Available online at www.sciencedirect.com
ScienceDirect
Physics Letters B 650 (2007) 428–431
PHYSICS LETTERS B
www.elsevier.com/locate/physletb

A simple derivation of the overlap Dirac operator

C.D. Fosco [a,*], G. Torroba [b], H. Neuberger [b]

[a] *Centro Atómico Bariloche and Instituto Balseiro, Comisión Nacional de Energía Atómica, R8402AGP Bariloche, Argentina*
[b] *Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855, USA*

Received 18 April 2007; accepted 22 May 2007

Available online 26 May 2007

Editor: M. Cvetič

Abstract

We derive the vector-like four-dimensional overlap Dirac operator starting from a five-dimensional Dirac action in the presence of a delta-function space–time defect. The effective operator is obtained by first integrating out all the fermionic modes in the fixed gauge background, and then identifying the contribution from the localized modes as the determinant of an operator in one dimension less. We define physically relevant degrees of freedom on the defect by introducing an auxiliary defect-bound fermion field and integrating out the original five-dimensional bulk fields.

1. Introduction

The overlap Dirac operator [1,2] has a somewhat unusual form. On the lattice, its determinant gives the exact effective action that is associated with a pair of left-handed fermions in conjugate representations located on two infinitely separated, parallel domain walls, embedded in five dimensions, and subjected to the same four-dimensional gauge field background [3]. The five-dimensional gauge field is independent of the coordinate orthogonal to the domain walls and has no component in that (fifth) direction [4–6]. Thus, the five-dimensional gauge field is defined by a four-dimensional one. The five-dimensional space is $R \times \mathcal{M}$ where $\mathcal{M}$ is four-dimensional, and for definiteness, compact and latticized (hence, IR and UV regularized).

The exact effective action is defined by a limiting procedure whereby R is replaced by a finite length segment L, imposing specific boundary conditions at the endpoints of L. A "bulk" action is defined from the system $S \times \mathcal{M}$ where S is a circle, which is of the same length as L but now one has translational invariance along S. The effective action is then obtained by dividing the fermion determinant associated with $R \times \mathcal{M}$ by the fermion determinant corresponding to the $S \times \mathcal{M}$ system, and taking the length of L to infinity [7].

The original derivation was different [1], since it dealt first with a single domain wall embedded in five dimensions, which produced a formula for the action associated with a single left-handed fermion. The partition function for the left-handed fermion had an incompletely defined phase, and had to be viewed as a section of a line bundle over the space of four-dimensional gauge fields. The product of this section with its conjugate produced the determinant of the overlap operator, a real valued functional of the four-dimensional gauge field which is invariant under gauge transformations.

In either of the two derivations, the chiral components of the Dirac fermion live on two separate copies of $\mathcal{M}$. Although there is no obstacle to the eventual identification of the two copies, physically, it would be more appealing to have the two Weyl components living on the same $\mathcal{M}$ from the beginning.

In this Letter we present a non-rigorous derivation of the overlap formula, from a starting point where the two chiral components of the Dirac field live on the same four-dimensional manifold $\mathcal{M}$. We say that the derivation is non-rigorous because we do not employ an explicit UV regularization, although $\mathcal{M}$ can be taken to be compact. As we shall see, this approach makes it possible to understand the formula for the overlap Dirac operator in just a few lines.

* Corresponding author.
E-mail address: fosco@cab.cnea.gov.ar (C.D. Fosco).

0370-2693
doi:10.1016/j.physletb.2007.05.045

2. The setup

Thinking in terms of the setup reviewed at the beginning of this note, we wish to replace the segment L with the two Weyl fields living at its opposite endpoints, by first gluing the two ends to each other, then cutting L in the middle, and finally letting the two (new) endpoints created by the cutting go to $\pm\infty$, respectively. So, we again have an $R \times \mathcal{M}$ manifold. Denoting the coordinate along R by s, we assume that $s = 0$ corresponds to the point where the original endpoints of L have been glued. The coordinates along $\mathcal{M}$ are denoted collectively by x. To keep the discussion more general, we replace the dimension 4 of $\mathcal{M}$ by an arbitrary even number d.

We must have a singularity, or a defect, at $s = 0$, since the two Weyl fields cannot, in general, be glued continuously to each other. Nevertheless, the massless Dirac fermion is expected to be bound to that singularity, i.e., to be localized on the single copy of $\mathcal{M}$ at $s = 0$. The defect can be realized by adding a suitable singular mass term to the $(d+1)$-dimensional Dirac operator $\mathcal{D}_{d+1}$, so that the eigenfunctions of $\mathcal{D}_{d+1}$ have a discontinuity at $s = 0$.

The dimension of s is fixed by the choice of the structure of $\mathcal{D}_{d+1}$, where first-order derivatives in s and x appear. Moreover, since the singular mass term must have support at $s = 0$, we conclude that a mass parameter of the form $\xi\delta(s)$, with a dimensionless parameter ξ, is of the right form. For $\mathcal{D}_{d+1}$ to have an eigenvalue problem, we must assign a value to $\Psi(x,0)$, depending linearly on $\Psi(x,0-)$ and $\Psi(x,0+)$. To preserve the $s \to -s$ symmetry, which is tied to four-dimensional charge conjugation invariance, we set $\Psi(x,0) = \frac{1}{2}[\Psi(x,0-) + \Psi(x,0+)]$.

The $(d+1)$-dimensional operator has thus the form:

$$\mathcal{D}_{d+1} = \gamma_s\partial_s + \not{D} + m + \xi\delta(s), \tag{2.1}$$

where $\not{D} = \not{\partial} + \not{A}$ is the d-dimensional Dirac operator in the presence of a gauge field whose components A_μ depend only on x and our Lie algebra generators are such that $A_\mu(x) = -A_\mu(x)^\dagger$. We now can determine ξ by ensuring that $\mathcal{D}_{d+1}$ has exact zero modes when the d-dimensional gauge field on $\mathcal{M}$ is from a non-trivial bundle and as a consequence, for the chosen background, $\not{D}$ has chiral zero modes $\psi_d(x)$. We wish to ensure that there exist then zero modes of $\mathcal{D}_{d+1}$ of the form $\Psi(x,s) = \phi(s)\psi_d(x)$. We find that the choice $\xi = -2\,\mathrm{sign}(m)$ does the job, with a discontinuous $\phi(s)$ which is zero on the $s > 0$ or $s < 0$ halves of R. On the other half it goes as $\exp(-|ms|)$. For definiteness, we pick $m > 0$ and $\xi = -2$.

3. Evaluation of the effective action

We now evaluate $\Gamma_\xi(A)$, the effective action resulting from the functional integral corresponding to the action $S_f = \int d^dx\,ds\,\bar{\Psi}\mathcal{D}_{d+1}\Psi$. We separate the singular part out, writing $\mathcal{D}_{d+1} = \mathcal{D} + M(s)$ with $M(s) = \xi\delta(s)$. The path integral

$$e^{-\Gamma_\xi(A)} = \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\, e^{-S_f(\bar{\Psi},\Psi;A)}, \tag{3.1}$$

gives the fermion determinant:

$$\det{}_{d+1}\big[\mathcal{D} + M(s)\big] \equiv e^{-\Gamma_\xi(A)}. \tag{3.2}$$

Hence:

$$\begin{aligned}\Gamma_\xi(A) &= -\ln\det{}_{d+1}\big[\mathcal{D} + M(s)\big]\\ &= -\mathrm{Tr}_{d+1}\ln\big[\mathcal{D} + M(s)\big].\end{aligned} \tag{3.3}$$

Γ_ξ does not vanish in the absence of a defect, i.e., when $\xi = 0$ and all modes are de-localized in the s direction. In order to assign an effective Dirac operator $\mathcal{O}$ to the additional modes showing up when $\xi \neq 0$, which are localized at the defect, we define the effective action $\Gamma_\mathcal{O}$ by subtracting Γ_0 from Γ_ξ:

$$\begin{aligned}\Gamma_\mathcal{O}(A) &\equiv \Gamma_\xi(A) - \Gamma_0(A)\\ &= -\mathrm{Tr}\ln\big[I + \mathcal{D}^{-1}M(s)\big],\end{aligned} \tag{3.4}$$

where the trace is over all of the space–time coordinates and indices.

The $\xi = 0$ piece is linear in L, the length of the s coordinate, and vanishes for $L = 0$. This reflects translational invariance at $\xi = 0$. It is given by:

$$\Gamma_0(A) = -L\left[\mathrm{tr}(1)(c_0 + c_1\ln\Lambda)\Lambda + \frac{1}{2}\mathrm{Tr}_d\Big(\sqrt{-\not{D}^2 + m^2}\Big)\right], \tag{3.5}$$

where Λ is a UV cutoff, c_1 and c_2 are constants, and Tr_d denotes the Dirac and functional trace in d-dimensional space. The symbol 'tr' denotes the Dirac trace. The L dependence of Γ_0 makes it possible to separate $\Gamma_\mathcal{O}$ out from Γ_ξ.

3.1. Perturbative derivation

We first calculate $\Gamma_\mathcal{O}$ as a series in powers of ξ:

$$\Gamma_\mathcal{O}(A) = \sum_{n=1}^{\infty}\Gamma_\mathcal{O}^{(n)}(A), \tag{3.6}$$

where

$$\Gamma_\mathcal{O}^{(n)}(A) = \frac{(-1)^n}{n}\mathrm{Tr}_{d+1}\big[\big(\mathcal{D}^{-1}\hat{M}\big)^n\big]. \tag{3.7}$$

The operator $\hat{M}$ has the matrix elements:

$$\langle s,x|\hat{M}|s',x'\rangle = \xi\delta^d(x-x')\delta(s)\delta(s').$$

Therefore, the integration over the s coordinates can be carried out explicitly, leaving us with

$$\Gamma_\mathcal{O}^{(n)}(A) = \frac{(-1)^n}{n}\xi^n\,\mathrm{Tr}_d\big[\mathcal{K}^n\big], \tag{3.8}$$

with a trace over internal indices and over the x coordinates. $\mathcal{K}$ is the operator that results from evaluating the kernel of $\mathcal{D}^{-1}$ from $s = 0$ to $s = 0$:

$$\mathcal{K} = \int_{-\infty}^{+\infty}\frac{dk_s}{2\pi}\frac{1}{i\gamma_s k_s + \not{D} + m}. \tag{3.9}$$

Writing

$$\mathcal{K} = \int_{-\infty}^{+\infty}\frac{dk_s}{2\pi}\left(\frac{-i\gamma_s k_s - \not{D} + m}{k_s^2 - \not{D}^2 + m^2}\right)$$

$$= (-\not{D} + m) \int_{-\infty}^{+\infty} \frac{dk_s}{2\pi} \frac{1}{k_s^2 - \not{D}^2 + m^2}, \tag{3.10}$$

we then perform the integral over k_s, after having thrown away the piece odd in k_s, to obtain:

$$\mathcal{K} = \frac{1}{2} V, \tag{3.11}$$

where V denotes the unitary operator:

$$V = \frac{-\not{D} + m}{\sqrt{-\not{D}^2 + m^2}}. \tag{3.12}$$

Inserting this into the expression for $\Gamma_{\mathcal{O}}$, we see that:

$$\Gamma_{\mathcal{O}}(A) = -\operatorname{Tr}_d \left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\xi \mathcal{K})^n \right], \tag{3.13}$$

or

$$\Gamma_{\mathcal{O}}(A) = -\operatorname{Tr}_d \ln\left[1 + \frac{\xi}{2} V\right] = -\ln \det_d \left[1 + \frac{\xi}{2} V\right]. \tag{3.14}$$

We now identify the effective d-dimensional operator $\mathcal{O}$:

$$e^{-\Gamma_{\mathcal{O}}(A)} \equiv \det_d \mathcal{O}, \tag{3.15}$$

where

$$\mathcal{O} = 1 + \frac{\xi}{2} V. \tag{3.16}$$

With $\xi = -2$ (3.16) reduces to the usual overlap operator [1], but here with a more convenient choice for the sign of V,

$$\mathcal{O} = 1 - V. \tag{3.17}$$

For a trivial gauge field on $\mathcal{M} = R^4$ and for small momenta, with $m > 0$, $V \simeq 1 - \not{D}/m$, leading to $\mathcal{O} \simeq \not{D}/m$.

One could worry about the validity of the expansion (3.8) in powers of ξ, given that at the end we set $\xi = -2$. We see that the expansion really is in the combination $\xi\mathcal{K}$ which is unitary and convergence is not a problem except for eigenstates of V with eigenvalue 1, where we know we should get the logarithm of zero.

3.2. Auxiliary field derivation

To gain further insight into the nature of this system, we now provide an alternative derivation, which makes use of auxiliary fields to take the defect into account. These auxiliary fields are localized at the defect (by definition), and together with $\Psi(x, 0)$ they can be used to make up two linear combinations of fields on $\mathcal{M}$: one which has a chirally symmetric propagator and another one which does not propagate at all. Those fields do mix, and this is how one can consistently maintain exact chirality for "valence" fermions but not for "sea" fermions which are governed by $\mathcal{O}$, which does not anti-commute with γ_s. This is consistent, because $\mathcal{O}$ obeys the Ginsparg–Wilson relation [8], which says that the propagator $\frac{1}{1-V}$ is chiral up to a contact term on $\mathcal{M}$.

The auxiliary fields arise naturally when we linearize the singular part of the action, by introducing Grassmann-valued (Dirac) fields χ and $\bar{\chi}$,[1] living in d dimensions:

$$e^{-\int d^d x\, ds\, \bar{\Psi} M(s) \Psi} = \frac{\int \mathcal{D}\chi \mathcal{D}\bar{\chi}\, e^{-\int d^d x\, (\frac{1}{\xi} \bar{\chi}(x)\chi(x) - i[\bar{\chi}(x)\Psi(x,0) + \bar{\Psi}(x,0)\chi(x)])}}{\int \mathcal{D}\chi \mathcal{D}\bar{\chi}\, e^{-\int d^d x \frac{1}{\xi} \bar{\chi}(x)\chi(x)}}. \tag{3.18}$$

We have

$$e^{-\Gamma_{\mathcal{O}}(A)} = \int \mathcal{D}\chi \mathcal{D}\bar{\chi}\, e^{-S_d(\bar{\chi}, \chi; A)}, \tag{3.19}$$

where

$$e^{-S_d(\bar{\chi}, \chi; A)} = \mathcal{N}_\xi (\det \mathcal{D})^{-1} \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi}\, e^{-\int d^d x\, ds\, \bar{\Psi} \mathcal{D} \Psi} \times e^{i \int d^d x\, [\bar{\chi}(x)\Psi(x,0) + \bar{\Psi}(x,0)\chi(x)]} \times e^{-\frac{1}{\xi} \int d^d x\, \bar{\chi}\chi} \tag{3.20}$$

and $\mathcal{N}_\xi \equiv \det \xi$. We now integrate over the original Dirac field:

$$e^{-S_d(\bar{\chi}, \chi; A)} = e^{-\int d^d x \int d^d x'\, \bar{\chi}(x) \mathcal{O}(x, x') \chi(x')}, \tag{3.21}$$

where

$$\mathcal{O} \equiv I + \frac{\xi}{2} V, \tag{3.22}$$

with V as defined in (3.12); $\mathcal{N}_\xi$ was dropped. We see that:

$$e^{-\Gamma_{\mathcal{O}}(A)} = \det_d \mathcal{O}, \tag{3.23}$$

where $\mathcal{O}$ becomes the overlap Dirac operator when $\xi \equiv -2$, exactly as in the perturbative derivation.

4. Degrees of freedom at the defect

The propagator of the auxiliary fields,

$$\langle \chi(x) \bar{\chi}(y) \rangle = \mathcal{O}^{-1}(x, y) \equiv \langle x | \mathcal{O}^{-1} | y \rangle, \tag{4.1}$$

has a pole at zero momentum when $\xi = -2$, describing a massless dynamical Dirac fermion. On the other hand, the original fields, $\Psi(x, 0)$ and $\bar{\Psi}(x, 0)$, have the propagator:

$$\langle \Psi(x, 0) \bar{\Psi}(y, 0) \rangle = \langle x | \frac{V}{2(1 - V)} | y \rangle. \tag{4.2}$$

We now form linear combinations of $\Psi(x, 0)$ and $\chi(x)$ of the form $\Phi_\alpha(x) = \Psi(x, 0) - \imath\alpha\chi(x)$ and the same combinations for the independent fields $\bar{\Phi}_\alpha(x) = \bar{\Psi}(x, 0) - \imath\alpha\bar{\chi}(x)$. We find that for $\alpha = -\frac{1 \pm \sqrt{2}}{2}$, Φ_α has a propagator proportional to $\frac{1-V}{1+V}$ which anticommutes with γ_s on account of $\gamma_s V \gamma_s = V^\dagger$. On the other hand, for $\alpha = \frac{1}{2}$, Φ_α has a propagator proportional to $\delta^d(x - y)$, so is non-dynamical. The various Φ_α fields mix. Physically, on the defect there is an ordinary massless fermion and also a non-propagating fermionic degree of freedom that mixes with it.

[1] We follow the conventions of [9].

5. Conclusion

The form of the overlap operator in d dimensions and the essential features that allow the preservation of exact chiral symmetry in the presence of a UV regulator are understandable with relative ease by considering states bound to a delta-function mass defect in $d+1$ dimensions, with a particular strength and sign. The δ-function mass defect plays the role of the domain walls in Kaplan's [3] original formulation, each domain wall being that introduced by Callan and Harvey for Weyl fermions [10]. The domain walls can be brought infinitesimally close to each other because the mass parameter on the short segment separating them is taken to infinity.

Acknowledgements

C.D.F. acknowledges support by ANPCyT and CONICET, under grants PICT 17-20434 and PIP 5231, respectively. G.T. is supported as a Research Assistant in the Rutgers Department of Physics. H.N. acknowledges partial support by the DOE under grant number DE-FG02-01ER41165 at Rutgers University.

References

[1] H. Neuberger, Phys. Lett. B 417 (1998) 141, hep-lat/9707022.
[2] H. Neuberger, Phys. Lett. B 427 (1998) 353, hep-lat/9801031.
[3] D.B. Kaplan, Phys. Lett. B 288 (1992) 342, hep-lat/9206013.
[4] D.B. Kaplan, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 597.
[5] R. Narayanan, H. Neuberger, Nucl. Phys. B 443 (1995) 305, hep-th/9411108.
[6] Y. Shamir, Nucl. Phys. B 406 (1993) 90, hep-lat/9303005.
[7] H. Neuberger, Phys. Rev. D 57 (1998) 5417, hep-lat/9710089.
[8] P.H. Ginsparg, K.G. Wilson, Phys. Rev. D 25 (1982) 2649.
[9] J. Zinn-Justin, Int. Ser. Monogr. Phys. 113 (2002) 1.
[10] C.G. Callan, J.A. Harvey, Nucl. Phys. B 250 (1985) 427.