary 2-block shows that there is an irreducible 2-rational character θ of height 0 in B. Let g be an element of a defect class for B and let m be an involution that centralizes g. We have $\bar{\theta}(gm) = \bar{\theta}(g) \neq 0$. The block orthogonality relations show that

$$\sum_{\chi} \chi(h_i)\,\bar{\chi}(gm) = 0, \qquad 1 \leqslant i \leqslant u,$$

the sum extending over the irreducible characters in B. Let W be the Galois group of $Q_{|G|}$ over Q_b. Then for σ in W we have $\chi^{\sigma}(h_i) = \chi(h_i)$ and $\bar{\chi}^{\sigma}(gm) = \bar{\chi}(gm)$. Moreover the W-conjugates of characters in B are themselves in B. Thus following the argument given in Theorem 3.3 we have

$$\sum \chi'(h_i)\,\bar{\chi}'(gm) = 0, \qquad 1 \leqslant i \leqslant u,$$

the sum extending now over irreducible 2-rational characters in B. We see from this that $\chi_1, \ldots, \chi_u$ cannot be the only 2-rational irreducible characters in B. For if this is so, the fact that $\det \chi_i(h_j) \neq 0$ forces the conclusion that $\bar{\chi}(gm) = 0$ for any irreducible 2-rational character χ in B. This contradicts our finding that $\bar{\theta}(gm) \neq 0$. Thus B has at least $u + 1$ irreducible 2-rational characters.

Returning to the subject of real 2-blocks, we show next that a real 2-block has a real defect class. The identity class will serve as a real defect class for the principal block, so we can restrict our attention to blocks other than the principal block.

THEOREM 5.5. *Let B be a real nonprincipal 2-block. Then B has a real non-identity defect class.*

Proof. Let χ be a real-valued character of height 0 in B and let d be the defect of B. Let θ be the generalized character constructed from χ in the manner described in Theorem 5.1. Since χ occurs in θ with odd multiplicity we see that

$$c = (2^d/|G|) \sum \chi(g)^2$$

is an odd integer, the sum ranging over 2-regular elements in G. If $g_1, \ldots, g_t$ are representatives of the 2-regular classes in G, we have

$$c = (2^d\chi(1)/|G|) \sum_{i=1}^{t} \omega(g_i)\,\chi(g_i).$$

Since $2^d\chi(1)/|G|$ is a 2-local integer, we must have $\sum_{i=1}^{t} \omega(g_i)\,\chi(g_i) \neq 0$.

We know that $\omega(g)\,\chi(g) = 0$ unless g belongs to a defect class for B. Moreover, $\omega(g)\,\chi(g) = \omega(g^{-1})\,\chi(g^{-1})$ as χ is real-valued. Thus, pairing each nonreal defect class with its inverse, we obtain

$$\sum_{j=1}^{s} \omega(g_j)\,\chi(g_j) \neq 0,$$

where the sum now extends only over real defect classes. Thus B has at least one real defect class. We have to show that this is not the identity class.

Let us suppose on the contrary that the identity class is the only real defect class for B. It follows that B has maximal defect and that χ has odd degree. Since B is not the principal block, 1_G does not belong to B and so $(\theta, 1_G) = 0$. Thus we have

$$\sum \chi(g) = 0,$$

the sum extending over 2-regular elements of G. Thus we have

$$\sum_{i=1}^{t} \omega(g_i) = 0.$$

A term $\omega(g_i)$ satisfies $\omega(g_i) = 0$ unless g_i has maximal defect. If g_i has maximal defect and is not the identity, g_i is not real. Thus we obtain

$$\sum_{i=1}^{t} \omega(g_i) = \omega(1) \neq 0.$$

This contradicts our previous deduction. Thus there is a nonidentity real defect class for B, and the defect of B is not maximal. This completes the proof.

If B is a real 2-block and g is an element belonging to a real defect class for B, we can think of a Sylow 2-subgroup of $C^*(g)$ as an extended defect group of B. It seems reasonable to ask if the extended defect groups, corresponding to different real defect classes, form a single conjugacy class of 2-subgroups. We have been unable to answer this question, but examination of some examples suggests that the conjugacy may indeed occur.

We have seen that a real 2-block always contains a real-valued irreducible character of height 0 and F–S invariant 1. Necessary and sufficient conditions for a real 2-block to contain an irreducible character of height 0 and F–S invariant -1 can be found from the results of Section 4.

THEOREM 5.6. *Let B be a real 2-block with defect group V. Let h be an element of a real defect class for B with defect group V. Let U be a Sylow 2-subgroup of $C^*(h)$ containing V. Then B contains a real-valued irreducible character of height* 0 *and F–S invariant* -1 *if and only if U/V' does not split over V/V'.*

Proof. Let $h = g_1, \ldots, g_r$ be representatives of the real 2-regular classes satisfying the hypotheses of Theorem 4.1. There exist real-valued irreducible characters $\chi_1, \ldots, \chi_r$, each of F–S invariant -1, satisfying $\det \chi_i(h_j) \neq 0$, $1 \leq i, j \leq r$. We will show that at least one of the χ_i is a character of height 0 in B.

Let ω be the central character associated with B. The argument used in Theorem 5.5 shows that an irreducible character χ has height 0 in B if and only if

$$\sum_{i=1}^{s} \omega(g_i)\,\chi(g_i) \not\equiv 0,$$

where the sum extends over representatives of the s real 2-regular classes. Now if g is a real 2-regular element, 4.3 shows that $\chi_i(g) = 0$, $1 \leqslant i \leqslant r$, unless g is conjugate to one of $g_1, \ldots, g_r$. Thus

$$\sum_{i=1}^{s} \omega(g_i)\,\chi_j(g_i) = \sum_{i=1}^{r} \omega(g_i)\,\chi_j(g_i), \qquad 1 \leqslant j \leqslant r.$$

It follows that if none of the $\chi_1, \ldots, \chi_r$ has height 0 in B, then we have

$$\sum_{i=1}^{r} \omega(g_i)\,\chi_j(g_i) \equiv 0, \qquad 1 \leqslant j \leqslant r.$$

Since $\det \chi_j(g_i) \not\equiv 0$, $\omega(g_i) \equiv 0$, $1 \leqslant i \leqslant r$. However, $g_1 = h$ belongs to a defect class for B and so $\omega(g_i) \not\equiv 0$. Thus at least one of the χ_i has height 0 in B, as required.

Conversely, if B has a real-valued irreducible character χ of height 0 and F–S invariant -1, $\chi(h) \not\equiv 0$, since h belongs to a defect class for B. Theorem 4.3 shows that U/V' does not split over V/V'.

6. Real-Valued Characters and Normal Subgroups

In this section we use well-known results and terminology from the Clifford theory of irreducible characters. Let N be a normal subgroup of the group G and let θ be an irreducible character of N. We say that θ belongs to a real G-orbit of characters of N if there is some g in G with $\theta^g = \bar{\theta}$. We define the extended stabilizer T of θ to be the subgroup $T = \{g \text{ in } G : \theta^g = \theta \text{ or } \bar{\theta}\}$.

Consider now a real-valued irreducible character χ of G. By Clifford's theorem, the irreducible constituents of χ_N are the members of some G-orbit of characters of N. Since χ is real-valued, this orbit must be real. Let θ be an irreducible constituent of χ_N, appearing with multiplicity r, and let S be the stabilizer of θ. Again by Clifford's theorem, there is an irreducible character ϕ of S such that $\phi^G = \chi$ and $\phi_N = r\theta$. Moreover, if ψ is any irreducible character of S that satisfies $(\psi_N, \theta) \neq 0$, ψ^G is irreducible, and if ψ_1, ψ_2 are any two such characters of S, $\psi_1{}^G = \psi_2{}^G$ only if $\psi_1 = \psi_2$ [7, 6.11, p. 82]. Using the above notation, we have the following result, whose proof is straightforward and therefore omitted.

LEMMA 6.1. *If T is the extended stabilizer of θ, ϕ^T is a real-valued irreducible character of T. Moreover, if σ is any irreducible character of T with $(\sigma_N, \theta) \neq 0$, σ^G is irreducible and is real-valued only if σ is real-valued.*

The number of real orbits of irreducible characters of N can easily be characterized, as the next lemma shows. The proof follows easily from Brauer's permutation lemma.

LEMMA 6.2. *Let N be a normal subgroup of the group G. The number of real G-orbits of irreducible characters of N equals the number of real classes of G that belong to N.*

We wish now to obtain more detailed information about the real-valued irreducible characters of a group and the real orbits of irreducible characters of a normal subgroup of odd order. The methods that we use are related to problems in the theory of real 2-blocks. We begin with a lemma of a general nature.

LEMMA 6.3. *Let M be a subgroup of the group G and let θ be a real-valued irreducible character of M. Suppose that $\chi = \theta^G$ is an irreducible character of G. Then if h is a real element of odd order in G that satisfies $\chi(h) \neq 0$, M contains a conjugate of a Sylow 2-subgroup of $C^*(h)$.*

Proof. Let U be a Sylow 2-subgroup of $C^*(h)$ and let

$$G = Ug_1M + \cdots + Ug_tM$$

be the decomposition of G into U, M double cosets. Let $|Ug_iM| = a_i|M|$. We will show that there is some a_i which is odd, and thus equals 1. This will imply that a conjugate of U is contained in M.

We have

$$\chi(h) = (1/|M|) \sum_{x \in G} \theta(x^{-1}hx),$$

where $\theta(x^{-1}hx) = 0$ if $x^{-1}hx$ is not in M. If $x = ug_im$, u in U, m in M, we have $x^{-1}hx = m^{-1}g_i^{-1}h^rg_im$, where $r = 1$ or -1. Thus if $x^{-1}hx$ is in M, we have

$$\theta(x^{-1}hx) = \theta(g_i^{-1}hg_i),$$

since θ is real-valued. We obtain

$$\chi(h) = \sum a_i\theta(g_i^{-1}hg_i),$$

where the sum extends over those indices for which $g_i^{-1}hg_i$ is in M. Since $\chi(h) \not\equiv 0$, we see that there is at least one index i for which a_i is odd. This gives the required result.

We consider next a normal subgroup N of odd order in G. We seek to relate the Sylow 2-subgroup of the extended centralizer of a real element in N to the Sylow 2-subgroup of the extended stabilizer of an irreducible character of N in a real G-orbit. Our methods are based on a paper of Wada [8].

THEOREM 6.4. *Let N be a normal subgroup of odd order of the group G. Let $n_1, \ldots, n_r$ be nonconjugate real elements in N. Let U_i be a Sylow 2-subgroup of $C^*(n_i)$ and V_i a Sylow 2-subgroup of $C(n_i)$ contained in U_i. Suppose that $V = V_1$ is conjugate to each V_i, $1 \leqslant i \leqslant r$. Then there exist r real-valued irreducible characters $\chi_1, \ldots, \chi_r$ of G that belong to r distinct 2-blocks, each having defect group V. The χ_i satisfy $\det \chi_i(n_j) \not\equiv 0$, $1 \leqslant i, j \leqslant r$. Moreover, for each χ_i there is an irreducible constituent θ_i of $(\chi_i)_N$ whose extended stabilizer has Sylow 2-subgroup U_i.*

Proof. We define class functions θ_{ij} on G by

$$\theta_{ij} = \sum_k \chi_k(n_i)\, \chi_k(n_j)\, \chi_k/\chi_k(1),$$

where the sum extends over all irreducible characters χ_k of G. Let W be a Sylow 2-subgroup of G. Since all the conjugates of the n_i belong to N, $\theta_{ij}(w) = 0$ for any nonidentity element w of W [5, 2.15, p. 16]. We also have $\theta_{ij}(1) = |C(n_i)|\, \delta_{ij}$. Let ρ be the character of the regular representation of W. Our observations show that

$$\theta_{ij} = |C(n_i)|\, \rho\, \delta_{ij}/|W|$$

on W. Thus if 1_W is the trivial character of W, we have

$$\begin{aligned} |W|\, ((\theta_{ij})_W, 1_W) &= |C(n_i)|\, \delta_{ij} \\ &= |W| \sum_k \chi_k(n_i)\, \chi_k(n_j)((\chi_k)_W, 1_W)/\chi_k(1). \end{aligned}$$

We obtain

$$|G : W|\, \delta_{ij} = \sum_k \omega_k(n_i)\, \chi_k(n_j)\, a(\chi_k),$$

where ω_k is the central character associated with χ_k and $a(\chi_k) = ((\chi_k)_W, 1_W)$.

Since the n_i are real, we have $\chi_k(n_i) = \bar{\chi}_k(n_i)$ and $\omega_k(n_i) = \bar{\omega}_k(n_i)$. Thus we obtain

$$|G : W|\, \delta_{ij} \equiv \sum_m \omega_m(n_i)\, \chi_m(n_j),$$

where the sum is taken only over real-valued characters χ_m with $a(\chi_m)$ odd. We can also neglect those characters χ_m for which $\omega_m(n_i) \equiv 0$, $1 \leqslant i \leqslant r$, since these characters contribute nothing to the sum modulo P. Let us suppose that there are t irreducible real-valued characters χ_m with $a(\chi_m)$ odd and not all $\omega_m(n_i) \equiv 0$.

We can assume that these characters are $\chi_1, \dots, \chi_t$. Thus if Y denotes the $r \times t$ matrix $\omega_m(n_i)$, $1 \leqslant i \leqslant r$, $1 \leqslant m \leqslant t$, and if X denotes the $t \times r$ matrix $\chi_m(n_j)$, $1 \leqslant m \leqslant t$, $1 \leqslant j \leqslant r$, we have

$$| G : W | I_r \equiv YX.$$

Since $| G : W |$ is odd, both X and Y have rank r modulo P.

By a suitable renumbering of the n_i and the χ_i, we can assume that the r characters $\chi_1, \dots, \chi_r$ satisfy det $\chi_i(n_j) \not\equiv 0$, $1 \leqslant i, j \leqslant r$, and $\chi_i(n_i) \not\equiv 0$, $1 \leqslant i \leqslant r$. If we have $\omega_i(n_i) \equiv 0$, it follows that $| K_i |/\chi_i(1) \equiv 0$, where K_i is the class of G containing n_i. Since the same power of 2 divides each $| K_j |$, $1 \leqslant j \leqslant r$, each $\omega_i(n_j) \equiv 0$, $1 \leqslant j \leqslant r$. However, we have omitted all characters from the sum that satisfy this congruence. Thus we must have $\omega_i(n_i) \not\equiv 0$ as well. From the theory of blocks, we see that χ_i belongs to a block B_i with defect group V and that χ_i has height 0 in B_i. Since the characters $\chi_1, \dots, \chi_r$ are independent on N, it is not hard to see that they belong to r different blocks $B_1, \dots, B_r$.

Let θ_i be an irreducible constituent of $(\chi_i)_N$. Since χ_i is real-valued, θ_i belongs to a real orbit of irreducible characters of N. By Lemma 6.1, there is a real-valued irreducible character σ_i of the extended stabilizer T_i of θ_i that satisfies $\sigma_i{}^G = \chi_i$. We know that $\chi_i(n_i) \not\equiv 0$ and so 6.3 shows that a conjugate of U_i is contained in T_i. We also observe that if $(\theta_i, (\chi_i)_N) = c$, the degree of σ_i is $2c\theta_i(1)$. Since θ_i and all its conjugates occur c times in $(\chi_i)_N$, we have $\chi_i(n_i) = c\alpha$, for some algebraic integer α. However, $\chi_i(n_i) \not\equiv 0$ and so c is odd. Now

$$\chi_i(1) = | G : T_i | \sigma_i(1) = 2 | G : T_i | c\theta_i(1)$$

and thus the 2-part of $\chi_i(1)$ is twice the 2-part of $| G : T_i |$. However we know that the 2-part of $\chi_i(1)$ is the 2-part of $| G : V |$. It follows that a Sylow 2-subgroup of T_i has the order of U_i and thus some conjugate of U_i is a Sylow 2-subgroup of T_i. By changing if necessary to some conjugate of θ_i, we can assume that U_i is a Sylow 2-subgroup of T_i, as required. This completes the proof.

7. Real-Valued Characters of 2-Nilpotent Groups

In this final section we combine results from Sections 2 and 6 to investigate the real-valued irreducible characters of 2-nilpotent groups. As a consequence, we obtain some new solutions to Brauer's problem of characterizing the number of irreducible characters with F–S invariant 1. We assume a knowledge of Glauberman's character correspondence for relatively prime operator groups, an exposition of which is given in [7, Chap. 13].

Let G be a 2-nilpotent group and let N be a normal 2-complement of G. Let θ be an irreducible character of N that belongs to a real G-orbit. Let S be the stabilizer of θ and let T be its extended stabilizer. Since N is a normal Hall subgroup of G, θ can be extended to a character ϕ of S. The extension ϕ can be chosen so that ϕ is 2-rational and also so that $\det \phi = 1$ for all 2-elements of S (this means that if F is a representation of S with character ϕ, $\det F(h) = 1$ for all 2-elements h of S). We say that ϕ is the canonical extension of θ to S, for it is the unique extension of θ with these properties. It is clear that $\bar{\phi}$ is the canonical extension of $\bar{\theta}$. Since for g in $T - S$, $\theta^g = \bar{\theta}$, we must have $\phi^g = \bar{\phi}$ by the uniqueness of $\bar{\phi}$. It follows that ϕ^T is an irreducible real-valued character of T and $\chi = \phi^G$ is an irreducible real-valued character of G. We will call ϕ^T the canonical real extension of $\theta + \bar{\theta}$ to T Our next result gives more numerical details about the character ϕ^T.

LEMMA 7.1. *Let μ denote the canonical real extension of $\theta + \bar{\theta}$ to T. Then μ has* F–S *invariant* 1. *Moreover if $x \in T - S$ and has order* 2,

$$\sum_N \mu(nx)^2 = 2 \mid N \mid;$$

if x has order a power of 2 greater than 2, we have

$$\sum_N \mu(nx)^2 + \mu(nx^r)^2 = 4 \mid N \mid,$$

where $x^{2r} = x^2$, $x^r \neq x$.

Proof. Let U denote the cyclic subgroup of T generated by x and let V be the subgroup of S generated by x^2. Let ϕ denote the canonical extension of θ to S. Since ϕ_V is rational-valued and has odd degree, it must contain a rational-valued irreducible character of V with odd multiplicity. The determinant condition on ϕ forces this character to be the trivial character of V. Since $\mu_U = (\phi_V)^U$, it follows that μ_U contains the trivial character of U an odd number of times. An argument used in Theorem 5.1 shows that μ has F–S invariant 1 (indeed, μ has Schur index 1 over the rationals).

We will now assume that $T = NU$, $S = NV$. Let λ be an irreducible character of S/N. The character $\phi\lambda$ is an irreducible character that extends θ, and $(\phi\lambda)^T$ is an irreducible character of T; $(\phi\lambda)^T$ is real-valued if and only if λ is real-valued. Moreover, if λ_1, λ_2 are distinct irreducible characters of S/N, $(\phi\lambda_1)^T$ and $(\phi\lambda_2)^T$ are distinct. Our next intention is to show that if λ is a nontrivial real-valued character of S/N, then $(\phi\lambda)^T$ has F–S invariant -1.

We have $\sum_T \mu(t^2) = \mid T \mid$, as $\epsilon(\mu) = 1$. As the kernel of λ has index 2 in S, we have

$$(\phi\lambda)^T(t^2) = \phi^T(t^2) = \mu(t^2),$$

if t is in S. If t is in $T - S$, t can be written in the form $t = x^r n$, where r is an odd integer and n is in N. In this case,

$$(\phi\lambda)^T(t^2) = (\phi(t^2) + \bar{\phi}(t^2))\, \lambda(x^{2r}) = -\mu(t^2),$$

as $x^{2r}N$ is a generator of S/N. Thus

$$\sum_T (\phi\lambda)^T(t^2) = \sum_S \mu(s^2) - \sum_{T-S} \mu(t^2).$$

Since $\mu_S = \phi + \bar{\phi}$ is the sum of two irreducible characters that are not real-valued, $\sum_S \mu(s^2) = 0$. It follows that $\sum_{T-S} \mu(t^2) = | T |$ and thus $\sum_T (\phi\lambda)^T(t^2) = -| T |$. This proves that the F–S invariant of $(\phi\lambda)^T$ is -1 if λ is a nontrivial real-valued character of S/N.

Let us first suppose that x has order 2. The previous arguments have shown that

$$\sum \mu(t^2) = | T | = 2 | N |.$$

As any element of $T - S$ has the form nx, we obtain

$$\sum_{n\in N} \mu(nx)^2 = 2 | N |,$$

as required. We suppose next that x has order 4 or more. We define a class function σ on T by

$$\sigma = \sum \bar{\lambda}(x^2)(\phi\lambda)^T,$$

where the sum extends over all irreducible characters λ of S/N. We have

$$(1/| T |) \sum_T \sigma(t^2) = \sum_\lambda \bar{\lambda}(x^2)\, \epsilon(\phi\lambda)^T.$$

We know that $(\phi\lambda)^T$ is not real-valued if λ is not real-valued, whereas $\epsilon(\phi^T) = 1$ and $\epsilon(\phi\lambda)^T = -1$, when λ is a nontrivial real-valued character of S/N. Since for such a character λ we have $\lambda(x^2) = -1$, we obtain $\sum \sigma(t^2) = 2 | T |$.

Let $t = nx^r$ be an arbitrary element of T, where n is in N. We have

$$\begin{aligned} \sigma(t^2) &= \sum_\lambda \bar{\lambda}(x^2)(\phi(t^2) + \bar{\phi}(t^2))\, \lambda(x^{2r}) \\ &= \mu(t^2) \sum_\lambda \bar{\lambda}(x^2)\, \lambda(x^{2r}). \end{aligned}$$

If $x^2 \neq x^{2r}$, the orthogonality relations for characters of S/N give

$$\sum \bar{\lambda}(x^2)\, \lambda(x^{2r}) = 0.$$

If $x^2 = x^{2r}$, we obtain

$$\sum \bar{\lambda}(x^2)\, \lambda(x^{2r}) = | S : N |.$$

Thus if $t = nx^r$, $\sigma(t^2) = 0$ if $x^2 \neq x^{2r}$, and $\sigma(t^2) = | S : N | \mu(t^2)$ if $x^2 = x^{2r}$. Summing, we obtain

$$\sum_T \sigma(t^2) = | S : N | \sum_N \mu(nx)^2 + \mu(nx^r)^2,$$

where $x^{2r} = x^2$, $x^r \neq x$. As we already know that $\sum \sigma(t^2) = 2 | T |$, we see that

$$\sum \mu(nx)^2 + \mu(nx^r)^2 = 2 | T |/| S : N | = 4 | N |.$$

LEMMA 7.2. *Let $x \in T - S$ have order a power of 2 greater than 2 and let x^r satisfy $x^{2r} = x^2$, $x \neq x^r$. Then if*

$$\sum_N \phi(nx)^2 + \phi(nx^r)^2 = \alpha(x) + i\beta(x),$$

we have $\alpha(x) = 2 | N |$ and $\beta(x) = \beta(x^{-1})$.

Proof. We have $\mu(nx)^2 = \phi(nx)^2 + \bar{\phi}(nx)^2$. Thus Lemma 7.1 gives

$$\alpha(x) + i\beta(x) + \alpha(x) - i\beta(x) = 4 | N |.$$

We obtain $\alpha(x) = 2 | N |$. As $x \in T - S$, we have $\phi(xtx^{-1}) = \bar{\phi}(t)$ for all t in S. Thus for n in N

$$\phi(nx)^2 = \phi(x(x^{-1}n^{-1})^2x^{-1}) = \phi(n^{-1}x^{-1})^2.$$

However

$$\sum_N \phi(n^{-1}x^{-1})^2 = \sum_N \phi(nx^{-1})^2,$$

since n^{-1} runs over the elements of N as n does. Thus

$$\sum_N \phi(nx^{-1})^2 + \phi(nx^{-r})^2 = \sum_N \phi(nx)^2 + \phi(nx^r)^2$$

and so $\alpha(x^{-1}) + i\beta(x^{-1}) = \alpha(x) + i\beta(x)$. This yields $\beta(x) = \beta(x^{-1})$, as required.

Let us now review what we have proved so far. We take θ to be an irreducible character of N in a real G-orbit. S denotes the stabilizer of θ and T denotes the extended stabilizer. If ϕ is the canonical extension of θ to S, ϕ^T is a real-valued irreducible character of T and $\epsilon(\phi^T) = 1$. Moreover $\chi = \phi^G$ is a real-valued irreducible character of G and $\epsilon(\chi) = \epsilon(\phi^T) = 1$. We also know that if σ is any irreducible character of S/N, considered as a character of S, $(\phi\sigma)^G$ is irreducible. Thus $(\phi\sigma)^T$ is irreducible and 6.1 tells us that $(\phi\sigma)^G$ is real-valued if and only if $(\phi\sigma)^T$ is real-valued. Let U be a Sylow 2-subgroup of T and V a Sylow 2-subgroup of S contained in U. We can consider σ to be an irreducible character of V. Our next result shows that the F–S invariant of $(\phi\sigma)^T$ (and hence of $(\phi\sigma)^G$) is determined by the value of $\eta(\sigma)$.

LEMMA 7.3. *Let σ be an irreducible character of V satisfying $\sigma^u = \bar{\sigma}$ for u in $U - V$. Interpret σ as an irreducible character of S whose kernel contains N. Then $(\phi\sigma)^T$ is an irreducible character of T and $\epsilon(\phi\sigma)^T = \eta(\sigma)$.*

Proof. Let χ denote $(\phi\sigma)^T$ and let $w = \epsilon(\chi)$. Since χ_S is the sum of two irreducible characters of S that are not real-valued, we have

$$\sum_S \chi(t^2) = 0.$$

We can write an element t of $T - S$ in the form $t = nx$ where n is in N, x is in $U - V$. Thus

$$\chi(t^2) = \phi(nx)^2\sigma(x^2) + \bar{\phi}(nx)^2\bar{\sigma}(x^2).$$

We obtain

$$\begin{aligned} |T|\, w \quad \sum_{T-S} \chi(t^2) &= \sum_{u\in U-V} \left(\sum_N \phi(nu)^2 + \bar{\phi}(nu)^2\right) \sigma(u^2) \\ &+ \sum_{x\in U-V} \left(\sum_N \phi(nx)^2 + \phi(nx^r)^2\right) \sigma(x^2)/2 \\ &+ \sum_{x\in U-V} \left(\sum_N \bar{\phi}(nx)^2 + \bar{\phi}(nx^r)^2\right) \bar{\sigma}(x^2)/2. \end{aligned}$$

Here, the first sum extends over involutions u in $U - V$, whereas the second and third sums extend over half the elements x of $U - V$ of order greater than 2, x being paired with x^r, where $x^{2r} = x^2$, $x^r \neq x$. Using Lemmas 7.1 and 7.2 we obtain

$$\begin{aligned} |T|\, w &= \sum 2\sigma(u^2)\,|N| + \sum (\alpha(x) + i\beta(x))\,\sigma(x^2)/2 \\ &+ \sum (\alpha(x) - i\beta(x))\,\bar{\sigma}(x^2)/2 \\ &= |T|\,\eta(\sigma) + \sum i\beta(x)(\sigma(x^2) - \bar{\sigma}(x^2)). \end{aligned}$$

Now if $x \in U - V$ and has order greater than 2, x^{-1} has the same property. Thus corresponding to a term $\beta(x)(\sigma(x^2) - \bar{\sigma}(x^2))$, we have a term $\beta(x^{-1})(\sigma(x^{-2}) - \bar{\sigma}(x^{-2}))$. Since $\beta(x) = \beta(x^{-1})$ from 7.2 and $\sigma(x^{-2}) = \bar{\sigma}(x^2)$, the terms $\beta(x)(\sigma(x^2) - \bar{\sigma}(x^2))$ cancel out in pairs. We obtain $\epsilon(\phi\sigma)^T = \eta(\sigma)$, as required.

This result may require some interpretation. It is not hard to see, using Lemma 6.2 and Theorem 6.4, that the subgroup U appearing in Lemma 7.3 is the Sylow 2-subgroup of the extended centralizer of some real element h. of N, and V is the Sylow 2-subgroup of the centralizer of h. The expression $\eta(\sigma)$ is that which determines the F–S invariants of the characters of the R-elementary subgroup $H = AU$, where A is the cyclic subgroup generated by h. We are in a position to begin to calculate the number of real-valued irreducible characters of G of a given invariant in terms of the same quantities evaluated for subgroups of the type H.

THEOREM 7.4. *Let G be a 2-nilpotent group and let $h_1, \ldots, h_t$ be representatives of the nontrivial 2-regular real classes of G (we assume that $t \geq 1$, otherwise there is nothing to prove). Let U_i be a Sylow 2-subgroup of $C^*(h_i)$ and let $H_i = A_i U_i$, where A_i is the cyclic subgroup generated by h_i. Suppose that for any nontrivial linear character λ of A_i, H_i has exactly r_i real-valued irreducible characters ρ with $(\rho_{A_i}, \lambda) \neq 0$ and $\epsilon(\rho) = 1$ (r_i is independent of λ). Suppose also that a Sylow 2-subgroup of G has exactly r real-valued irreducible characters with F–S invariant 1. Then G has exactly $r + \sum_{i=1}^{t} r_i$ real-valued irreducible characters with F–S invariant 1.*

Proof. Let N be a normal 2-complement of G. The real-valued irreducible characters χ of G have the form $\chi = (\phi\sigma)^G$, where ϕ is the canonical extension of an irreducible constituent θ of χ_N to its stabilizer, S, and σ is an irreducible character of S/N satisfying $\sigma^t = \bar{\sigma}$ for t in $T - S$, T being the extended stabilizer of θ. Moreover $\epsilon(\chi) = \epsilon(\phi\sigma)^T$. Conversely, an irreducible character θ of N in a real G-orbit gives rise to real-valued irreducible characters χ of G of the form $(\phi\sigma)^G$. Thus we need to find the number of real-valued characters of the form $(\phi\sigma)^G$ and to calculate their invariants.

By Lemma 6.2 there are $t + 1$ distinct real orbits $W_0, \ldots, W_t$ of characters of N. Here, W_0 is taken to consist of the trivial character of N. The r real-valued irreducible characters of G/N with invariant 1 are associated with W_0. Let $\theta_1, \ldots, \theta_t$ be representative characters in the orbits $W_1, \ldots, W_t$. By Theorem 6.4, it can be assumed that if T_i is the extended stabilizer of θ_i, then $T_i = NU_i$. Moreover, if S_i is the stabilizer of θ_i, $S_i = NV_i$. We will now work with a fixed i and drop the indices, putting $S_i = S$, $\theta_i = \theta$, etc.

If σ is an irreducible character of V, with $\sigma^t = \bar{\sigma}$ for t in $U - V$, we can interpret σ as an irreducible character of S, and if ϕ is the canonical extension of θ to S, $(\phi\sigma)^T$ is an irreducible real-valued character of T. By 7.3,

$$\epsilon(\phi\sigma)^T = \eta(\sigma).$$

However in $H = AU$, if λ is a nontrivial irreducible character of A, $(\lambda\sigma)^H$ is a real-valued irreducible character of H, with

$$\epsilon(\lambda\sigma)^H = \eta(\sigma).$$

Thus $(\lambda\sigma)^H$ and $(\phi\sigma)^T$ have the same invariant and so $(\lambda\sigma)^H$ and $(\phi\sigma)^G$ have the same invariant. We have set up a correspondence between real-valued irreducible characters ρ of H satisfying $(\rho_A, \lambda) \neq 0$ and real-valued irreducible characters χ of G satisfying $(\chi_N, \theta) \neq 0$. This correspondence preserves the F–S invariants of the characters concerned. By examining each subgroup H_i in turn, our formula for the number of irreducible characters of G with the given invariant can be seen to hold. This completes the proof.

Theorem 7.4 is really only a reduction of a problem concerning the 2-nilpotent group G to a problem concerning certain 2-subgroups of G. This suggests that Brauer's problem of characterizing in group-theoretic terms the number of irreducible characters with a given invariant is ultimately a problem about characters of 2-groups. If we select some special types of Sylow 2-subgroup of G, we can give solutions of Brauer's problem that are intelligible in group-theoretic terms. We begin by proving a lemma.

LEMMA 7.5. *Let h be a nonidentity real 2-regular element of a group and let U be a Sylow 2-subgroup of $C^*(h)$. Let A be the cyclic subgroup generated by h and let $H = AU$. Suppose that U is Abelian. If h is strongly real, all real-valued irreducible characters of H have* F–S *invariant* 1. *If h is weakly real and λ is a nontrivial irreducible character of A, the number of irreducible characters χ of H satisfying $(\chi_A, \lambda) \neq 0$ and $\epsilon(\chi) = 1$ is $(1 + w)/2$, where w is the number of involutions in V.*

Proof. Let V be a subgroup of index 2 in U that centralizes h. If h is strongly real, H splits over AV and the result follows from [5, 11.7, p. 64]. If h is not strongly real, U does not split over V. By 2.2, the number of characters χ of H with the stated property is the number of nontrivial real-valued irreducible characters of V that extend to characters of U that are also real-valued. This is easily seen to be half the number of real-valued irreducible characters of V. Duality for the characters of Abelian groups shows that the latter number is $(w + 1)/2$.

We can give a solution to Brauer's problem for a 2-nilpotent group with an Abelian Sylow 2-subgroup.

THEOREM 7.6. *Let G be a 2-nilpotent group with an Abelian Sylow 2-subgroup. The number of real-valued irreducible characters χ of G satisfying $\epsilon(\chi) = 1$ is the sum of the number of strongly real classes in G and half the number of weakly real classes in G.*

Proof. Let $h_0 = 1, h_1, \ldots, h_t$ be representatives of the real 2-regular classes in G. Let V_i be a Sylow 2-subgroup of $C(h_i)$ and let U_i be a Sylow 2-subgroup of $C^*(h_i)$ containing V_i. Thus $V_0 = U_0$ is a Sylow 2-subgroup of G. If u is an involution of V_i, uh_i is a real element of G and it is strongly real if and only if h_i is strongly real. Moreover, if u and v are distinct involutions of V_i, uh_i and vh_i are not conjugate (this is because G is 2-nilpotent). We also observe that any real element of G is conjugate to some element of the form uh_i, where u commutes with h_i, $u^2 = 1$. Thus if V_i contains w_i involutions, the number of real classes of G is $\sum_{i=0}^{t}(1 + w_i)$. Moreover if we assume that the h_i are numbered so that

$h_1, \ldots, h_s$ are strongly real, but the remaining classes are weakly real, the number of strongly real classes in G is $\sum_{i=0}^{s}(1 + w_i)$.

Following Theorem 7.4 and Lemma 7.5, the number r_i from H_i is $1 + w_i$ if $1 \leq i \leq s$ and is $(1 + w_i)/2$ if $i > s$. Moreover the number r is $w_0 + 1$, where w_0 is the number of involutions in the Sylow 2-subgroup. Thus the number of irreducible real-valued characters χ with $\epsilon(\chi) = 1$ is

$$\sum_{i=0}^{s} (1 + w_i) + \sum_{i>s} (1 + w_i)/2.$$

This number is the sum of half the number of weakly-real classes and the number of strongly-real classes, as required.

A similar result holds if G is 2-nilpotent and its Sylow 2-subgroup is dihedral. In principle, the number of irreducible characters χ with $\epsilon(\chi) = 1$ could also be worked out for a 2-nilpotent group whose Sylow 2-subgroup is quaternion or semidihedral, but it is difficult to give group-theoretic interpretations to the numbers which emerge. We speculate that Theorem 7.6 may be true for any group with an Abelian Sylow 2-subgroup.

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