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A novel nonlinear value-at-risk method for modeling risk of option portfolio with multivariate mixture of normal distributions[☆]

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ABSTRACT

This paper proposes a novel nonlinear model for calculating Value-at-Risk (VaR) when the market risk factors of an option portfolio are heavy-tailed. A multivariate mixture of normal distributions is used to depict the heavy-tailed market risk factors and accordingly a closed form expression for the moment generating function that can reflect the change in option portfolio value can be derived. Moreover, in order to make use of the correlation between the characteristic function and the moment generating function, Fourier-Inversion method and adaptive Simpson rule with iterative algorithm of numerical integration into the nonlinear VaR model for option portfolio are applied for calculation of VaR values of option portfolio. VaR values of option portfolio obtained from different methods are compared. Numerical results of Fourier-Inversion method and Monte Carlo simulation method show that high accuracy VaR values can be obtained when risk factors have multivariate mixture of normal distributions than when they have normal distributions. Moreover, VaR values obtained by using the Fourier-Inversion method are not obviously different from VaR values obtained by using Monte Carlo simulation when market risk factors have normal distributions or multivariate mixture of normal distributions. However, the speed of computation is obviously faster when using Fourier-Inversion method, than when using Monte Carlo simulation method. Besides, Cornish Fisher method is faster and simpler than Monte Carlo simulation method or Fourier-Inversion method. However, this method does not offer high accuracy and cannot be used to calculate VaR values of option portfolio when market risk factors have heavy-tailed distributions.

1. Introduction

Option value depends on prices of the underlying assets and these changes constitute exactly the market risk factors. However, while change in value of a stock portfolio linearly depends on market risk factors, value of an option portfolio has a nonlinear dependence on market risk factors. Financial parameters such as Delta and Gamma can reflect the nonlinear relationship. So market risk factors are important for option portfolio risk.

In the past few decades, several nonlinear Value-at-Risk (VaR) models have been proposed for computation of VaR of option portfolios. These models have focused on relaxing the assumption that option portfolio value changes linearly with change in market risk factors while preserving computational tractability. These models refine the correlation between market risk factors and option portfolio value to include quadratic as well as linear terms and are known as Delta–Gamma–Theta-Normal VaR models. Morgan (1996) computed VaR of option portfolio using Johnson distributions family transformations method. Britten-Jones and Schaefer (1999) applied the theory of quadratic forms in normal variables to estimate higher moments of a quadratic portfolio using Solomon & Stephens approximate method. Hardle et al. (2002) evaluated the main methods of calculation of nonlinear VaR, such as the Johnson transformations, Cornish–Fisher, Monte Carlo and Fourier-Inversion methods, in terms of accuracy and speed. Their numerical experiments illustrated that the Johnson transformations and Cornish–Fisher method is fast but less accurate, the Monte Carlo method is accurate but computationally less efficient and the Fourier-Inversion method is the best choice for speed and accuracy. Castellacci and Siclari (2003) computed first higher moments of distribution of the change in option portfolio value, using the Cornish–Fisher method. Cui et al. (2013) applied Delta-Normal and Delta–Gamma–Theta-Normal VaR and parametric VaR approximations for nonlinear portfolio selection and investigated their respective computational aspects.

In these nonlinear VaR models, distributions of market risk factors are usually assumed to be conditionally normal for the convenience of modeling and numerical calculations. However, a large number of empirical studies have indicated that tails of most empirical distributions

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of market risk factors are heavier than those of the normal distributions, showing high kurtosis and heavy tails. For example, Heyde (1999), Hosking et al. (2000), Heyde and Kou (2004), Lu (2005) and Xu and Hou (2006) found that the high-frequency financial data often show high kurtosis and heavy tails with different approaches and different datasets. However, the tails in financial data are not so heavy as to produce infinite variance, though higher order moments (e.g., five or higher) may be infinite. This implies that the traditional normal distribution assumption cannot fully reflect the generic feature of practical data.

Furthermore, in the nonlinear VaR model, calculation of option portfolio with heavy-tailed market risk factors is more complex than the multivariate normality case, where the main issue is to obtain the moment generating function that can reflect the change in its value. Glasserman (2004) used multivariate t-distributions to represent heavy-tailed market risk factors and obtained indirectly a closed form expression for the moment generating function that can reflect the change in option portfolio value by transforming a heavy-tailed problem into a light-tailed problem. They used the structure of the multivariate t-distributions to obtain the moment generating function. On this basis, they developed a Fourier-Inversion method for computing the non-linear VaR of option portfolio. Albanese et al. (2004) described the heavy tails of market risk factors using multivariate t-distributions. They first derived the matrix transform of the change in option portfolio value from the Delta–Gamma–Theta model and then discretized the density function of the change in option portfolio value. Finally, approximate VaR values are calculated by using the convolution formula, Fourier-Inversion method and linear interpolation. Considering two different probability distributions of the market risk factors, multivariate normal and multivariate t-distributions, Albanese and Campolieti (2006) developed multivariate Monte Carlo simulation method for computing the probability density function for the change in option portfolio value as well as to estimate portfolio VaR at a given confidence level. They explored some of the differences between the use of a normal distribution and a heavy-tailed distribution model when computing VaR. This method allows computation of VaR for distribution where the returns can possess different degrees of freedom for different market risk factors. Johannes et al. (2009) followed Glasserman (2004) to derive a closed form expression for the moment generating function under the multivariate t-distributions case and performed some simulations where the Fourier-Inversion method is benchmarked against the Monte Carlo simulation method. Their numerical experiments showed that the Fourier-Inversion method is significantly faster than the Monte Carlo. They concluded that the Fourier-Inversion method is a highly competitive alternative for computing VaR of option portfolios. Sorwar and Dowd (2010) proposed a simulation-lattice method to estimate VaR for option position with market risk factor following the constant elasticity of variance (CEV) diffusion process, which exhibits tails heavier than the geometric Brownian motion. However, the method has focused on the univariate option. One of the limitations of the method is that it cannot deal with portfolio, i.e. multiple market risk factors.

Besides, Zhou (2002) studied the correlation between risk and return of a portfolio using multivariate mixture of normal distributions to represent the heavy-tailed characteristic of market risk factors and built a capital asset pricing model with heavy-tailed feature but the portfolio in the model did not include options. So the portfolio value changes linearly with market risk factors. This is evidently different from portfolio value changing nonlinearly with market risk factors.

In order to further expand and enrich the nonlinear VaR model for option portfolio with heavy-tailed market risk factors, unlike Glasserman (2004), this paper considers the case where the market risk factors have multivariate mixture of normal distributions. Because the multivariate mixture of normal distributions shares some attractive properties with normal distributions and it has heavy tails. This is important for combining a realistic model of market risk factors with nonlinear correlations between market risk factors and option portfolio value, which is our goal. Accordingly we derive a moment generating function that can reflect the change in option portfolio value. Moreover, to make use of the correlation between characteristic function and moment generating function, we apply the Fourier-Inversion method and adaptive Simpson rule with iterative algorithm of numerical integration for calculating nonlinear VaR of option portfolio. Finally, VaR values calculated by Fourier-Inversion method are compared with those computed by Monte Carlo simulation under multivariate mixture of normal distributions.

The main purpose of this paper is to propose a nonlinear VaR model for modeling the risk of option portfolio under multivariate mixture of normal distributions and to compare VaR values of option portfolio obtained from different methods. The rest of the paper is organized as follows. Section 2 presents the multivariate mixture of normal distributions, develops the quadratic approximation transform to describe the change in option portfolio values and derives a closed form expression for the moment generating function of the change in value of option portfolio with multivariate mixture of normal distributions. Then the Fourier-Inversion method is used to calculate VaR of option portfolio in Section 2. For verification purpose, some numerical examples and the corresponding analysis are given in Section 3. And Section 4 concludes the article.

2. Delta–Gamma–Theta approximation with multivariate mixture of normal distributions for VaR

In this section, a brief description of multivariate mixture of normal distributions is first presented. Then a quadratic approximation form (Delta–Gamma–Theta approximation) (Chen, 2005) is introduced to describe the change in option portfolio values. Accordingly, a closed form expression for the moment generating function of the change in option portfolio value with multivariate mixture of normal distributions is derived. Finally, the Fourier-Inversion method is used to calculate the VaR of option portfolio.

2.1. Multivariate mixture of normal distributions

In reality, many variables are multivariate and heavy-tailed. Usually, multivariate mixture of normal distributions is a kind of useful distributions to describe the heavy-tailed feature. One mixture of normal distributions can be expressed as p-dimensional random vector X, which may be a p-dimensional normal distributed random vector W with probability ε or a p-dimensional normal distributed random vector W' with probability $(1-\varepsilon)$, namely,

$$X =_d \varepsilon W + (1-\varepsilon)W' \tag{1}$$

where $X \in \Re^p$, $X = \left(X_1, \cdots, X_p\right)^T$, $=_d$ denotes equality in distribution, $W \sim N_p(0, \Sigma)$, $W' \sim N_p(0, \Sigma')$, Σ and Σ' are two covariance matrixes of $W$and W', respectively. Some other features of a mixture of normal distributions can be seen in Campbell and Zhou (1993). If $\Sigma' = \gamma\Sigma\,(\gamma > 1)$, then $|\Sigma'| = \gamma^p|\Sigma|$, $\quad (\Sigma')^{-1} = \frac{1}{\gamma}\Sigma$. Using this result, the density function of X can be obtained by the following formulae:

$$\begin{aligned} f(X) = {} & \varepsilon \frac{|\Sigma|^{-1/2}}{(2\pi)^{p/2}} \exp\left(-\frac{1}{2}X^T\Sigma^{-1}X\right) \\ & +(1-\varepsilon)\frac{|\Sigma|^{-1/2}}{(2\pi\gamma)^{p/2}} \exp\left(-\frac{1}{2\gamma}X^T\Sigma^{-1}X\right). \end{aligned} \tag{2}$$

Mean value of random vector X is equal to zero, i.e., $E(X) = 0$ and covariance matrix is $D(X) = [\varepsilon^2 + (1-\varepsilon)^2\gamma]\Sigma$. Eq. (2) shows that X is multivariate normality if $\varepsilon = 1$ and X follows multivariate mixture of normal distributions in case of $0 < \varepsilon < 1$. Furthermore, the lower parameter value ε and higher parameter value γ lead to heavier tails.

2.2. Delta–Gamma–Theta approximation for the change of option portfolio value

In this subsection, the change of option portfolio value is discussed in terms of quadratic approximation (Delta–Gamma–Theta approximation) in detail. We first express the change of option portfolio position as

$$\Delta V = \Delta V_1 + \cdots + \Delta V_i + \cdots + \Delta V_p, \tag{3}$$

where ΔV represents the change in option portfolio value over a specified horizon, ΔV_i is the change in the ith option value over a fixed horizon and $i = 1,2,\cdots,p$.

Usually, the option price is treated as a second-order Taylor series expansion. Accordingly a Delta–Gamma–Theta Model of quadratic approximation (Chen, 2005) can be represented by

$$\Delta V_i \approx \widetilde{\Delta}_i R_{i,t} + \frac{1}{2}\widetilde{\Gamma} R_{i,t}{}^2 + \widetilde{\Theta}_i, \tag{4}$$

where $\widetilde{\Delta}_i = d_i \Delta_i, \widetilde{\Gamma}_i = d_i \Gamma_i S_{i,t}, \widetilde{\Theta}_i = \frac{d_i \Theta_i \Delta t}{S_{i,t}}$, Δt denotes the time horizon of risk prediction, $S_{i,t}$ is price of the ith market risk factor at time t, market risk factor return $R_{i,t} = \frac{\Delta S_{i,t}}{S_{i,t}}$, $\Delta S_{i,t}$ is the change in market risk factors, d_i is the i th option position value, V_i is the i th option value at time t, Delta of an option $\Delta_i = \frac{\partial V_i}{\partial S_{i,t}}$ (the partial derivative of the ith option value with respect to price of the ith market risk factor), Gamma of an option $\Gamma_i = \frac{\partial^2 V_i}{\partial S_{i,t}^2}$ (the second partial derivative of the ith option with respect to price of the ith market risk factor) and Theta of an option $\Theta_i = \frac{\partial V_i}{\partial(\Delta t)}$ (the partial derivative of the ith option value with respect to the passage of time).

In terms of Eqs. (3) and (4), the change in option portfolio value can be represented by a quadratic approximation form as shown below.

$$\Delta V \approx \widetilde{\Delta}^T R + \frac{1}{2} R^T \widetilde{\Gamma} R + \sum_{i=1}^{p} \widetilde{\Theta}_i, \tag{5}$$

where a vector $\widetilde{\Delta} = \begin{bmatrix} \widetilde{\Delta}_1 \\ \widetilde{\Delta}_2 \\ \vdots \\ \widetilde{\Delta}_p \end{bmatrix}$ (corresponding $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$, … and $\widetilde{\Delta}_p$ are its elements), a diagonal matrix $\widetilde{\Gamma} = \begin{bmatrix} \widetilde{\Gamma}_1 & 0 & \vdots & 0 \\ & \widetilde{\Gamma}_2 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & \widetilde{\Gamma}_p \end{bmatrix}$ (corresponding $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots$ and $\widetilde{\Gamma}_p$ are its diagonal elements), a vector $\widetilde{\Theta} = \begin{bmatrix} \widetilde{\Theta}_1 \\ \widetilde{\Theta}_2 \\ \vdots \\ \widetilde{\Theta}_p \end{bmatrix}$ (corresponding $\widetilde{\Theta}_1$, $\widetilde{\Theta}_2$, … and $\widetilde{\Theta}_p$ are its elements), and vector of market risk factor return $R = (R_{1,t}, R_{2,t}, \cdots, R_{p,t})^T$.

2.3. Moment generating function of the change in option portfolio value with multivariate mixture of normal distributions

For convenience of computation, suppose $L = -\Delta V = V(S_{(t)},t) - V(S_{(t+\Delta t)}, t + \Delta t)$ is a loss function, then we have $L \approx -\widetilde{\Delta}^T R - \frac{1}{2} R^T \widetilde{\Gamma} R - \sum_{i=1}^{p} \widetilde{\Theta}_i$. Let $a = -\widetilde{\Delta}$, $A = -\frac{1}{2}\widetilde{\Gamma}$, $a_0 = -\sum_{i=1}^{p} \widetilde{\Theta}_i$, so the loss function L can be represented as

$$L \approx a^T R + R^T A R + a_0 \equiv a_0 + Q. \tag{6}$$

There are two closely related problems associated with tail distribution of loss function L. The first is the problem of estimating a probability $P(L > x)$ of the loss function L given the loss threshold x. The second is the inverse problem of finding a quantile x_α for which $P(L > x_\alpha) = \alpha$, given a probability α. The estimation of VaR is an instance of the second problem, typically with $\alpha = 1\%$ or 5%. However, calculating probabilities $P(L > x)$ is a prerequisite for computing quantiles, so we focus primarily on the first problem. Given values of $P(L > x)$ for several values of x in the vicinity of x_α, it is then straightforward to estimate the quantile itself using the linear interpolation method. For this purpose, our study concentrates on calculation of tail probability of loss function L, namely

$$\begin{aligned} P(L > x) &\approx P(a_0 + Q > x) \\ &= P(Q > x - a_0) \\ &= P\{Q - (x - a_0) > 0\} \end{aligned}$$

Let $Q_{x-a_0} = Q - (x - a_0)$, then the above equation may be written as

$$P(L > x) \approx P\left(Q_{x-a_0} > 0\right), \tag{7}$$

where $Q_{x-a_0} = Q - (x - a_0) = a^T R + R^T A R - (x - a_0)$ Assuming that risk is computed over one-day horizon, RiskMetrics group holds that the expected change of market risk factor can be regarded as 0 in the next 24 h (one trading day). Indeed, the density function of market risk factor return vector R described by multivariate mixture of normal distributions whose mean vector $E(R) = 0$ is the same as Eq. (2). In this case, market risk factor return R can be expressed as $R =_d \varepsilon W + (1 - \varepsilon) W'$ according to Eq. (1), where $W \sim N_p(0, \Sigma)$, $W' \sim N_p(0, \gamma\Sigma)$.

Define a new factor X by $X =_d \varepsilon Z + (1-\varepsilon) Z'$ and satisfy that $R = CX$, where $CC^T = \Sigma$, $Z \sim N_p(0, I)$, $Z' \sim N_p(0, \gamma I)$, I is a unit matrix. This ensures that $R = CX$ and $R =_d \varepsilon W + (1 - \varepsilon) W'$ have identical distributions. Then we may find a matrix C (how to find C is shown below) for which $C^T A C$ is diagonal, namely, $C^T A C = \Lambda$, where Λ denote a diagonal matrix and $\lambda_1, \cdots, \lambda_p$ are its diagonal elements.

Assume that there exists a matrix D such that $DD^T = \Sigma$ (e.g., D is derived from Cholesky Factorization of Σ). Then $D^T A D$ and $DD^T A = \Sigma A$ have the same eigenvalue $\lambda_1, \cdots, \lambda_p$. Furthermore, we have $D^T A D = U \Lambda U^T$, namely, $U^T D^T A D U = \Lambda$ (U is an orthogonal matrix whose columns of U are eigenvectors of $D^T A D$). So with $(DU)\ (DU)^T = DD^T = \Sigma$, we obtain the matrix $C = DU$ for which $C^T A C$ is diagonal.

Let $b^T = a^T C$, with $C^T A C = \Lambda$, then Q_{x-a_0} in Eq. (7) can be written as

$$Q_{x-a_0} = b^T X + X^T \Lambda X - (x - a_0) = \sum_{j=1}^{p} \left(b_j X_j + \lambda_j X_j^2 \right) - (x - a_0), \tag{8}$$

where random variable of mixture of normal distributions $X_j =_d \varepsilon Z_j + (1-\varepsilon) Z_j'$, Z_j and Z_j' are normal variables and independent ($j = 1, \cdots, p$) and $Z_j \sim N(0,1)$, $Z_j' \sim N(0, \gamma)$.

In order to calculate the quantile, a moment generating function of Q_{x-a_0} can be defined as

$$\begin{aligned} M(\theta) &= E\left[\exp\left(\theta Q_{x-a_0}\right)\right] \\ &= E\left\{\exp\left[\theta\left(\sum_{j=1}^{p}\left(b_j X_j + \lambda_j X_j^2\right) - (x - a_0)\right)\right]\right\} \\ &= \exp(-\theta(x - a_0)) \times \prod_{j=1}^{p} E\left\{\exp\left[\theta\left(b_j X_j + \lambda_j X_j^2\right)\right]\right\} \\ &= \exp(-\theta(x - a_0)) \times \prod_{j=1}^{p} E\left\{\exp\left[\theta\left(b_j X_j + \lambda_j X_j^2\right)\right]\right\} \\ &= \exp(-\theta(x - a_0)) \\ &\quad \times \prod_{j=1}^{p} E\left\{\exp\left[\theta\left(b_j\left(\varepsilon Z_j + (1-\varepsilon) Z_j'\right) + \lambda_j\left(\varepsilon Z_j + (1-\varepsilon) Z_j'\right)^2\right)\right]\right\} \\ &= \exp(-\theta(x - a_0)) \times \prod_{j=1}^{p} \frac{1}{\sqrt{1 - 2\theta\varepsilon^2\lambda_j - 2\theta(1-\varepsilon)^2\lambda_j\gamma^2}} \\ &\quad \exp\left(\sum_{j=1}^{p}\left\{\frac{\theta^2 b_j^2}{2\left(1 - 2\theta\varepsilon^2\lambda_j\right)}\left[\varepsilon^2 + \frac{(1-\varepsilon)^2\gamma^2}{1 - 2\theta\varepsilon^2\lambda_j - 2\theta(1-\varepsilon)^2\lambda_j\gamma^2}\right]\right\}\right). \end{aligned} \tag{9}$$

For a concrete proof of Eq. (9), see Appendix A.

The characteristic function of Q_{x-a_0}, which can be obtained using the relationship between characteristic function and moment generating function, can not only contain all information of all order moments (no information loss of distribution of Q_{x-a_0} statistically), but also can uniquely determine the distribution of Q_{x-a_0}. Numerical calculation in nonlinear VaR model for option portfolio is expanded around Q_{x-a_0}. In order to obtain the characteristic function, different methods can be used. In this study, Fourier-Inversion method is employed to derive the moment generating function, which is elaborated in the next section.

2.4. Computing VaR of option portfolios by Fourier-Inversion method

The essence of Fourier-Inversion method is to first derive the moment generating function of the change in option portfolio value and then tentatively calculate the VaR of option portfolio, using the relationship between characteristic function and moment generating function, Fourier-Inversion method for transforms of probability distribution and iterative algorithm of numerical integration. Since a characteristic function describes completely a probability distribution of random variable, in the statistical sense, there is no loss of information of distribution of the change in option portfolio value.

In order to further expand and enrich the nonlinear VaR model of option portfolio with heavy-tailed market risk factors, this paper applies multivariate mixture of normal distributions to represent heavy-tailed market risk factors and derives the moment generating function in terms of Eq. (9) that reflects the change in option portfolio value and tentatively calculates VaR of option portfolio in terms of the relationship between characteristic function and moment generating function, Fourier-Inversion method for distribution and adaptive Simpson rule with iterative algorithm of numerical integration.

According to the previous analysis, it is easy to find the tail probability of loss function L: $P(L>x) \approx P\left(Q_{x-a_0}>0\right) = 1-P\left(Q_{x-a_0} \le 0\right) = 1-F_{x-a_0}(0)$.

In fact, F_{x-a_0} can be obtained with inversion of the integral transform as

$$F_{x-a_0}(t)-F_{x-a_0}(t-y) = \frac{1}{\pi}\mathrm{Re}\left(\int_0^\infty \varphi_{x-a_0}(iu)\left[\frac{e^{iuy}-1}{iu}\right]e^{-iut}\right)du \qquad (10)$$
$$i = \sqrt{-1}$$

where the characteristic function of Q_{x-a_0} is given by $\varphi_{x-a_0}(iu) = E\left[\exp\left(iuQ_{x-a_0}\right)\right]$ with $i = \sqrt{-1}$. In applying this method a large y is chosen for which $F_{x-a_0}(t-y)$ can be assumed to be approximately zero. As the expectation and variance of Q_{x-a_0} can be easily computed, Chebychev's Inequality may be used to find a value of y for which $F_{x-a_0}(t-y)$ is appropriately small.

This integration can be estimated approximately by numerical methods (Abate and Whitt, 1992). Abate & Whitt (1992) described numerical calculations of transform inversion for probability distribution. When implementing this method, we tentatively calculate VaR of option portfolio by applying adaptive Simpson rule with iterative algorithm of numerical integration. Based on this idea, a large y is chosen with Chebychev's Inequality, such that $F_{x-a_0}(t-y) \approx 0$.

In order to obtain a large y with Chebychev's Inequality such that $F_{x-a_0}(t-y) \approx 0$, expectation m(1) and variance m(2) of Q_{x-a_0} need to be worked out by its moment generating function for y. The purpose of this calculation is two-fold. The first is to obtain n-order original moments by the moment generating function of Q_{x-a_0}, i.e. to calculate the successive derivatives of $\varphi_{x-a_0}(\theta)$ and to take the values when $\theta = 0$ in $\begin{matrix}\varphi'_{x-a_0}(\theta) = E\left[Q_{x-a_0}e^{\theta Q_{x-a_0}}\right] \\ \varphi''_{x-a_0}(\theta) = E\left[Q^2_{x-a_0}e^{\theta Q_{x-a_0}}\right]\end{matrix}, \ldots, \varphi^{(n)}_{x-a_0}(\theta) = E\left[Q^n_{x-a_0}e^{\theta Q_{x-a_0}}\right]$. The second is to compute the values of n-order original moments of Q_{x-a_0} when $\theta = 0$ so as to obtain n-order origin moment $\varphi^{(n)}_{x-a_0}(0) = E\left[Q^n_{x-a_0}\right]$ where $n \ge 1$. Using the above idea, m(1) and m(2) are obtained by using the first-order and second-order moments.

In summary, the main reason of choosing a large y with Chebychev's Inequality is that $F_{x-a_0}(t-y)$ can be extremely small. In detail, the large y with Chebychev Inequality is solved in the following steps:

(1) In terms of Chebychev's Inequality and the definition of VaR, we can have $\frac{D(y)}{\gamma^2} = \frac{m(2)}{\gamma^2} = \frac{\alpha}{10^n}$ (n is a positive integer), where α satisfies the loss function probability $P(L > x_\alpha) = \alpha$ and $\frac{D(y)}{\gamma^2}$ decreases with the increase of n.
(2) From the above equation, we can solve $\gamma = \sqrt{\frac{m(2)10^n}{\alpha}}$.
(3) According to Chebychev's Inequality, we have $P(|y-m(1)| \ge \gamma) \le \frac{D(y)}{\gamma^2}$ for $\forall\ \gamma > 0$. Accordingly the formula $y = \sqrt{\frac{m(2)10^n}{\alpha}} - m(1)$ for calculating y is obtained.

3. Numerical results and analysis

Most European warrants traded in Chinese markets are call warrants, so this paper selects samples of five tradable call warrants (European warrants): Baogang CWB1, Gangyue CWB1, Shangqi CWB1, Shihua CWB1 and Shenggao CWB1. Data of the five tradable call warrants and the corresponding underlying stocks' closing prices as on February 6, 2009 are presented in Table 1, which comes from RESSET/DB http://www.resset.cn. Assuming an investment institution holds these five warrants on February 6, 2009 (one unit of each warrant in the portfolio) the possible worst loss of the portfolio over the next one trading day is calculated with a given level of confidence.

Impact of any share ownership enlargement and dividend payment, strike price and executive proportion is taken into account, on the day nearest to the day of adjustment. Risk-free annual interest rate is assumed to be 0.0225. The five underlying stocks corresponding to the five call warrants are Baoganggufen (BGGF), Gangyuegaosu (GYGS), Shanghaiqiche (SHQC), Zhongguoshihua (ZGSH), and Shenggaosu (SGS). The samples are composed of daily closing prices from January 1, 2003 to February 6, 2009, i.e. 944 observations, except data corresponding to any events of trade suspension. The daily return of the underlying stock is obtained from the log-price difference. Fig. 1 shows the daily return series of the five underlying stocks. As Fig. 1 shows, there exists volatility clustering, which means large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes. Besides volatility clustering in every single series, correlation clustering among the five return series is also observed. This correlation can be explained by the covariance matrix of the five return series.

Table 1
Essential information of each warrant.

Warrant	Underlying stock closing price	Strike price	Period of validity	Annualized volatility	Risk-free annual interest rate	Executive proportion
Baogang CWB1	5.83	12.50	17 months	0.4628	2.25%	2:1
Gangyue CWB1	9.13	20.53	11 months	0.4846	2.25%	1:1
Shangqi CWB1	7.52	26.97	13 months	0.6109	2.25%	1:1
Shihua CWB1	8.55	19.43	11 months	0.5104	2.25%	2:1
Shenggao CWB1	5.25	13.48	9 months	0.5190	2.25%	1:1

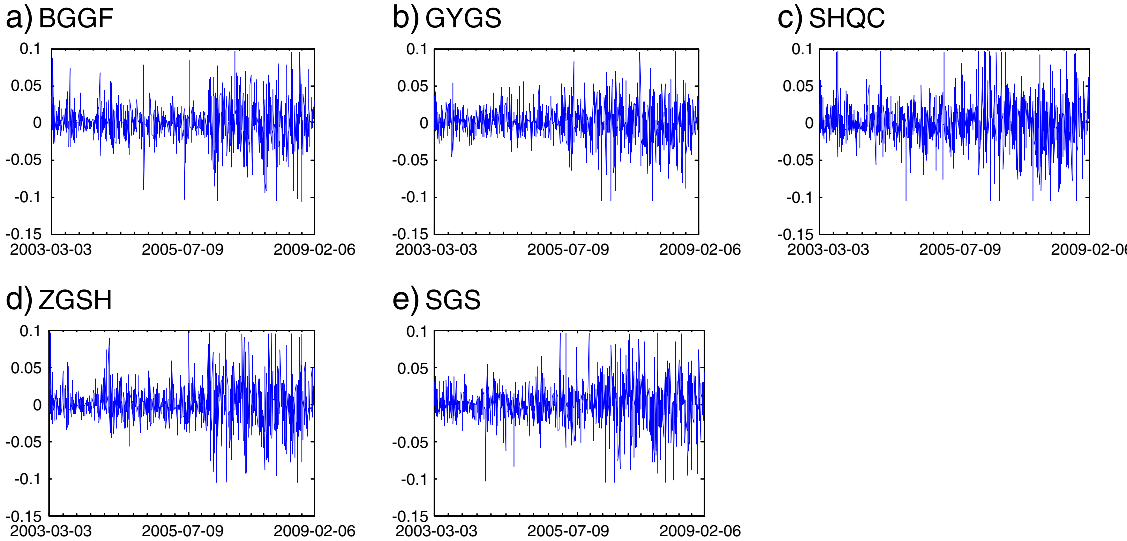

Fig. 1. Characteristic diagram of the daily return of underlying stocks.

Table 2
Statistics of the daily return of underlying stocks (with a confidence level of 0.95).

Sequence	Mean	Standard deviation	Skewness	Kurtosis	JB statistic
Baoganggufen	0.000019	0.026252	−0.208510	5.600565	272.5600
Gangyuegaosu	0.000542	0.023662	−0.321466	5.413240	245.0656
Shanghaiqiche	0.000723	0.030972	−0.061513	4.666099	109.6639
Zhongguoshihua	0.000665	0.028317	0.114319	5.283091	206.8620
Shenggaosu	−0.000095	0.027908	−0.197335	5.352420	223.5555

Note: the series with a kurtosis larger than 3 is treated as heavy-tailed.

The statistical properties of data of daily returns of the five underlying stocks are shown in Table 2; all average values of the five return series are close to zero, so it can be assumed that the expectation of each return series is zero. Skewnesses values of BGGF, GYGS, SHQC and SGS are less than zero with a left deviation while ZGSH is not obviously toward right deviation with a positive skewness. Besides, kurtosis of the five return series are greater than 3, and the JB statistic values are greater than the critical value of 5.9913 (with a 0.95 confidence level), so all the five return series are leptokurtic and heavy-tailed.

Based on the analysis of statistical properties of the above five return series, normality test of the five return series is also conducted, by Quantile–Quantile (Q–Q) plots, as shown in Fig. 2. Q–Q plot is a probability plot, which is a graphical method for comparing two probability distributions by plotting their quantiles against each other. A point (x, y) on the plot corresponds to one of the quantiles of the standard normal distribution (y-coordinate) plotted against the same quantile of the sample distribution (x-coordinate). If the sample follows the normal distribution, the points should be approximately fall in a straight line. The more points lie in the line, the more closely the sample distribution follows the normal distribution.

As Fig. 2 shows, part of the points of all the returns of the five underlying stocks are deviating from the trend lines, especially the ones at the ends swinging around the trend lines. So, we have a good reason to believe that the data reject the hypothesis of normal distribution. In view of this, the multivariate mixture of normal distributions is used

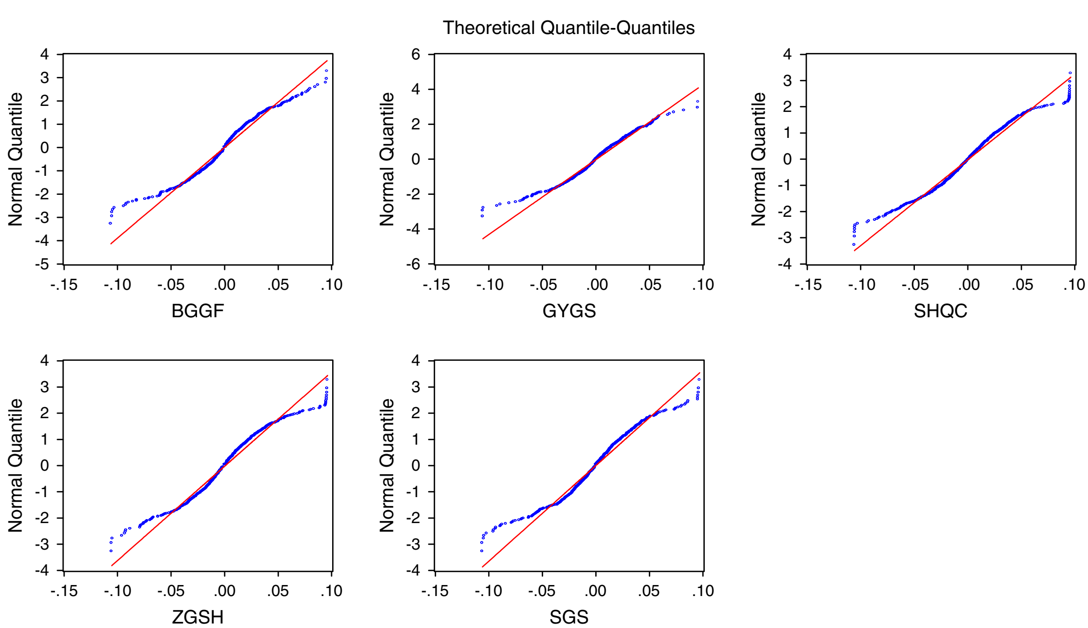

Fig. 2. Q–Q plots of the daily return of underlying stocks.

Table 3
Parameters Σ in multivariate mixture of normal distributions.

Σ	Baoganggufen	Gangyuegaosu	Shanghaiqiche	Zhongguoshihua	Shenggaosu
Baoganggufen	0.0002178	0.0000909	0.0001642	0.0001531	0.0001066
Gangyuegaosu	0.0000909	0.0002388	0.0001284	0.0000936	0.0001228
Shanghaiqiche	0.0001642	0.0001284	0.0003794	0.0001745	0.0001588
Zhongguoshihua	0.0001531	0.0000936	0.0001745	0.0002648	0.0001240
Shenggaosu	0.0001066	0.0001228	0.0001588	0.0001240	0.0002738

Table 4
Financial parameters $\widetilde{\Delta}$, $\widetilde{\Gamma}$, and $\widetilde{\Theta}$ of the five warrants.

Warrant	Mixture of normal distribution			Normal distribution		
	$\widetilde{\Delta}$	$\widetilde{\Gamma}$	$\widetilde{\Theta}$	$\widetilde{\Delta}$	$\widetilde{\Gamma}$	$\widetilde{\Theta}$
Baogang CWB1	0.4272	1.2184	−0.1375	0.3207	1.1059	−0.1017
Gangyue CWB1	0.8746	3.0780	−0.3773	0.3167	1.7921	−0.1325
Shangqi CWB1	0.2418	0.9243	−0.1777	0.0559	0.3285	−0.0407
Shihua CWB1	0.4645	1.4998	−0.2036	0.3029	1.2389	−0.1307
Shenggao CWB1	0.1743	0.8643	−0.1197	0.0687	0.4612	−0.0466

to depict the heavy-tailed features of return distributions of the five underlying stocks.

Besides, the three parameters $\varepsilon = 0.6316$, $\gamma = 8.9813$ and Σ in Eq. (2) are estimated using maximum likelihood estimation (MLE) based on EM algorithm. Accordingly, the estimated parameters Σ are shown in Table 3.

According to the data in Table 3 and the form of expression for covariance matrix of multivariate mixture of normal distributions $D(X) = [\varepsilon^2 + (1-\varepsilon)^2\gamma]\sum$ and the formula for annualized volatility of the underlying stocks' return $\sigma\sqrt{T}$, where σ is the daily volatility of the underlying stock return and for T it is assumed that there are 252 trading days per year. So the obtained annualized volatilities are shown in the fifth column in Table 1. Financial parameters $\widetilde{\Delta}$, $\widetilde{\Gamma}$ and $\widetilde{\Theta}$ of the five warrants are obtained by option pricing model under heavy-tailed market risk factors proposed by Chen (2006), which are shown in Table 4. Furthermore, for the convenience of comparison, $\widetilde{\Delta}$, $\widetilde{\Gamma}$ and $\widetilde{\Theta}$ of the five warrants under normal distribution market risk factors are obtained by the Black–Scholes model (Table 4).

According to the above data, we can obtain the parameters for moment generating function of Q_{x-a_0} in Eq. (9), such as $a_0 = -\sum_{i=1}^{p}\widetilde{\Theta}_i$, $a = -\widetilde{\Delta}$, $A = -\frac{1}{2}\widetilde{\Gamma}$, λ and b (Table 4). Accordingly, the corresponding parameters a_0, a, A, λ and b in moment generating function of Q_{x-a_0} under multivariate normal distribution market risk factors are also shown in Table 5.

Using MATLAB 7.6 software and the data in Table 5, we calculate one-day VaR of the current position of the portfolio with different confidence levels of 90%, 92.5%, 95%, 97.5% and 99% by Fourier-Inversion method when heavy-tailed market risk factors are multivariate mixture of normal distributions. Accordingly, the computational results are reported in Table 6. For the convenience of comparison, in Table 6 we also list VaR values obtained with Cornish–Fisher method when distributions of market risk factors assumed to be multivariate normal distributions, with Fourier-Inversion method when market risk factors have multivariate normal distributions, with Monte Carlo method of 300,000 simulations when heavy-tailed market risk factors are depicted by multivariate mixture of normal distributions and Monte Carlo method of 300,000 simulations when market risk factors have multivariate normal distributions. As Monte Carlo simulation is a good benchmark for obtaining a numerical estimate of VaR, in many numerical experiments, VaR obtained with this method is often used as the indicator of accuracy of the new proposed method. Hardle et al. (2002), Glasserman (2004) and Johannes et al. (2009) applied this method for benchmarking against their new proposed methods. In our numerical experiment, we also computed VaR using the Fourier-Inversion method benchmarked against the Monte Carlo simulation method.

Computation time required for calculation of VaR of the portfolio under all the above confidence levels using Cornish–Fisher method,

Table 5
Parameters in the moment generating function.

Warrant	Under multivariate mixture of normal distributions				Under multivariate normal distributions			
	$a_0 = 1.0159$				$a_0 = 0.4522$			
	a	A	λ	b	a	A	λ	b
Baogang CWB1	−0.4272	−0.6074	−0.0006227	0.026607	−0.3207	−0.5529	−0.0013627	−0.023760
Gangyue CWB1	−0.8746	−1.5390	−0.0001983	0.000997	−0.3167	−0.8961	−0.0002895	−0.001371
Shangqi CWB1	−0.2418	−0.4622	−0.0000537	−0.001899	−0.0559	−0.1642	−0.0001783	0.003059
Shihua CWB1	−0.4645	−0.7499	−0.0000629	−0.001060	−0.3029	−0.6194	−0.0000591	0.000660
ShenggaoC WB1	−0.1743	−0.4322	−0.0000863	−0.002178	−0.0687	−0.2306	−0.0000784	0.000626

Table 6
One-day VaR values of the current position obtained by three different methods.

Confidence level	Cornish–Fisher method	Fourier Inversion method		Monte Carlo method	
		Under multivariate mixture of normal distributions	Under multivariate normal distributions	Under mixture of normal distributions	Under normal distributions
One-day VaR at −99% level	−0.5086*	−1.0700	−0.5915	−1.1462	−0.5002
One-day VaR at −97.5% level	−0.4986	−1.0671	−0.5901	−1.1196	−0.4935
One-day VaR at −95% level	−0.4904	−1.0531	−0.5883	−1.0939	−0.4874
One-day VaR at −92.5% level	−0.4849	−1.0492	−0.5864	−1.0776	−0.4833
One-day VaR at −90% level	−0.4837	−1.0447	−0.5845	−1.0669	−0.4801

Note: VaR values obtained by Monte Carlo simulation method require 300,000 simulations and * means the obtained VaR value 0.5086 is under 99% confidence levels.

Table 7
Computational time required for calculation of VaR using Fourier-Inversion method and Monte Carlo method.

Method	Computational time (s)
Cornish Fisher method	38.6842
Fourier-Inversion method under multivariate mixture of normal distributions	73.5623
Fourier-Inversion method under multivariate normal distributions	65.7139
Monte Carlo method under multivariate mixture of normal distributions	8.3679e + 002
Monte Carlo method under multivariate normal distributions	7.6285e + 002

Fourier- Inversion method under multivariate mixture of normal distributions or multivariate normal distributions and Monte Carlo simulation method under multivariate mixture of normal distributions or multivariate normal distributions is listed in Table 7.

As can be seen from Table 6, VaR calculated with Cornish Fisher method, Fourier-Inversion method and Monte Carlo method under multivariate normal distributions are much less than VaR values obtained with Fourier-Inversion method and Monte Carlo simulation method under mixture of normal distributions. The result shows that the tail of the portfolio distribution under market risk factors having multivariate mixture of normal distributions is heavier than that under market risk factors having multivariate normal distributions, which causes VaR values under mixture of normal distributions to be obviously higher than those under multivariate normal distributions. On the other hand, it explains that the model under mixture of normal distributions can capture heavy tails in the joint distribution of market risk factors.

When market risk factors are heavy-tailed, they are represented by multivariate mixture of normal distributions. As can be seen from Table 6, it is easy to find that VaR values obtained with Fourier-Inversion method are not obviously different from VaR values obtained with Monte Carlo simulation. That means with the same parameters γ and ε, Fourier-Inversion method can obtain accuracy close to that of Monte Carlo simulation and solve the nonlinear problems with high accuracy. The main reasons leading to this result are two-fold. On the one hand, Fourier-Inversion method can make full use of the characteristic function which reflects the change in option portfolio value. On the other hand, the characteristic function can represent completely the distribution of random variables, in the statistical sense, and thus there is no loss of information of distribution of the change in option portfolio value.

However, the difference between computation time consumed in Fourier-Inversion and Monte Carlo simulation is significant. As can be seen from Table 7, Fourier-Inversion needs less computation time than Monte Carlo. In other words, with a little difference in terms of accuracy, Fourier-Inversion is obviously much faster and efficient than Monte Carlo simulation. Furthermore, Monte Carlo simulation method needs a large number of samples, which increases computational time, thus affecting computation efficiency.

In addition, computation time required when using Cornish Fisher method based on market risk factors having multivariate normal distributions is also obviously less than the Monte Carlo simulation method or Fourier-Inversion method. The method embodies moment matched idea and uses the first four moments of the distribution of the change in option portfolio value to fit its distribution. In view of the simple calculation process of the method, it can be computed faster and in a simpler manner than Monte Carlo simulation or Fourier-Inversion method. However, this method makes use of only the first four moments of the distribution of the change in option portfolio value, which implies limited information fits its distribution. This means that the method belongs to the field of partial risk measurement and does not offer high accuracy. Besides, the method cannot be used to calculate VaR values of option portfolio when market risk factors have heavy-tailed distributions.

4. Conclusions

When the distributions are heavy-tailed, the key to calculation of nonlinear VaR of option portfolio is to obtain the characteristic function that reflects the change in option portfolio value. Usually, the characteristic function can reflect the change in option portfolio value and it can describe completely the distribution of random variables of the change in a statistical sense. Use of the characteristic function causes no loss of information of distribution of the change in option portfolio value.

Assuming heavy-tailed market risk factors are described with multivariate mixture of normal distributions, the paper proposes a model for calculation of nonlinear VaR of option portfolio with multivariate mixture of normal distributions and derives the moment generating function that reflects the change in option portfolio value. In the proposed method, we often have to face a tradeoff between computation speed and accuracy when calculating VaR of complex derivatives portfolios. Usually, Monte Carlo simulation can solve the nonlinear problems with high accuracy. However, in order to achieve high accuracy, Monte Carlo simulation requires a large number of samples, which increases computation time and workload, thus affecting computation efficiency. Thus, on the basis of the characteristic function that reflects the change in option portfolio value, the paper develops the Fourier-Inversion method and the adaptive Simpson rule with iterative algorithm of numerical integration to calculate VaR values of option portfolio.

VaR values obtained with different methods are compared numerically. Numerical results from Fourier-Inversion method or Monte Carlo simulation show that these VaR values have high accuracy when market risk factors have multivariate mixture of normal distributions, compared with market risk factors having normal distributions. Moreover, VaR values obtained from Fourier-Inversion method are not obviously different from those obtained from Monte Carlo simulation, whether market risk factors have normal distributions or multivariate mixture of normal distributions. However, computation time required for Fourier-Inversion method is less than Monte Carlo simulation.

Besides, Cornish Fisher method is faster and simpler than Monte Carlo simulation or Fourier-Inversion method. However, the method does not offer high accuracy and cannot be used to calculate VaR values of option portfolio when market risk factors have heavy-tailed distributions.

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Appendix A. Proof of Eq. (9)

From the normal random variable $X \sim N(0,\sigma)$, we have

$$E\left[\exp\left(aX^2+bX\right)\right]=\frac{1}{\sqrt{1-2a\sigma^2}}\exp\left[\frac{b^2\sigma^2}{2(1-2a\sigma^2)}\right], \quad .a\sigma^2<1/2. \tag{A.1}$$

This is because $E[\exp(aX^2+bX)]$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(aX^2+bX\right)\exp\left(-\frac{X^2}{2\sigma^2}\right)dX\\
&= \frac{1}{\sqrt{2\pi}\sigma}\times\int_{-\infty}^{\infty}\exp\left(-\frac{X^2-2a\sigma^2X^2-2b\sigma^2X}{2\sigma^2}\right)dX\\
&= \frac{1}{\sqrt{2\pi}\sigma}\times\int_{-\infty}^{\infty}\exp\left(-\frac{(1-2a\sigma^2)X^2-2b\sigma^2X}{2\sigma^2}\right)dX\\
&= \frac{1}{\sqrt{2\pi}\sigma}\times\int_{-\infty}^{\infty}\exp\left[-\frac{\left(\sqrt{1-2a\sigma^2}X-\frac{b\sigma^2}{\sqrt{1-2a\sigma^2}}\right)^2-\frac{b^2\sigma^4}{1-2a\sigma^2}}{2\sigma^2}\right]dX\\
&= \frac{\exp\left[\frac{b^2\sigma^2}{2(1-2a\sigma^2)}\right]}{\sqrt{1-2a\sigma^2}}\times\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{\left(\sqrt{1-2a\sigma^2}X-\frac{b\sigma^2}{\sqrt{1-2a\sigma^2}}\right)^2}{2\sigma^2}\right]d\left(\sqrt{1-2a\sigma^2}X-\frac{b\sigma^2}{\sqrt{1-2a\sigma^2}}\right)\\
&\underset{t=\sqrt{1-2a\sigma^2}X-b\sigma^2/\sqrt{1-2a\sigma^2}}{=}\frac{\exp\left[\frac{b^2\sigma^2}{2(1-2a\sigma^2)}\right]}{\sqrt{1-2a\sigma^2}}\times\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{t^2}{2\sigma^2}\right)dt\\
&= \frac{1}{\sqrt{1-2a\sigma^2}}\exp\left[\frac{b^2\sigma^2}{2(1-2a\sigma^2)}\right].
\end{aligned}
$$

Since $X_j =_d \varepsilon Z_j+(1-\varepsilon)Z_j'+(1-\varepsilon)Z_j'\,(Z_j\sim N(0,1),\;\; Z_j'\sim N(0,\gamma))$, Z_j and Z_j' are independent, we can obtain the moment generating function Q_{x-a_0}

$$
\begin{aligned}
M(\theta) &= E\left[\exp\left(\theta Q_{x-a_0}\right)\right]\\
&= \exp(-\theta(x-a_0))E\left\{\exp\left[\theta\sum_{j=1}^{p}\left(b_jX_j+\lambda_jX_j^2\right)\right]\right\}\\
&= \exp(-\theta(x-a_0))\prod_{j=1}^{p}E\left\{\exp\left[\theta\left(b_jX_j+\lambda_jX_j^2\right)\right]\right\}
\end{aligned}
$$

$$
\prod_{j=1}^{p}E\left\{\exp\left[\theta\left(b_jX_j+\lambda_jX_j^2\right)\right]\right\} = \exp(-\theta(x-a_0))\times\prod_{j=1}^{p}E\left\{\exp\left[\theta\left(b_j\left(\varepsilon Z_j+(1-\varepsilon)Z_j'\right)+\lambda_j\left(\varepsilon Z_j+(1-\varepsilon)Z_j'\right)^2\right)\right]\right\} = \exp((-\theta(x-a_0))\times)
$$

$$
\begin{aligned}
&\prod_{j=1}^{p}E\left\{\exp\left[\theta\varepsilon b_jZ_j+\theta(1-\varepsilon)b_jZ_j'+\theta\varepsilon^2\lambda_jZ_j^2+2\theta\varepsilon(1-\varepsilon)\lambda_jZ_jZ_j'+\theta(1-\varepsilon)^2\lambda_jZ_j'^2\right]\right\}\\
&= \exp(-\theta(x-a_0))\times\prod_{j=1}^{p}E\left\{\exp\left[[\theta\varepsilon^2\lambda_jZ_j^2+\left(\theta\varepsilon b_j+2\theta\varepsilon(1-\varepsilon)\lambda_jZ'_j\right)Z_j+\theta(1-\varepsilon)^2\lambda_jZ_j'^2+\theta(1-\varepsilon)b_jZ_j'\right]\right\}.
\end{aligned}
\tag{A.2}
$$

According to Eq. (A.1), Z_j and Z'_j are independent and $Z_j\sim N(0,1)$ $(j=1,2,\cdots,p)$, then Eq. (A.2) can be rewritten as

$$
M(\theta) = \exp(-\theta(x-a_0))\times\prod_{j=1}^{p}\frac{1}{\sqrt{1-2\theta\varepsilon^2\lambda_j}}
$$

$$
\prod_{j=1}^{p}E\left\{\exp\left[\frac{\left(\theta\varepsilon b_j+2\theta\varepsilon(1-\varepsilon)\lambda_jZ_j'\right)^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}+\theta(1-\varepsilon)^2\lambda_jZ_j'^2+\theta(1-\varepsilon)b_jZ_j'\right]\right\} = \exp(-\theta(x-a_0))\times\prod_{j=1}^{p}\frac{1}{\sqrt{1-2\theta\varepsilon^2\lambda_j}}
$$

$$
\prod_{j=1}^{p}E\left\{\exp\left[\frac{4\theta^2\varepsilon^2(1-\varepsilon)^2\lambda_j^2Z_j'^2+4\theta^2\varepsilon^2(1-\varepsilon)b_j\lambda_jZ_j'+\theta^2\varepsilon^2b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}+\theta(1-\varepsilon)^2\lambda_jZ_j'^2+\theta(1-\varepsilon)b_jZ_j'\right]\right\} = \exp(-\theta(x-a_0))
$$

$$
\prod_{j=1}^{p}\frac{\exp\left[\frac{\theta^2\varepsilon^2b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}\right]}{\sqrt{1-2\theta\varepsilon^2\lambda_j}}\prod_{j=1}^{p}E\left\{\exp\left[\left(\frac{2\theta^2\varepsilon^2(1-\varepsilon)^2\lambda_j^2}{1-2\theta\varepsilon^2\lambda_j}+\theta(1-\varepsilon)^2\lambda_j\right)Z_j'^2+\left(\frac{2\theta^2\varepsilon^2(1-\varepsilon)b_j\lambda_j}{1-2\theta\varepsilon^2\lambda_j}+\theta(1-\varepsilon)b_j\right)Z_j'\right]\right\}
$$

$$= \exp(-\theta(x-a_0)) \prod_{j=1}^{p} \frac{\exp\left[\frac{\theta^2\varepsilon^2 b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}\right]}{\sqrt{1-2\theta\varepsilon^2\lambda_j}} \prod_{j=1}^{p} E\left\{\exp\left[\left\{\frac{2\theta^2\varepsilon^2(1-\varepsilon)^2\lambda_j^2}{1-2\theta\varepsilon^2\lambda_j}+\frac{\theta(1-\varepsilon)^2\lambda_j-2\theta^2\varepsilon^2(1-\varepsilon)^2\lambda_j^2}{1-2\theta\varepsilon^2\lambda_j}\right\}Z_j'^2+\frac{2\theta^2\varepsilon^2(1-\varepsilon)b_j\lambda_j+\theta(1-\varepsilon)b_j-2\theta^2\varepsilon^2(1-\varepsilon)b_j\lambda_j}{1-2\theta\varepsilon^2\lambda_j}Z_j'\right]\right\} \tag{A.3}$$

$$= \exp(-\theta(x-a_0)) \prod_{j=1}^{p} \frac{\exp\left[\frac{\theta^2\varepsilon^2 b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}\right]}{\sqrt{1-2\theta\varepsilon^2\lambda_j}} \prod_{j=1}^{p} E\left\{\exp\left[\frac{\theta(1-\varepsilon)^2\lambda_j}{1-2\theta\varepsilon^2\lambda_j}Z_j'^2+\frac{\theta(1-\varepsilon)b_j}{1-2\theta\varepsilon^2\lambda_j}Z_j'\right]\right\}.$$

Using Eq. (A.1) and seeing that $Z_j' \sim N(0,\gamma)$, Eq. (A.3) can be rewritten as $M(\theta)$

$$= \exp(-\theta(x-a_0)) \prod_{j=1}^{p} \frac{\exp\left[\frac{\theta^2\varepsilon^2 b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}\right]}{\sqrt{1-2\theta\varepsilon^2\lambda_j}} \prod_{j=1}^{p} \frac{\exp\left\{\frac{\theta^2(1-\varepsilon)^2 b_j^2\gamma^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)^2\left[1-2\frac{\theta(1-\varepsilon)^2\lambda_j}{1-2\theta\varepsilon^2\lambda_j}\gamma^2\right]}\right\}}{\sqrt{1-2\frac{\theta(1-\varepsilon)^2\lambda_j}{1-2\theta\varepsilon^2\lambda_j}\gamma^2}}$$

$$= \exp(-\theta(x-a_0)) \prod_{j=1}^{p} \frac{\exp\left[\frac{\theta^2\varepsilon^2 b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}\right]}{\sqrt{1-2\theta\varepsilon^2\lambda_j}} \prod_{j=1}^{p} \frac{\exp\left\{\frac{\theta^2(1-\varepsilon)^2 b_j^2\gamma^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)\left[1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2\right]}\right\}}{\sqrt{\frac{1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2}{1-2\theta\varepsilon^2\lambda_j}}}$$

$$= \exp(-\theta(x-a_0))\times$$

$$\prod_{j=1}^{p} \frac{\exp\left[\frac{\theta^2\varepsilon^2 b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}\right]}{\sqrt{1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2}} \times \exp\left\{\frac{\theta^2(1-\varepsilon)^2 b_j^2\gamma^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)\left[1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2\right]}\right\}$$

$$= \exp(-\theta(x-a_0)) \times \prod_{j=1}^{p} \frac{\exp\left(\frac{\theta^2 b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}\right)}{\sqrt{1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2}} \times \exp\left[\varepsilon^2+\frac{(1-\varepsilon)^2\gamma^2}{1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2}\right]$$

$$= \exp(-\theta(x-a_0)) \times \left(\prod_{j=1}^{p} \frac{1}{\sqrt{1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2}}\right) \times \exp\left(\sum_{j=1}^{p}\left\{\frac{\theta^2 b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}\left[\varepsilon^2+\frac{(1-\varepsilon)^2\gamma^2}{1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2}\right]\right\}\right.$$

$$= \exp(-\theta(x-a_0)) \times \prod_{j=1}^{p} \frac{1}{\sqrt{1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2}} \times \exp\left(\sum_{j=1}^{p}\left\{\frac{\theta^2 b_j^2}{2\left(1-2\theta\varepsilon^2\lambda_j\right)}\left[\varepsilon^2+\frac{(1-\varepsilon)^2\gamma^2}{1-2\theta\varepsilon^2\lambda_j-2\theta(1-\varepsilon)^2\lambda_j\gamma^2}\right]\right\}\right).$$

The above result is exactly equal to Eq. (9) of the moment generating function Q_{x-a_0}.

References

Abate, J., Whitt, W., 1992. The Fourier-series method for inverting transforms of probability distributions. Queueing Syst. 10, 5–88.

Albanese, C., Campolieti, G., 2006. Advanced. Derivatives Pricing And Risk Management. Elsevier Academic Press, Diego California.

Albanese, C., Jackson, K., Wiberg, P., 2004. A new Fourier transform algorithm for value-at-risk. Quant. Finance 4, 328–338.

Britten-Jones, M., Schaefer, S.M., 1999. Non-linear value-at-risk. Eur. Finance Rev. 2 (2), 161–187.

Campbell, R.T., Zhou, G., 1993. International asset pricing with alternative distributional specifications. J. Empir. Finance 1 (5), 107–131.

Castellacci, G., Siclari, M.J., 2003. The practice of Delta–Gamma VaR: implementing the quadratic portfolio model. Eur. J. Oper. Res. 150, 529–545.

Chen, R.D., 2005. Risk measure of FX options based on Delta–Gamma–Theta model. Syst. Eng. Theory Pract. 25 (7), 55–60.

Chen, R.D., 2006. Foreign currency options pricing model with heavy-tailed exchange rate return. Oper. Res. Manag. Sci. 15 (3), 137–140.

Cui, X.T., Zhu, S.S., Sun, X.L., Li, D., 2013. Nonlinear portfolio selection using approximate parametric Value-at-Risk. J. Bank. Finance 37 (10), 2124–2139.

Glasserman, P., 2004. Monte Carlo Methods in Financial Engineering. Springer-Verlag, New York.

Hardle, W., Kleinow, T., Stahl, G., 2002. Applied Quantitative Finance. Springer-Verlag, New York.

Heyde, C.C., 1999. A risky asset model with strong dependence through fractal activity time. J. Appl. Probab. 36, 1234–1239.

Heyde, C.C., Kou, S., 2004. On the controversy over tail weight of distributions. Oper. Res. Lett. 32, 399–408.

Hosking, J.R.M., Bonti, G., Siegel, D., 2000. Beyond the lognormal. Risk 13 (5), 59–62.

Johannes, V.S., Jeffrey, T.L., Anna, S.-H., 2009. Value-at-Risk computation by Fourier inversion with explicit error bonds. Finance Res. Lett. 6 (2), 95–105.

Lu, F.Y., 2005. Analysis on fat tail characteristics of stock market returns in China. Syst. Eng. Theory Methodol. Pract. 14 (4), 350–352.

Morgan, J.P., 1996. RiskMetrics — Technical Document, 4th edition. (http//www.RiskMetrics.com).

Sorwar, G., Dowd, K., 2010. Estimating financial risk measures for options. J. Bank. Finance 34 (8), 1982–1992.

Xu, X.S., Hou, C.Q., 2006. A portfolio selection model conditional on non-normal stable distributions: mean-scale parameter model. Syst. Eng. Theory Pract. 9, 1–9.

Zhou, G.F., 2002. Financial Econometrics: Empirical Analysis of Asset Pricing. Peking University Press, Beijing.