ELSEVIER
Topology and its Applications 95 (1999) 1–30
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Explorations in Nielsen periodic point theory for the double torus

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Received 20 August 1996; received in revised form 17 May 1997, 4 November 1997

Abstract

For $f : X \to X$, with X a compact manifold, Nielsen periodic point theory involves the calculation of f-homotopy invariant lower bounds for $|\text{fix}(f^n)|$ and for the number of periodic points of minimal period n. In this paper we combine the covering space approach to Nielsen periodic point theory with an algebraic method of Fadell and Husseini to study the behavior of the Nielsen periodic classes of maps on $T^2\#T^2$, the surface of genus two. Nil and solvmanifolds have basic properties for Nielsen periodic classes that make the calculation of these lower bounds possible. In this paper we accomplish two objectives. We show firstly that virtually all of these basic properties for the periodic classes fail in general on $T^2\#T^2$ as well as on a collection of manifolds of arbitrarily high dimension. Secondly, despite these difficulties, we develop and apply techniques involving linear algebra, combinatorial group theory, number theory, and geometric facts from the theory of surface homeomorphisms, to make some calculations of the Nielsen periodic numbers. In our final example the combinatorial structure of the essential Nielsen periodic classes is fully displayed in a manner which relies on some of the classic identities involving the Fibonacci and Lucas numbers. © 1999 Published by Elsevier Science B.V. All rights reserved.

Keywords: Fox calculus; Generalized Lefschetz number; Nielsen number; Nielsen periodic numbers; Reidemeister trace; Surface

AMS classification: 55M20

Dedicated to Edward Fadell in honor of his 70th birthday.

1. Introduction

Suppose that f is a self map on a compact manifold X. If X is a nilmanifold or solvmanifold, then the computation of the sequence $\{N(f^n)\}_{n=1}^{\infty}$ and the relationship of

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PII: S0166-8641(97)00277-0

this sequence to the Nielsen periodic numbers of f is well understood. However, for many other spaces, such as the double torus or handcuff space $T^2\#T^2$, the relationship between the Nielsen periodic numbers and the ordinary Nielsen numbers for various iterates is much more complicated, making these numbers difficult to compute. We explore these complications in a variety of examples of self maps on $T^2\#T^2$ as well as analogues of these self maps on higher dimensional spaces. These examples illustrate the differences between the periodic structure on nil and solvmanifolds and on the manifolds studied here.

Given $f : X \to X$ on a finite polyhedron or a compact manifold X, basic Nielsen theory involves finding a lower bound, the Nielsen number $N(f)$, for the number of fixed points of any map g homotopic to f (see [2,14,18]). We partition the fixed points of f into equivalence classes. Then $N(f)$ is the number of fixed point classes which persist in some sense of equivalence under any homotopy of f. Such classes are called essential. Alternatively, $N(f)$ can be obtained by partitioning $\pi_1(X)$ into algebraic classes and assigning an index to each class. $N(f)$ is then the number of essential classes (those with nonzero index). The Nielsen periodic numbers $N\Phi_n(f)$ (see [11]) and $NP_n(f)$ (see [10]) are homotopy invariant lower bounds for the number of periodic points of f with period dividing n and for the number of periodic points with minimal period n, respectively. These periodic numbers are invariant with respect to homotopies of f (i.e., not of f^n). Both $N\Phi_n(f)$ and $NP_n(f)$ are computed by analyzing the ordinary Nielsen classes of each f^n and the relationship of these classes to one another as induced by the natural inclusions of $\mathrm{fix}(f^m)$ into $\mathrm{fix}(f^n)$ when $m|n$.

The Reidemeister trace of f [6,13], which has also been called the generalized Lefschetz number of f, is an algebraic object from which the essential Nielsen classes and their indices, and thus the Nielsen number, can be determined. The Reidemeister trace is a formal sum over $\mathbb{Z}$ of algebraic classes. Provided this trace is in reduced form, the number of nonzero terms is equal to $N(f)$. To calculate the Reidemeister trace for the map f on a closed surface, one can use the method of Fadell and Husseini [5], which involves the Fox calculus [3]. There are two fundamental reasons why such computations become so problematic for the Nielsen periodic numbers. The first is that in order to draw conclusions from the Reidemeister trace it must be in a reduced form, i.e., each algebraic class must appear in the sum at most once. Because there is no known procedure that can always be used to determine whether two classes in $\pi_1(T^2\#T^2)$ expressed with different representatives are equal, we must use a variety of algebraic and geometric techniques to reduce the Reidemeister trace. The second reason for complications in the calculation of the periodic numbers on $T^2\#T^2$ arises from the process of iteration. The equivalence relation that produces the algebraic classes is different for each iterate of f. While the algebraic classes for various iterates are related by certain boosting functions (see Section 2), it can be difficult to know when an algebraic class for some f^n reduces to a class for a lower iterate. The software package Magma [1] can often find reductions of classes and demonstrate equivalence between two classes in a way that can always be checked by hand, whereas proofs of irreducibility and the distinctness of certain algebraic classes are much harder to come by.

A nilmanifold is a generalization of a torus, $\mathbb{R}^n/\mathbb{Z}^n$, where $\mathbb{R}^n$ has the structure of a nilpotent Lie group and $\mathbb{Z}^n$ is a subgroup. A solvmanifold is also a coset space of the form $\mathbb{R}^n/\Gamma$, where $\mathbb{R}^n$ need only be solvable and Γ need not be discrete. In [8] a number of basic properties for the essential Nielsen periodic classes on nil and solvmanifolds are established. These include essential reducibility (i.e., essential classes which reduce do so to essential classes), length = depth (i.e., a class is irreducible at level n iff its orbit contains n distinct classes), the injectivity of the inclusions ($\mathrm{fix}(f^m) \subseteq \mathrm{fix}(f^n)$) when applied to essential classes, and the uniqueness of roots of essential classes. [1] In addition to having these properties, most maps on nil and solvmanifolds are weakly Jiang (i.e., either all Reidemeister classes are essential or none are). When this is the case, the other properties always make it possible to express the $NP_n(f)$ and $N\Phi_n(f)$ in terms of the numbers $\{N(f^m)\colon m|n\}$. One of the main objectives of this paper, in addition to describing techniques for computation on $T^2\#T^2$, is to show that on $T^2\#T^2$ as well as other manifolds of arbitrarily high dimension, all of these properties fail to hold in general. Despite these observations, our last example will demonstrate what is possible when many of them do hold. This suggests that in future work, as well as trying to make sense of this strange behavior, one might search for general conditions on a map to assure that the techniques for computing $NP_n(f)$ and $N\Phi_n(f)$ implied by these properties can be used on the double torus. However, since general algorithms for the computation of $N(f^n)$ itself are not known for the double torus, such formulae which express $NP_n(f)$ and $N\Phi_n(f)$ in terms of Nielsen numbers are only part of the story.

The paper is organized as follows. Section 2 contains the required prerequisites of Nielsen periodic theory, covering spaces, and the Reidemeister trace. We describe and motivate the basic properties that hold for essential periodic classes on nil and solvmanifolds (see [8]). This provides a starting point for comparison with the double torus and certain higher dimensional analogues.

In Section 3 we present our four examples for maps f on the double torus. In this section we also develop new techniques for the use of abelianized Reidemeister classes in determining the length and depth of essential classes. While the abelianization methods of [5] are useful in all our examples, the new techniques described here are used in Example 3 to show a case where length $\neq$ depth for essential classes.

Example 1 shows the failure of essential reducibility, and of the weakly Jiang property, and provides a case in which $N\Phi_n(f) \neq \sum_{m|n} NP_m(f)$. Example 2 shows the failure of injectivity of the boosting functions on essential classes and the failure of uniqueness of essential roots. This provides a situation in which $N\Phi_n(f) \neq N(f^n) \neq 0$. [2] These first two examples provide a short and surprising introduction to what can, in some sense, go

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[1] Heath and Keppelmann [8] also discuss the property of essential reducibility to the GCD. As is done in our Example 4, this property is usually used to prove the uniqueness of roots of essential classes. Since we show that such uniqueness fails to hold in general, we have not attempted to find a counterexample for essential reducibility to the GCD.

[2] The papers [10,11] also contain examples for which these two basic formulas fail to hold. However, as these are in situations where the fundamental group is finite, they are of a significantly different nature than what is presented here.

wrong in the calculation of Nielsen periodic point numbers for self-maps on the double torus.

After Example 3 we observe that for any self-map f of $T^2\#T^2$ and any $k > 1$ the self-map $f \times g$ on $(T^2\#T^2) \times S^k$, where g is a map of degree 2, will have an algebraic periodic point structure that is isomorphic to that of f. Thus our counterexamples and examples can be reproduced in any dimension greater than 3.

In Example 4 we demonstrate a situation in which, although not all of the basic properties can be verified, it is still possible to obtain the formula

$$N\Phi_n(f) = N(f^n) = \sum_{m|n} NP_m(f) \quad \text{for all } n.$$

This is done with the application of a Nielsen and Lefschetz number inequality for surface homeomorphisms by Jiang and Guo from [15]. It is here that there is a surprising combinatorial structure involving Fibonacci numbers for the essential classes. We prove that $N(f^{2n+1}) = L_{2n+1}$, where $\{L_i\}_{i=1}^{\infty}$ is the sequence of Lucas numbers, the companion sequence to the Fibonacci numbers (see [12]). Thus $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$. The techniques presented here allow us to complete the calculation in Example 4 of [4]. We also take this opportunity to correct a typographical error in Example 3 of [4].

In Section 4 we conclude with a conjecture and several ideas for new techniques that could be developed in this subject, especially if Thurston's classification of surface diffeomorphisms is considered. We hope that the discussions and examples presented here will promote further study in what we have found to be a very interesting subject. We will call this study, which involves combinatorial group theory, number theory, and linear algebra, combinatorial Nielsen theory.

2. Preliminaries

There are two equivalent approaches to Nielsen fixed point theory. These two approaches define the Nielsen classes differently. Since both approaches are used in our primary references, we feel that it is important to give a complete description of the equivalence between the two. One approach involves comparing loops in X and the second involves comparing lifts of $f : X \to X$ to the universal cover $p : \widetilde{X} \to X$. (For more details the reader is encouraged to consult [14,18].) In addition to these two approaches, there is also a great variety of notation in the literature for the Reidemeister action. [3] While most of the following is standard, we hope this sketch provides some insight for the reader; especially in allowing for an appreciation of the compatibility of the covering space approach with what has been done previously for periodic points. We feel the covering space approach provides a much cleaner relationship between the fixed points of different iterates in that we can now view the inclusions $\mathrm{fix}(f^m) \subseteq \mathrm{fix}(f^n)$ on the algebraic level as arising from similar inclusions of fixed points for the iterates of lifts of f (see [14]).

[3] For example, one can choose the Reidemeister action to be given by $\alpha \cdot \gamma = \alpha\gamma\phi(\alpha^{-1})$. This is done in [8–11] where $\hat{\iota}_{m,n}(\alpha) = \alpha f_{\#,m(w)}(\alpha) \cdots f_{\#,(n-m)(w)}(\alpha)$. One can also have $[\alpha]$ indicate $p(\mathrm{fix}(\tilde{f}\alpha))$. Our choice is compatible with [5].

2.1. Nielsen theory

Here we present material from [14,18]. Suppose that $f : X \to X$ is a map on a finite polyhedron or compact manifold. In what follows $\operatorname{fix}_N(f)$ will denote the collection of geometric Nielsen fixed point classes of f. The equivalence relation that determines these classes can be described in two ways. First, for $x, y \in \operatorname{fix}(f)$, $x \sim_f y$ iff there is a path δ from x to y so that (rel endpoints) $\delta \sim f(\delta)$. Equivalently, $x \sim_f y$ iff there is a lift $\tilde{f}'$ of f such that $x, y \in p(\operatorname{fix}(\tilde{f}'))$. As we will see, this equivalence corresponds to an action of $\pi_1(X)$ on itself. For a homomorphism $\psi : G \to G$ on any group G, the orbit, or Reidemeister class, of the Reidemeister action that contains $g \in G$ is denoted by $[g] = \{\psi(h)gh^{-1}\colon h \in G\}$. The symbol $\mathcal{R}(\psi)$ will denote the set of Reidemeister classes. [4]

We begin by considering the path approach. Fix coordinates by choosing an $x_0 \in X$ and a path ω from x_0 to $f(x_0)$. Then f induces the homomorphism $f_{\#,\omega} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ by the rule that $f_{\#,\omega}(\alpha)$ is the loop class containing $\omega f(\alpha)\omega^{-1}$. We define a function $\rho : \operatorname{fix}_N(f) \to \mathcal{R}(f_{\#,\omega})$ as follows. For any $x \in \operatorname{fix}(f)$ we associate the Nielsen class of x with the element in $\mathcal{R}(f_{\#,\omega})$ containing $\omega f(C)C^{-1}$ where C is any path from x_0 to x. It can be checked that if $\widehat{C}$ is another path from x_0 to x then $\omega f(\widehat{C})\widehat{C}^{-1}$ will belong to the same Reidemeister class in $\mathcal{R}(f_{\#,\omega})$. It should also be verified that this correspondence respects the Nielsen equivalence relation on $\operatorname{fix}(f)$ defined above. Thus the Reidemeister classes of $f_{\#,\omega}$ can be thought of as the fixed point classes of f as long as we recognize that some Reidemeister classes will correspond to empty fixed point classes.

For the covering space approach we will let $\mathcal{D}$ denote the collection of covering transformations of the universal covering space $\widetilde{X}$ of X, with covering projection $p : \widetilde{X} \to X$. These are homeomorphisms $\widetilde{X} \to \widetilde{X}$ which project to the identity on X. We recall that $\mathcal{D}$ is isomorphic to $\pi_1(X, x_0)$, the homotopy classes of loops based at x_0. The isomorphism $\Theta : \pi_1(X, x_0) \to \mathcal{D}$ is defined as follows. Each $\beta \in \pi_1(X, x_0)$ gives rise to a permutation of $p^{-1}(x_0)$ and hence a covering transformation $\Theta(\beta) : \widetilde{X} \to \widetilde{X}$ determined by letting, for each $y \in p^{-1}(x_0)$, $\Theta(\beta)(y)$ be the endpoint of the lift of β that begins at y.

In the covering space approach we will again need to fix coordinates. To do this we fix a lift $\tilde{f}$ of f and a base point $\tilde{x}_0 \in p^{-1}(x_0)$. Then every lift of f can be written uniquely in the form $\alpha\tilde{f}$ for some covering transformation α. The homomorphism $\phi : \mathcal{D} \to \mathcal{D}$, induced by f, is then specified by requiring that for each covering transformation α we let $\phi(\alpha)$ be the unique covering transformation which satisfies $\phi(\alpha)\tilde{f} = \tilde{f}\alpha$. In order to guarantee compatibility between the coordinate choices for the covering space and path approaches we will require that $\tilde{f}(\tilde{x}_0)$ be the endpoint of the lift $\tilde{\omega}$ of ω which begins at $\tilde{x}_0$. The following diagram will commute

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_{\#,\omega}} & \pi_1(X, x_0) \\ \Theta \downarrow & & \downarrow \Theta \\ \mathcal{D} & \xrightarrow{\phi} & \mathcal{D} \end{array} .$$

[4] This notation is standard, as is the use of the similar symbol $\mathcal{R}(f)$ to denote the number of orbits.

This means that in essence we may use Θ to identify $f_{\#,\omega}$ with ϕ, $R(f_{\#,\omega})$ with $R(\phi)$, and $\mathcal{D}$ with $\pi_1(X, x_0)$. When nonempty, $p(\mathrm{fix}(\beta^{-1}\tilde{f})) = p(\mathrm{fix}(\hat{\beta}^{-1}\tilde{f}))$ iff $\hat{\beta} \in [\beta] \in \mathcal{R}(\phi)$. In analogy with what we did for paths, $\hat{\rho}:\mathrm{fix}_N(f) \to \mathcal{R}(\phi)$ in this context is given by specifying that $\hat{\rho}([x])$ corresponds to the unique Reidemeister class $[\alpha] \in \mathcal{R}(\phi)$ for which $x \in p(\mathrm{fix}(\alpha^{-1}\tilde{f}))$. Note that here α corresponds to Θ of the loop class containing $\omega f(C)C^{-1}$ from the path approach. This yields compatibility between the two approaches since as functions from $\mathrm{fix}_N(f)$ to $\mathcal{R}(\phi)$ we have that $\Theta\rho = \hat{\rho}$. Henceforth we will not mention Θ, and we will not distinguish between ρ and $\hat{\rho}$.

2.2. Nielsen periodic point theory

For $\psi : G \to G$ and $g \in G$ we will use $[g]^k$ to denote the class of $\mathcal{R}(\psi^k)$ which contains g. Similarly, the element of $\mathrm{fix}_N(f^k)$ containing x will be denoted by $[x]^k$. We let $\rho_k : \mathrm{fix}_N(f^k) \to \mathcal{R}(f^k_{\#,k(\omega)})$ be the function ρ defined above for f^k. When iterating the map f, whether the covering space or path approach is used, it is important to pick coordinates for the iterates which are compatible. As in [10,11] for for the path approach given $\omega : x_0 \to f(x_0)$ we let, for each natural number n, $n(\omega) = \omega f(\omega) f^2(\omega) \cdots f^{n-1}(\omega)$ be the path of choice between x_0 and $f^n(x_0)$. Now suppose that $x \in \mathrm{fix}(f^m)$ for some $m|n$. Let C be a path from x_0 to x. Then the Reidemeister class for x with f^m is given (using ρ_m in place of $\rho_1 = \rho$) by $\rho_m([x]^m) = [m(\omega) f^m(C) C^{-1}]^m$. The relation between $\beta = \rho_m([x]^m)$ and $\rho_n([x]^n)$ is described by the change of level boosting function in the following definition.

Definition 2.1. Suppose that $\psi : G \to G$ is a homomorphism. Then for positive integers m, n with $m|n$ define $\iota_{m,n} : G \to G$ by

$$\iota_{m,n}(\beta) = \psi^{n-m}(\beta)\psi^{n-2m}(\beta) \cdots \psi^m(\beta)\beta.$$

It is not hard to check that $\iota_{m,n}$ induces a function (with the same name) from $\mathcal{R}(\psi^m) \to \mathcal{R}(\psi^n)$. In our case, $G = \pi_1(X, x_0)$ and $\psi = \phi$.

For the covering space approach we must find a choice of coordinates for f^n that is compatible with the choice of x_0 and $n(\omega)$ made above for the path approach. This algebraic approach appears in [14], but we extend it here to nonabelian fundamental groups. We note that since the lift of ω which begins at $\tilde{x}_0$ will end at $\tilde{f}(\tilde{x}_0)$, we have that the lift of $n(\omega)$ beginning at $\tilde{x}_0$ will end at $\tilde{f}^n(\tilde{x}_0)$. Our choice of lift $\tilde{f}$ for f naturally gives rise to $\tilde{f}^n$ for f^n and thus is compatible with the choice of coordinates for the path approach.

The following lemma allows us to view the inclusion of Nielsen classes from one iterate to another as equivalent to the inclusion of entire fixed point sets of one lift into those of its iterate. Suppose that $m|n$ and that $x \in p(\mathrm{fix}(\alpha^{-1}\tilde{f}^m))$. Then $x \in p(\mathrm{fix}((\alpha^{-1}\tilde{f}^m)^{n/m}))$. Expanding this composition and moving all the α^{-1} past all $\tilde{f}^m$ by the relation $\tilde{f}^m\alpha^{-1} = \phi^m(\alpha^{-1})\tilde{f}^m$ yields the following.

Lemma 2.2. *Suppose that $f:X\to X$ and m, n are natural numbers with $m|n$. The following diagram commutes*

$$\begin{array}{ccc} \mathrm{fix}_N(f^m) & \xrightarrow{\rho_m} & \mathcal{R}(\phi^m) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle \iota_{m,n}} \\ \mathrm{fix}_N(f^n) & \xrightarrow{\rho_n} & \mathcal{R}(\phi^n) \end{array},$$

where i^* is induced by inclusion. Additionally, when $k|m|n$, we have $\iota_{m,n}\iota_{k,m}=\iota_{k,n}$.

Now, having given a complete correspondence between the path approach and the covering space approach for partitioning each $\mathrm{fix}(f^n)$, we will, for the remainder of this paper, use solely the covering space approach. This is due to our heavy reliance on the covering space approach of the Reidemeister trace (see [13,4–6]). From now on we will use π in place of $\pi_1(X,x_0)$.

2.3. Nielsen periodic numbers

Classical Nielsen theory follows the partitioning of $\mathrm{fix}(f)$ into Nielsen classes by the use of an integer valued fixed point index for each $\boldsymbol{F}\in\mathrm{fix}_N(f)$. (For example, see [2, 14].) A class is essential iff its index is nonzero. We refer to the index of an algebraic class as being zero if the class is empty. Otherwise, the index of an algebraic class is the same as the index of the unique Nielsen class that corresponds to it under ρ. The Nielsen number $N(f)$, a homotopy invariant, is then the number of essential classes in $\mathrm{fix}_N(f)$. The classical lower bound property given by

$$N(f)\leqslant\min\bigl\{|\,\mathrm{fix}(g)|:\ g\sim f\bigr\}$$

follows from the fact that for any g homotopic to f there is a one-to-one correspondence between the essential geometric classes for f and those for g.

As stated above, the study of the Nielsen periodic numbers (see [10,11]) is a study of what happens to Nielsen classes and their essentiality under iteration. More specifically, for an algebraic class $[\alpha]^m$ of f^m, we must consider the depth $d([\alpha]^m)$ and length $l([\alpha]^m)$ of $[\alpha]^m$. The *depth* of $[\alpha]^m$ is the smallest $k|m$ so that $[\alpha]^m$ is in the image of $\iota_{k,m}:\mathcal{R}(\phi^k)\to\mathcal{R}(\phi^m)$. We say that $[\alpha]^m$ is *irreducible* when its depth is m. In this case if $[\alpha]^m$ is also essential then we know that the points of $\mathrm{fix}(f^m)$ that are in $p(\mathrm{fix}(\alpha^{-1}\tilde{f}^m))$ are all of minimal period m. If $[\beta]^m$ is an irreducible class and if $\iota_{m,n}([\beta]^m)=[\alpha]^n$, then $[\beta]^m$ is said to be a *root* of $[\alpha]^n$.

The following is a useful fact that allows us to determine reducibility of a Reidemeister class by studying a representative of the class. It justifies our abuse of notation in thinking of the $\iota_{m,n}$ as functions on either π or $\mathcal{R}(\phi)$.

Lemma 2.3. *The class $[\alpha]^n$ reduces to level $m|n$ iff there is a $\gamma\in\pi$ such that*

$$\iota_{m,n}(\gamma)=\phi^{n-m}(\gamma)\cdots\phi^{2m}(\gamma)\phi^m(\gamma)\gamma=\alpha.$$

Proof. Suppose $[\alpha]^n$ reduces to level m. Then there are $\delta, \beta \in \pi$ such that $\iota_{m,n}(\delta) = \phi^n(\beta)\alpha\beta^{-1}$. Thus $\iota_{m,n}(\phi^m(\beta^{-1})\delta\beta) = \alpha$. The other direction follows from the fact that $\iota_{m,n}$ is well defined on Reidemeister classes. □

The basic principle that underlies all of Nielsen periodic point theory is that periodic points of minimal period m occur in orbits of length m. In other words, if x has minimal period m, then so does each element of the orbit $\{x, f(x), \ldots, f^{m-1}(x)\}$. Algebraic length is a notion that measures the extent to which this basic principle is reflected by the algebra. We note that ϕ induces a well defined action on $\mathcal{R}(\phi^m)$. The *length* $l = l([\alpha]^m)$ of $[\alpha]^m$ is the number of distinct Reidemeister classes in the algebraic orbit

$$\langle [\alpha]^m \rangle = \{[\alpha]^m, [\phi(\alpha)]^m, [\phi^2(\alpha)]^m, \ldots, [\phi^{m-1}(\alpha)]^m\} \quad \text{of } [\alpha]^m.$$

Because $[\alpha]^m = [\phi^m(\alpha)]^m$ we know that $l([\alpha]^m) \leqslant m$. The length of a class is obviously well defined on orbits. The following shows that when the boosting functions are injective, the length of an orbit is independent of the level at which a given class is considered.

Lemma 2.4. *Suppose that* $\iota_{m,n} : \mathcal{R}(\phi^m) \to \mathcal{R}(\phi^n)$ *is injective. Given* $\alpha, \beta \in \pi$*, if* $[\beta]^n = \iota_{m,n}([\alpha]^m)$ *then* $l([\beta]^n) = l([\alpha]^m)$.

Proof. Let l be minimal so that there is a $\gamma \in \pi$ with $\phi^m(\gamma)\phi^l(\alpha)\gamma^{-1} = \alpha$. Since $[\beta]^n = [\iota_{m,n}(\phi^l(\alpha))]^n = [\phi^l(\beta)]^n$, we know that $k = l([\beta]^n) \leqslant l$. Also,

$$[\beta]^n = \phi^k([\beta]^n) = [\phi^k(\iota_{m,n}(\alpha))]^n = [\iota_{m,n}(\phi^k(\alpha))]^n = \iota_{m,n}([\phi^k(\alpha)]^m).$$

By hypothesis $\phi^k([\alpha]^m) = [\alpha]^m$ so $k \geqslant l$ and thus $k = l$ as claimed. □

Since the index of $[\alpha]^n$ and of $[\phi(\alpha)]^n$ are equal [10], the property of being essential is also a property of orbits. Likewise, depth is well defined on orbits [10]. The algebra does exactly reflect the geometry in the sense that if the orbit $\langle [\alpha]^m \rangle$ is essential and irreducible then we know that it must contribute a positive multiple of m points of minimal period m to any map homotopic to f.

As discussed in [10,11] there are two Nielsen type periodic numbers: $NP_n(f)$ and $N\Phi_n(f)$. The number $NP_n(f)$ is defined to be n times the number of essential irreducible orbits of f^n. As described above, this is an f-homotopy invariant lower bound for the number of periodic points of f that have minimal period exactly n. The number $N\Phi_n(f)$, which is significantly more complicated to define than $NP_n(f)$, is an f-homotopy invariant lower bound for $|\operatorname{fix}(f^n)|$ (as opposed to $N(f^n)$ which would be an f^n-homotopy invariant for $|\operatorname{fix}(f^n)|$). A set of n-representatives for f is a collection of algebraic orbits from various levels $m|n$ with the property that any essential orbit at any level $m|n$ will reduce to an orbit in this set. The height of a set of n-representatives is the sum of the depths of all its members. The number $N\Phi_n(f)$ is then the minimal height over all sets of n representatives for f. Of course, $N(f^n)$ is always a lower bound for $N\Phi_n(f)$ since $N\Phi_n(f)$ restricts one to homotopies of f^n induced from homotopies of f. The definition of $N\Phi_n(f)$ is designed to count periodic points of period n by considering all algebraic orbits at all levels $m|n$. Those orbits which are essential at some level $m|n$ will contribute to $N\Phi_n(f)$ even if their images under boosting at level n are inessential. The choice of a *minimal* set of n-representatives seeks to avoid duplication and forces the counting of each of these essential orbits at the level of their depth. (I.e., classes at different levels which represent the same geometric points should only be counted once according to the minimal period of those points.)

By "the computation of $NP_n(f)$ and $N\Phi_n(f)$" one can mean several things. In previous work such as that of [8,9] it is shown that for self-maps on compact solvmanifolds there is a standard procedure by which the numbers $\{N(f^m)\colon m|n\}$ can be used to express $NP_n(f)$ and $N\Phi_n(f)$. Thus the computation of the periodic numbers is possible using the computation of the ordinary Nielsen numbers of the iterates of f (which for solvmanifolds are also very well understood and computable [17,21,22]). Although strides forward are being made, in the case of surfaces the computation of ordinary Nielsen numbers is far less well understood. In this sense then, calculating the periodic numbers in terms of the ordinary Nielsen numbers of the iterates, although far from what this paper can accomplish, would be somewhat less satisfying than for solvmanifolds. Despite this, we will indicate a number of powerful techniques which, in some cases, can determine $NP_n(f)$, $N\Phi_n(f)$ and $N(f^n)$ for self-maps on the surface $T^2\#T^2$.

As a point of reference, we begin with a basic understanding of how the Nielsen classes of the various iterates of f fit together on nil and solvmanifolds to compute the periodic numbers. We list the relevant properties below. Although not quite described in this way, these properties were proved for all maps on tori in [11] and were extended by fibre techniques to nil and solvmanifolds (except for (P5) below) in [8]. All maps on nilmanifolds are weakly Jiang and for any map f on a solvmanifold S there are simple criteria, involving the Nielsen numbers on the fibers in a Mostow fibration for S, that determine whether f is weakly Jiang. However, even when the map is not weakly Jiang, the Nielsen periodic numbers are still quite computable on nil and solvmanifolds as shown in [9]. For us the properties below will represent, when they are valid, the simplest way in which the periodic numbers can be formed. This puts us in an excellent position to appreciate just how complicated the situation can be on $T^2\#T^2$ or on certain other manifolds of arbitrarily high dimension.

(P1) **Essential reducibility**. *If $\iota_{m,n}([\beta]^m) = [\alpha]^n$ with $[\alpha]^n$ essential, then $[\beta]^m$ is also essential*. This is important in allowing one to relate the $NP_n(f)$ and $N\Phi_n(f)$ by the formula

$$N\Phi_n(f) = \sum_{m|n} NP_m(f).$$

Möbius inversion (see [11]) then also allows one to write $NP_n(f)$ in terms of $\{N\Phi_m(f)\colon m|n\}$.

(P2) **Injectivity of $\iota_{m,n}$ on essential classes**. *If $\iota_{m,n}([\gamma]^m) = [\alpha]^n = \iota_{m,n}([\beta]^m)$ where $[\alpha]^n$, $[\beta]^m$, and $[\gamma]^m$ are all essential, then $[\beta]^m = [\gamma]^n$*. Such a condition allows us to track with combinatorial arguments the Nielsen classes at various levels (see (P5) below).

(P3) **Length $=$ depth on essential classes**. *If $[\alpha]^n$ is essential and irreducible then its length is n.* This indicates that the basic geometric principle that points of minimal period n come in orbits of n points is exactly reflected by the algebra. In this case, $NP_n(f)$ is the number of essential irreducible classes at level n, and $NP_n(f) \leqslant N(f^n)$.

(P4) **Uniqueness of roots for essential classes**. *If $[\alpha]^n$ is essential, then there is a unique irreducible class $[\beta]^m$ for some $m|n$ for which $\iota_{m,n}([\beta]^m) = [\alpha]^n$.* This means that $N\Phi_n(f)$ can be computed by simply adding the depths of the roots of all orbits which are essential at any level $m|n$. As is shown in [8] this is actually a consequence of (P1), (P2) and a property called essentially reducible to the GCD. An essential class $[\alpha]^n$ is essentially reducible to the GCD if whenever $[\alpha]^n$ reduces to essential classes at levels s and k, then $[\alpha]^n$ also reduces to an essential class at the level $gcd(s,k)$.

(P5) **Weakly Jiang**. *The map f^n is weakly Jiang provided that either $N(f^n) = 0$ or else every element of $\mathcal{R}(f^n)$ is essential.* This means that when $N(f^n) \neq 0$ the image of any $\iota_{m,n}$ consists only of essential classes. In conjunction with all of the properties above, this then implies that

$$N\Phi_n(f) = N(f^n) \quad \text{when } N(f^n) \neq 0.$$

This, along with the formula from (P1) is the desired connection between the Nielsen numbers and the periodic numbers. Let $M(f,n)$ be the set of maximal divisors m of n for which $N(f^m) \neq 0$. If $N(f^n) = 0$ and if we know that for all $m \in M(f,n)$ that f^m is weakly Jiang, then we get from all of the above properties that

$$N\Phi_n(f) = \sum_{\emptyset \neq \mu \subseteq M(f,n)} (-1)^{\#\mu - 1} N\bigl(f^{\xi(\mu)}\bigr),$$

where $\xi(\mu)$ is the gcd of all numbers in μ. In conjunction with the formula in (P1) we can then also compute the $NP_n(f)$.

2.4. Calculating the Reidemeister trace on $T^2\#T^2$

Let $\pi = \pi_1(T^2\#T^2) = \langle a,b,c,d : R\rangle$, with $R = aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Because $T^2\#T^2$ is a $K(\pi,1)$, every endomorphism $\phi : \pi \to \pi$ is induced by a self-map f on $T^2\#T^2$. Thus we may consider endomorphisms rather than continuous maps.

The Reidemeister trace of f^n, $R(f^n, \tilde{f}^n)$ (for the chosen lift $\tilde{f}^n$), previously known as $L(f^n, \tilde{f}^n)$, is an element of the free $\mathbb{Z}$-module $\mathbb{Z}(\mathcal{R}(\phi^n))$. This trace incorporates information about both the Nielsen classes and their indices into a single algebraic object.

Let $\widetilde{X}$ be the universal cover of $T^2\#T^2$. To define $R(f^n, \tilde{f}^n)$ (see [13,6]) we consider the $\mathbb{Z}$-homomorphisms $\tilde{f}^n_* : C_*(\widetilde{X},\mathbb{Z}) \to C_*(\widetilde{X},\mathbb{Z})$ induced by $\tilde{f}^n$ on the cellular chains of $\widetilde{X}$. Let $\tau_n : \mathbb{Z}[\pi] \to \mathbb{Z}(\mathcal{R}(\phi^n))$ be defined by extending linearly the function that for each $\alpha \in \pi$ is given by $\tau_n(\alpha) = [\alpha]^n$. Then $R(f^n, \tilde{f}^n)$ is defined to be $\sum_q (-1)^q \tau_n(\text{trace}(\tilde{f}^n_q)) \in \mathbb{Z}(\mathcal{R}(\phi^n))$. When $R(f^n, \tilde{f}^n)$ has been reduced so that each Reidemeister class appears at most once (no easy task, as the reader will see in Example 4), then the coefficient of each Reidemeister class is its index. Thus, in this reduced Reidemeister trace, the essential classes are exactly the classes with nonzero coefficient, and $N(f^n)$ is the number of such terms.

Let F be the free group $\langle a, b, c, d\rangle$. The Fox calculus (see [3]), provides a partial derivative function from $\mathbb{Z}[F]$ to $\mathbb{Z}[F]$ for each generator of π. Once a partial derivative is calculated, the result is immediately interpreted as an element of $\mathbb{Z}[\pi]$. Let $x_1 = a$, $x_2 = b$, $x_3 = c$, and $x_4 = d$. The Fox derivatives are defined by

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}, \quad \frac{\partial 1}{\partial x_j} = 0, \quad \text{and}$$

$$\frac{\partial uv}{\partial x_j} = \frac{\partial u}{\partial x_j} + u\frac{\partial v}{\partial x_j}, \quad \text{for } u, v \in F.$$

These definitions imply that for any $w \in F$, $\partial w^{-1}/\partial x_j = -w^{-1}\partial w/\partial x_j$. Let $\phi_F : F \to F$ be a homomorphism that for each $i = 1, \ldots, 4$ takes x_i to a word in F from the coset determined by $\phi(x_i)$. Many different homomorphisms ϕ_F will induce a given ϕ. Different choices of ϕ_F give different Reidemeister traces but the same Nielsen number. In this paper, having chosen ϕ_F for ϕ we will always use the iterates $(\phi_F)^n$ (or just ϕ_F^n) in our study of f^n. Fadell and Husseini prove in [5] that, for $f : T^2\#T^2 \to T^2\#T^2$,

$$R(f^n, \tilde{f}^n) = \tau_n\left(1 - \frac{\partial\phi_F^n(a)}{\partial a} - \frac{\partial\phi_F^n(b)}{\partial b} - \frac{\partial\phi_F^n(c)}{\partial c} - \frac{\partial\phi_F^n(d)}{\partial d} + A_n\right),$$

where A_n is the contribution to $R(f^n, \tilde{f}^n)$ due to the trace of $\tilde{f}_2^n$. An algorithm developed in [4] can be used to calculate A_n. The algorithm involves writing $\phi_F^n(R)$ in the form $\prod_{i=1}^r y_i R^{\lambda_i} y_i{}^{-1}$ where $r \in \mathbb{Z}^+$, $\lambda_i \in \mathbb{Z}$, and $y_i \in F$ for each i. Then $A_n = \sum_i \lambda_i y_i \in \mathbb{Z}[\pi]$. Even though this expression for $\phi_F^n(R)$ is not unique, A_n is uniquely determined [5].

When ϕ is an automorphism, the element $\phi_F(R)$ must be a conjugate of R or R^{-1} (see p. 49 of [20]), which implies that A_n, for every n, will be a monomial in $\mathbb{Z}[\pi]$. When ϕ is not an automorphism, there is always a choice for ϕ_F for which $\phi_F(R) = 1$ so $A_1 = 0$ (see [20]), and hence $A_n = 0$ for every n when we use ϕ_F^n. The latter situation occurs in our first three examples.

3. Techniques and examples

As we have mentioned, the key difficulty in using the Reidemeister trace for calculations of ordinary Nielsen numbers is in the simplification of the sums in $\mathbb{Z}(\mathcal{R}(\phi))$. That is, we must know how to decide whether two Reidemeister classes (expressed in terms of different representatives) are equal. As we will see from the examples in this paper, abelianization can play a significant role in this process. Let $s_a, s_b s_c, s_d : F \to \mathbb{Z}$ denote the exponent sum homomorphisms for a, b, c and d, respectively. Since every such homomorphism sends R to zero, these induce homomorphisms (with the same names) from π to $\mathbb{Z}$. We define

$\mathrm{Ab}:\pi \to \mathbb{Z}^4$ by $\mathrm{Ab}(\alpha) = \bar{\alpha} = (s_a(\alpha), s_b(\alpha), s_c(\alpha), s_d(\alpha))$. The idea of abelianization is illustrated by the following commutative diagram:

$$\begin{array}{ccc} \langle a,b,c,d:R\rangle & \xrightarrow{\ \phi\ } & \langle a,b,c,d:R\rangle \\ \Big\downarrow \mathrm{Ab} & & \Big\downarrow \mathrm{Ab} \\ \mathbb{Z}^4 & \xrightarrow{\ \bar{\phi}\ } & \mathbb{Z}^4 \end{array} .$$

The homomorphism $\bar{\phi}$ is represented by the 4×4 matrix formed from abelianizing ϕ (e.g., $\bar{\phi}_{2,3} = s_b(\phi(c))$). With $w \in \pi$ we will use $\overline{w}$ to denote $\mathrm{Ab}(w)$. Two elements $[w]^1, [z]^1 \in \mathcal{R}(\phi)$ are then distinct if the cosets $\overline{w} + (\bar{\phi} - I)(\mathbb{Z}^4)$ and $\bar{z} + (\bar{\phi} - I)(\mathbb{Z}^4)$ are distinct. Unfortunately, abelianization can never be used to prove that two Reidemeister classes are equal. If we are lucky and all the summands in $R(f, \tilde{f})$ project to distinct cosets, then this is not a concern.

Our first three examples involve endomorphisms on π of the form $\phi(a) = \phi(d)$ and $\phi(b) = \phi(c)$. Then the natural choice for ϕ_F gives $\phi_F^n(R) = 1$ for all n so that $A_n = 0$ for all n as mentioned at the end of Section 2. This does not occur in Example 4 where a discussion of the A_n has been relegated to the proof of Proposition 3.6.

Example 1. We begin our survey of what can go wrong for the periodic numbers with an example that, among other things, has $N\Phi_n(f) \neq \sum_{m|n} NP_m(f)$. Those readers who are acquainted with other examples of this in the literature, such as maps on a wedge of spheres (e.g., see Example 3.1 in [10]), may be surprised to find the same inequality in this more natural setting of closed manifolds. This example also shows the failure of several other of the properties listed in Section 2.3.

Suppose, on $T^2 \# T^2$ with $\pi = \langle a,b,c,d : R\rangle$, that $\phi:\pi \to \pi$ is given by $\phi(a) = \phi(d) = ab^{-1}c$, $\phi(b) = \phi(c) = b^{-2}$. Let ϕ_F be the homomorphism on F that has the same definition as that just given for ϕ on π.

We have $A_n = 0$ for all n as above, and

$$R(f, \tilde{f}) = \tau_1\bigl(1 - 1 + b^{-1} + b^{-2}\bigr) = [b^{-1}]^1 + [b^{-2}]^1.$$

These two classes are distinct and are both distinct from $[1]^1$ by abelianization, and thus $[1]^1$ is not essential. To aid the reader we will outline why this is true. Now

$$\bar{\phi} - I = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & -3 & -2 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \bar{b}^i = \begin{bmatrix} 0 \\ i \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \bar{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The statement amounts to observing that

$$\{\bar{b}^{-1} - \bar{b}^{-2}, \bar{1} - \bar{b}^{-1}, \bar{1} - \bar{b}^{-2}\} = \{(0,1,0,0)^T, (0,2,0,0)^T\}.$$

While the elements of this set do belong to $(\bar{\phi} - I)(\mathbb{R}^4)$, they are not contained in $(\bar{\phi} - I)(\mathbb{Z}^4)$.

At level 2 we have $\phi^2(a) = \phi^2(d) = ab^{-1}c$, $\phi^2(b) = \phi^2(c) = b^4$. Thus

$$R(f^2, \tilde{f}^2) = \tau_2\big(1 - 1 - 1 - b - b^2 - b^3\big) = -[1]^2 - [b]^2 - [b^2]^2 - [b^3]^2.$$

Note that $[b^3]^2 = [\phi^2(b)b^{-1}]^2 = [1]^2$. Using the same technique of abelianization we can see that the reduced form of the Reidemeister trace at level 2 is

$$R(f^2, \tilde{f}^2) = -2[1]^2 - [b]^2 - [b^2]^2.$$

We note that $[\iota_{1,2}(1)]^2 = [1]^2$ so that the essential class $[1]^2$ reduces to the inessential class $[1]^1$. Thus $T^2\#T^2$ does not have essential reducibility. Since $N(f) = 2 \neq 0$ and $[1]^1$ is inessential, f is not weakly Jiang. We also note that $[\iota_{1,2}(b^{-1})]^2 = [b]^2$, and $[\iota_{1,2}(b^{-2})]^2 = [b^2]^2$. A minimal set of 2-representatives is $\{[1]^1, [b^{-1}]^1, [b^{-2}]^1\}$, so $N\Phi_2(f) = 3$. The summation formula from (P1) in Section 2.3 fails since $N\Phi_2(f) = 3 \neq NP_1(f) + NP_2(f) = 2 + 0$. We do get that $N\Phi_2(f) = 3 = N(f^2)$.

As we have seen in Example 1, it is important for several reasons to be able to distinguish Reidemeister classes. This is required not only to reduce the Reidemeister trace but also to determine the lengths of the orbits of the classes. In addition, we need to know whether a class is reducible in order to determine its depth. Since essentiality, length, and depth are properties of orbits (see [10]), it is instrumental in these calculations to recognize how the terms in the reduced form of $R(f^n, \tilde{f}^n)$ combine into orbits. We would like now to indicate the role played by abelianization in this process. First, some new definitions are required.

Definition 3.1. Suppose that X is a finite polyhedron with $\pi_1(X)$ having abelianization $\overline{\pi_1(X)} \cong \mathbb{Z}^r$. Let $\phi : \pi_1(X) \to \pi_1(X)$ be a homomorphism and $[\alpha]^n \in \mathcal{R}(\phi^n)$. We say that $[\alpha]^n$ is *abelian reducible* to level $m|n$ provided that there is a $\vec{v} \in \mathbb{Z}^r$ such that for $\bar{\iota}_{m,n} = I + \bar{\phi}^m + \bar{\phi}^{2m} + \cdots + \bar{\phi}^{n-m}$ we have that $\bar{\iota}_{m,n}(\vec{v}) = \bar{\alpha}$. The *abelian depth* of $[\alpha]^n$ is the smallest $d|n$ such that $[\alpha]^n$ abelian reduces to level d. The *abelian length* of $[\alpha]^n$ is the smallest $l|n$ such that $(\bar{\phi}^l - I)\bar{\alpha} \in (\bar{\phi}^n - I)(\mathbb{Z}^r)$.

We note that, just as with ordinary length and depth, the notions of abelian length and abelian depth are well defined on orbits. Linear algebra and two fundamental inequalities, abelian length $\leqslant$ (ordinary) length and abelian depth $\leqslant$ (ordinary) depth, give the next result. The ideas of the proof are the same as those for the theory of n-toral maps in [10].

Theorem 3.2. *Assume the notation and setup of Definition* 3.1. *If n is such that* $\det(\bar{\phi}^n - I) \neq 0$, *then the abelian length of any class $[\alpha]^n$ is the same as its abelian depth. In particular, if for all $m|n$ with $m < n$ we know that $\bar{\iota}_{m,n}^{-1}(\bar{\alpha}) \notin \mathbb{Z}^r$, then $[\alpha]^n$ is* (*ordinary*) *irreducible and has* (*ordinary*) *length equal to* (*ordinary*) *depth equal to n.*

Proof. Since for each $m|n$ we have $(\bar{\iota}_{m,n})(\bar{\phi}^m - I) = \bar{\phi}^n - I$, it follows that $\bar{\iota}_{m,n}$ is invertible over $\mathbb{R}$. So the statement that $\bar{\iota}_{m,n}^{-1}(\bar{\alpha}) \notin \mathbb{Z}^r$ is equivalent to saying that $[\alpha]^n$ is abelian irreducible (i.e., of abelian depth n). By the inequality mentioned above this means that the (ordinary) depth of $[\alpha]^n$ must also be n. Of course, the length of any class at level n is never more than n. Thus once we show that the abelian length of $[\alpha]^n$ is also n we will have by the other inequality that the (ordinary) length of $[\alpha]^n$ is n.

So suppose then that there is an $l < n$ and a $\vec{v} \in \mathbb{Z}^r$ for which $(\bar{\phi}^l - I)(\bar{\alpha}) = (\bar{\phi}^n - I)(\vec{v})$. Applying $\bar{\iota}_{l,n}$ to both sides of this equation, noting that $\bar{\iota}_{l,n}(\bar{\phi}^n - I) = (\bar{\phi}^n - I)\bar{\iota}_{l,n}$, and then canceling $\bar{\phi}^n - I$ from both sides gives that $\vec{v}$ is an abelian reduction of $[\alpha]^n$ to level l. This contradicts the fact that the abelian depth of $[\alpha]^n$ is n. $\Box$

These abelianization techniques will be used in Examples 3 and 4. Theorem 3.2 gives a large number of cases where ordinary length and depth will be equal. However, as Example 3 shows, this is not true in general for essential classes.

Example 2. In this example the boosting functions are not injective on essential classes and, as a result, some essential classes do not have unique roots.

Suppose, on $T^2\#T^2$ with $\pi = \langle a, b, c, d : R\rangle$, that $\phi : \pi \to \pi$ is given by $\phi(a) = \phi(d) = a^{-1}$, $\phi(b) = \phi(c) = b^{-1}$. As before, we let ϕ_F have the same definition.

We have $A_n = 0$ for all n as on p. 11, last paragraph of Section 2, and

$$R(f, \tilde{f}) = [1]^1 + [a^{-1}]^1 + [b^{-1}]^1.$$

These three classes are distinct by abelianization. (In this case, because $\bar{\phi} - I$ is invertible over $\mathbb{R}$, we can simply check that $(\bar{\phi} - I)^{-1}$ applied to the abelianized difference of any two of the above classes does not produce a vector in $\mathbb{Z}^4$.)

At level 2 we have $\phi^2(a) = \phi^2(d) = a$, and $\phi^2(b) = \phi^2(c) = b$. Thus

$$R(f^2, \tilde{f}^2) = \tau_2(1 - 1 - 1) = -[1]^2.$$

Note that $[\iota_{1,2}(1)]^2 = [\iota_{1,2}(a^{-1})]^2 = [\iota_{1,2}(b^{-1})]^2 = [1]^2$ so that $[1]^1$, $[a^{-1}]^1$, and $[b^{-1}]^1$ all act as essential roots for $[1]^2$. Thus $T^2\#T^2$ does not have unique roots and $\iota_{1,2}$ is not injective on essential classes. As in Example 1, a minimal set of 2-representatives consists of the three essential classes at level 1, and hence $N\Phi_2(f) = 3$. While $N\Phi_2(f) \neq N(f^2) = 1$, we do have that $N\Phi_2(f) = 0 + 3 = NP_2(f) + NP_1(f)$.

Example 3. For our last counter example on $T^2\#T^2$ we present a situation in which length does not equal depth for essential classes. This emphasizes the importance of counting essential orbits rather than just essential classes when computing $NP_n(f)$ and $N\Phi_n(f)$.

Suppose, on $T^2\#T^2$ with $\pi = \langle a, b, c, d : R\rangle$, that $\phi : \pi \to \pi$ is given by $\phi(a) = \phi(d) = b^{-2}$ and $\phi(b) = \phi(c) = b^{-1}a^{-1}$, with ϕ_F defined as usual. Then $\phi^2(a) = \phi^2(d) = abab$ and $\phi^2(b) = \phi^2(c) = ab^3$. As before, $A_n = 0$ for all n. We see that $R(f, \tilde{f}) = [1]^1 + [b^{-1}]^1$ and $R(f^2, \tilde{f}^2) = -2[ab]^2 - [a]^2 - [ab^2]^2$. Both at level 2 and at level 1, the classes appearing in the Reidemeister trace have coefficients that are either all positive or all negative. Thus in this case we do not need to distinguish classes to know that they are all essential. The fact that $\phi^2(a)\phi(b) = \phi^2(b)\phi(a) = ab$ implies, by applying ϕ, that $\phi(ab) = \phi^3(a)\phi^2(b) = \phi^3(a)\phi^2(b)\phi(a)(\phi(a))^{-1} = \phi^3(a)ab(\phi(a))^{-1}$. Thus $[ab]^2 = \phi([ab]^2)$. Hence the essential class $[ab]^2$ has length one. We now use the abelianization techniques developed in Theorem 3.2 to prove that $[ab]^2$ has depth two.

The abelianization of ϕ is given by the matrix

$$\bar{\phi} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -2 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To determine whether $[ab]^2$ is abelian reducible to level 1, we consider the image of the matrix

$$\bar{\iota}_{1,2} = \bar{\phi} + I = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We note that $\bar{\iota}_{1,2}$ is invertible over $\mathbb{R}$ and in fact, since

$$\bar{\iota}_{1,2}{}^{-1}(\overline{ab}) = (-1/2, -3/2, 0, 0)^T$$

is not in $\mathbb{Z}^4$, we know that $[ab]^2$ is irreducible and thus has depth two even though its length is one.

To calculate $N\Phi_2(f)$ we need to know more. Note that all of the classes appearing in $R(f, \tilde{f})$ and $R(f^2, \tilde{f}^2)$ can be shown to be distinct by abelianizing. The class $[a]^2$ reduces to $[b^{-1}]^1$ because $\phi(b^{-1})b^{-1} = a$. We must also determine whether $[ab^2]^2$ reduces to level 1. Since $\bar{\iota}_{1,2}{}^{-1}(\overline{ab^2}) = (-1, -2, 0, 0)^T$ belongs to $\mathbb{Z}^4$ we cannot use abelianization to determine this. It was no problem for the computer algebra system Magma (see [1]) to find a reduction and show us that $\iota_{1,2}([b^{-1}a^{-1}b^{-1}]^1) = [ab^2]^2$. Since $[b^{-1}a^{-1}b^{-1}]^1 = [\phi(b)b^{-1}]^1 = [1]^1$ we know that in fact $\iota_{1,2}([1]^1) = [ab^2]^2$. Thus $\{[1]^1, [b^{-1}]^1, [ab]^2\}$ is a minimal set of 2-representatives and hence $N\Phi_2(f) = 4$. At level 2, the only essential irreducible orbit is $\langle [ab]^2 \rangle = \{[ab]^2\}$ so that $NP_2(f) = 2$. Because $NP_1(f) = N(f) = 2$ we do have that $N\Phi_2(f) = NP_1(f) + NP_2(f)$ although $N\Phi_2(f) \neq N(f^2) = 3$.

The extension to manifolds of higher dimension. The properties given in Section 2 that hold for periodic classes on nil and solvmanifolds do not hold for manifolds in general. We have provided counterexamples in dimension 2 and will now extend these to every dimension greater than 3. We thank Robert F. Brown for the discussions that led to these extensions.

Let $X^{m+2} = (T^2 \# T^2) \times S^m$ with $m \geqslant 2$. For each of the three counterexamples given above, with f a self-map on $T^2 \# T^2$, we define a self-map on X^{m+2} given by $h = f \times g$, where $g : S^m \to S^m$ is a map of degree 2. The canonical isomorphism

$$\eta : \pi_1(T^2 \# T^2) \to \pi_1\big((T^2 \# T^2) \times S^m\big) \cong \pi_1(T^2 \# T^2) \times \{0\}$$

makes the following diagram commute:

$$\begin{array}{ccc} \pi_1(T^2\#T^2) & \xrightarrow{\eta} & \pi_1(T^2\#T^2)\times\pi_1(S^m) \\ {\scriptstyle f_\#}\downarrow & & \downarrow{\scriptstyle h_\#} \\ \pi_1(T^2\#T^2) & \xrightarrow{\eta} & \pi_1(T^2\#T^2)\times\pi_1(S^m) \end{array}.$$

Now η induces a bijective correspondence between the Reidemeister classes of f and those of h. Because S^m is simply connected and $L(g^n) = 1 + (-1)^m 2^n \neq 0$ for every n, we know that each g^n has exactly one essential Nielsen class. Thus by the product theorem for the index (see [2, p. 60]) we see that η respects Nielsen classes and their essentiality. Furthermore, the diagram remains commutative if $f_\#$ and $h_\#$ are replaced by their corresponding boosting functions $\iota_{m,n}$. Therefore the Nielsen periodic class structure for $\{h^n\}_{n=1}^\infty$ will be identical to that of $\{f^n\}_{n=1}^\infty$, and any of the basic properties that do not hold in Examples 1–3 above will not hold for these corresponding product maps.

Example 4. We now close the paper with a lengthy example which shows that, in some cases at least, the techniques of [8] do remain valid and computations are possible.

The previous examples demonstrate situations where the useful formulas and reasoning for calculating $NP_n(f)$ and $N\Phi_n(f)$ which are valid for nil and solvmanifolds cannot be applied. This makes the computation of $NP_n(f)$ and $N\Phi_n(f)$ difficult in the general case. Despite this, we would now like to present an example where the computation of these numbers is possible for every n. In fact, for this example we do get that $N(f^n) = N\Phi_n(f) = \sum_{m|n} NP_m(f)$ where $N(f^n) = L_n - ((-1)^n + 1)$. (Here $\{L_n\}_{n=1}^\infty$ is the sequence of Lucas numbers where $L_1 = 1$, $L_2 = 3$, and for $r > 2$, $L_r = L_{r-1} + L_{r-2}$.) Möbius inversion then gives that

$$NP_n(f) = \sum_{\tau \subset P(n)} (-1)^{|\tau|} N(f^{n:\tau}),$$

where $P(n)$ is the collection of prime divisors of n and $n:\tau$ is $n/(\prod_{p\in\tau} p)$. For the main results of this example the reader is referred to Propositions 3.6 (which uses Definition 3.4) and 3.7.

Suppose, on $T^2\#T^2$ with $\pi = \langle a, b, c, d : R\rangle$, that $\phi : \pi \to \pi$ is the automorphism given by $\phi(a) = b^{-1}a^{-1}$, $\phi(b) = ab^2$, $\phi(c) = d$, $\phi(d) = c$. It is known that any automorphism on π is induced by a self-homeomorphism on $T^2\#T^2$. In [15] it is proven that for any self-homeomorphism of a closed surface $N(f) = \min\{|\operatorname{fix}(g)|:\ g \sim f\}$. Since not all homotopies of f^n can necessarily be realized by homotopies of f, this does not prove that $NP_n(f)$ and $N\Phi_n(f)$ are equal to their respective minimum numbers. However, since these periodic numbers are based on the the counting of essential classes at various levels, there is strong reason to believe that these lower bounds are sharp.

We list the results here and provide proofs at the end of this section. The following properties of this automorphism allow us to determine the structure of the essential algebraic periodic orbits.

Claim 3.3 (Example 4). *For any* $n \geqslant 3$,

(1) $\phi^n(a) = b^{-1}\phi(a)\phi^2(a)\phi^3(a)\cdots\phi^{n-2}(a)$,
(2) $\phi^n(a) = \phi^{n-1}(a)\phi^{n-2}(a)$,
(3) $\phi^n(b) = \phi^{n+2}(a^{-1})$, *and*
(4) $\tau_n\left(\frac{\partial\phi^n(b)}{\partial b}\right) = -\tau_n\left(\frac{\partial\phi^{n+2}(a)}{\partial b}b\right)$.

Definition 3.4 (Example 4). For $n \geqslant 1$, let $S_n, T_n, U_n \subset \pi$ be given by the following.

$$T_n = \bigl\{\phi^{v_s}(a)\phi^{v_{s-1}}(a)\cdots\phi^{v_2}(a)\phi^{v_1}(a) \in \pi\colon \text{ for all } i,\ 2 \leqslant v_i \leqslant n+1,\\ v_i - v_{i-1} \geqslant 2,\ \text{ and }\ v_1 \neq 2 \text{ whenever } v_s = n+1\bigr\}.$$

For n even,

$$U_n = \left\{\phi^{v_s}(a)\cdots\phi^{v_1}(a)\colon\ s = \frac{n}{2},\ v_i = v_{i-1}+2,\ \text{for } i > 1,\ v_1 \in \{2,3\}\right\}.$$

Note that $T_1 = \emptyset$, $T_2 = U_2 = \{\phi^2(a), \phi^3(a)\}$, and in general $U_n \subset T_n$ when n is even. Let S_n be given by

$$S_n = \begin{cases} T_n - U_n & \text{for } n \text{ even},\\ T_n & \text{for } n \text{ odd}.\end{cases}$$

This implies that $S_1 = S_2 = \emptyset$.

We will represent the product $\phi^{v_s}(a)\phi^{v_{s-1}}(a)\cdots\phi^{v_2}(a)\phi^{v_1}(a) \in \pi$ by the vector $(v_s, v_{s-1}, \ldots, v_2, v_1)$. For example, for $n = 6$ we have

$$T_6 = \bigl\{(2),(3),(4),(5),(6),(7),(4,2),(5,2),(6,2),(5,3),\\ (6,3),(7,3),(6,4),(7,4),(7,5),(6,4,2),(7,5,3)\bigr\}$$

and $U_6 = \{(6,4,2),(7,5,3)\}$. For n even, U_n will always have exactly two elements.

Since the only relation for π is R, which involves both c and d, the subgroup of π generated by $\{a,b\}$ is free.[5] Furthermore, since $\phi^n(\langle a,b\rangle)$ belongs to $\langle a,b\rangle$, we need only consider a 2×2 matrix when determining the abelianizations of $\phi^n(a)$ and $\phi^n(b)$. We note that the abelianization of $\left(\phi|_{\langle a,b\rangle}\right)^n$, which we denote by $\overline{\phi}^n$, is the matrix

$$\overline{\phi}^n = \begin{bmatrix} -F_{n-2} & F_n \\ -F_n & F_{n+2}\end{bmatrix},$$

where F_n is the nth Fibonacci number. (That is, $F_{-1} = 1$, $F_0 = 0$, $F_1 = 1$, and $F_n + F_{n+1} = F_{n+2}$. Thus $2F_n - F_{n-2} = F_n + F_{n-1} + F_{n-2} - F_{n-2} = F_{n+1}$.)

We claim that two vector representations of elements in T_n give distinct elements of π if and only if the two vectors are not equal. To see this we note that the abelianization of $\phi^{v_s}(a)\cdots\phi^{v_1}(a) \in T_n$ is $\bar{a}^\alpha\bar{b}^\beta$ where $\alpha = -\sum_{i=1}^s F_{v_i-2}$ and $\beta = -\sum_{i=1}^s F_{v_i}$. The claim follows from the theory of Zeckendorf decompositions (see p. 281 of [7] for an English

[5] This is the Freiheitssatz, see [20, p. 104].

summary and [19] and [23] for the original papers), which states that each natural number is uniquely represented by a Fibonacci sum of the form $\sum_{i=1}^{s} F_{v_i}$ with $\vec{v} = (v_s, \ldots, v_1)$ such that $v_1 \geqslant 2$ and $v_{i+1} - v_i \geqslant 2$ for $1 \leqslant i < s$.

Proposition 3.5 (Example 4). *For all n, $|S_n| + 1 = L_n - (1 + (-1)^n)$, where $L_n = F_{n+2} - F_{n-2}$ is the nth Lucas number.*

Proposition 3.6 (Example 4). *For any map f that induces ϕ and for any $n \geqslant 1$,*

(1) *The Reidemeister trace can be written as*

$$R(f^n; \tilde{f}^n) = \begin{cases} -3[1]^n - \sum_{s \in S_n} [s]^n & \text{for } n \text{ even}, \\ -[1]^n - \sum_{s \in S_n} [s]^n & \text{for } n \text{ odd}. \end{cases}$$

(2) *The Lefschetz number is $L(f^n) = -L_n$.*
(3) *The expression for $R(f^n, \tilde{f}^n)$ given in* (1) *above is reduced.*
(4) *The Nielsen number is $N(f^n) = L_n - ((-1)^n + 1) = q^n + \overline{q}^n - ((-1)^n + 1)$, where q is the golden ratio $(1 + \sqrt{5})/2$ and $\overline{q}$ is $(1 - \sqrt{5})/2$.*

Proposition 3.7 (Example 4). *For any map f that induces ϕ, length equals depth for all essential classes at any level. The boost $\iota_{m,n}$ maps essential classes injectively to essential classes for all n and $m|n$. Each essential class $[x]^n$ has exactly one essential root.* (*It may also have inessential roots.*) *If $m, k|n$ and $[\alpha]^m$ and $[\beta]^k$ are essential and $\iota_{m,n}([\alpha]^m) = \iota_{k,n}([\beta]^k)$, then both $[\alpha]^m$ and $[\beta]^k$ reduce to a common essential class at level gcd(m, k). While f^n is never weakly Jiang, it is true that $N(f^n) = N\Phi_n(f) = \sum_{m|n} NP_n(f)$ for all n.*

We note that, for this example, Proposition 3.7 provides a proof of Properties (P2) and (P3) from Section 2 as well as the property of being essentially reducible to the GCD. (See Property (P4) in Section 2.) Since we will not be able to gather much information about the reductions that might exist between essential and inessential periodic classes, it will not be possible to prove essential reducibility in general or that every class has a unique root. Despite this, we are still able, with this weaker set of properties, to deduce the usual formulas relating the periodic numbers.

We now provide proofs of the above statements.

Proof of Claim 3.3. From $\phi^3(a) = \phi(b^{-1}) = b^{-2}a^{-1} = b^{-1}\phi(a)$, statement (1) follows by induction. Statement (2) follows from statement (1) by multiplying the formulation for $n - 1$ in statement (1) by $\phi^{n-2}(a)$ on the right. To prove statement (3), we note that $\phi^2(a) = b^{-1}$, so $\phi^n(b) = \phi^n(\phi^2(a^{-1})) = \phi^{n+2}(a^{-1})$.

Statement (4) is proven as follows: For any $n \geqslant 3$,

$$\frac{\partial \phi^n(b)}{\partial b} = \frac{\partial \phi^n(\phi^2(a)^{-1})}{\partial b} = -\phi^n\big(\phi^2(a)^{-1}\big)\frac{\partial \phi^{n+2}(a)}{\partial b}.$$

This follows from statement (3) and the fact that, for any $w \in \langle a, b\rangle$, $\partial w^{-1}/\partial b = -w^{-1}\partial w/\partial b$ (see Section 2.4). We also have, by the Reidemeister action at level n of $\phi^2(a)$ on the above,

$$\tau_n\left(-\phi^{n+2}(a^{-1})\frac{\partial\phi^{n+2}(a)}{\partial b}\right) = -\tau_n\left(\frac{\partial\phi^{n+2}(a)}{\partial b}\phi^2(a^{-1})\right) = -\tau_n\left(\frac{\partial\phi^{n+2}(a)}{\partial b}b\right).$$

This completes the proof of Claim 3.3. □

Proofs of our results are possible because essential algebraic classes have representatives in S_n for which it is easy to see the orbit structure as described in the following two paragraphs. We will now use the elements of S_n and their vector representations interchangeably.

In order to study the structure of orbits of periodic point classes, we interpret the action of ϕ on the Reidemeister classes of elements in S_n by an action γ_n^k on S_n given by

$$\gamma_n^k(v_s, \ldots, v_1) = \begin{cases} (v_s+1, \ldots, v_i+1, \ldots, v_1+1) & \text{if } v_s \leqslant n, \\ (v_{s-1}+1, \ldots, v_{i-1}+1, \ldots, v_1+1,\ 2) & \text{if } v_s = n+1. \end{cases}$$

This is equivalent to first adding one to each coordinate, then reducing mod n any value larger than $n+1$ and rewriting the resulting numbers in decreasing order. To justify this interpretation we next show that for all $x \in S_n$, $[\gamma_n^k(x)]^n = [\phi(x)]^n$. If $v_s \leqslant n$, this is because $\gamma_n^k(x) = \phi(x)$. For $v_s = n+1$ this follows from the fact that $[\phi(x)]^n = [\phi^{n+2}(a)\phi^{v_{s-1}+1}(a)\cdots\phi^{v_1+1}(a)]^n$ which, when $\phi^2(a)^{-1}$ acts on $\phi(x)$ according to the Reidemeister action at level n, gives $[\phi(x)]^n = [\phi^{v_{s-1}+1}(a)\cdots\phi^{v_1+1}(a)\phi^2(a)]^n$.

In terms of the vector representations, if $\vec{v} = (v_s, \ldots, v_1)$ at level m represents $x \in \pi$, then $\iota_{m,n}([x]^m)$ is represented by $(\vec{v} + (n-m)) \circ (\vec{v} + (n-2m)) \circ \cdots \circ (\vec{v} + 2m) \circ (\vec{v} + m) \circ (\vec{v})$. Here $\vec{v} + k = (v_s + k, \ldots, v_i + k, \ldots, v_1 + k)$ and $\circ$ denotes the concatenation of vectors. We observe that $\iota_{m,n}$ will carry elements of S_m directly to elements of S_n.

Let $x \in S_n$. We say that the class $[x]^n$ has *visible length* k for some $k \leqslant n$ if k is the smallest positive integer with $\gamma_n^k(x) = x$. We will shortly prove that visible length and actual length are the same for elements of S_n. Because $\gamma_n^n(x) = x$, the division algorithm implies that $k|n$.

Claim 3.8. *Given $x \in S_n$ and $1 < k$ with $k|n$, $\gamma_n^k(x) = x$ if and only if x is in the set $\iota_{k,n}(S_k)$.*

Proof. Let us first establish some notation and a useful correspondence. For each r and $\vec{w} \in S_r$ there corresponds a unique subset $\mathcal{S}_{\vec{w}}^r$ of $\{0, 1, \ldots, r-1\}$ obtained by reducing each coordinate of $\vec{w}$ modulo r. Via this injective function, γ_r^i acts on $\mathcal{S}_{\vec{w}}^r$ by adding i to all elements of $\mathcal{S}_{\vec{w}}^r$ and reducing modulo r. For $r|s$, $\iota_{r,s}$ acts on $\mathcal{S}_{\vec{w}}^r$ to produce $\mathcal{S}_{\iota(\vec{w})}^s \subset \{0, \ldots, s-1\}$ by producing a union of s/r copies of $\mathcal{S}_{\vec{w}}^r$ with each copy having a different multiple of r (between 0 and $s/r - 1$) added to it and then reducing all elements modulo s.

For example, for $\vec{w} = (5,3) \in S_5$ we have that $\mathcal{S}^5_{\vec{w}} = \{0,3\}$. Also, $\iota_{5,20}(\mathcal{S}^5_{\vec{w}}) = \mathcal{S}^{20}_{\iota(\vec{w})} = \{0,3,5,8,10,13,15,18\}$. This corresponds to the fact that $\gamma^5_{20}\iota_{5,20}(\vec{w}) = (20,18,15,13,10,8,5,3) \in S_{20}$. Then $\gamma^5_{20}(\mathcal{S}^{20}_{\iota(\vec{w})})$ is the subset $\{0,3,5,8,10,13,15,18\}$. Note that $\gamma^5_{20}(\iota_{5,20}(\vec{w})) = \iota_{5,20}(\vec{w})$.

Now we prove the forward implication. Suppose that $\vec{v} \in S_n$. The fact that $\gamma^k_n(\vec{v}) = \vec{v}$ means that $\mathcal{S}^n_{\vec{v}} + k \equiv \mathcal{S}^n_{\vec{v}}$ modulo n. Let β be the smallest element of $\mathcal{S}^n_{\vec{v}}$. Let $\mu \in \mathcal{S}^n_{\vec{v}}$ be the unique element equivalent to $\beta + k$ modulo n. There is a unique $\vec{w} \in S_k$ for which $\mathcal{S}^k_{\vec{w}}$ consists of all elements from $\mathcal{S}^n_{\vec{v}}$ that are less than μ. Thus $\vec{w}$ is such that $\iota_{k,n}(\vec{w}) = \vec{v}$.

In our example, $\gamma^5_{20}(\{0,3,5,8,10,13,15,18\}) = \{0,3,5,8,10,13,15,18\}$. In this case $\beta = 0$, $\mu = 5$, $\mathcal{S}^5_{\vec{w}} = \{0,3\}$, and thus $\vec{w} = (5,3)$ as stated above.

For the reverse direction of the proof we first note that $\iota_{k,n}\gamma_k = \gamma_n\iota_{k,n}$. If $y \in S_k$ then $\gamma^k_k(y) = y$. Thus $\iota_{k,n}(y) = \iota_{k,n}(\gamma^k_k(y)) = \gamma^k_n(\iota_{k,n}(y))$. □

Proof of Proposition 3.5. Because $S_1 = S_2 = \emptyset$ and $L_1 = 1$, $L_2 = 3$, the result follows for $n = 1, 2$.

For $n \geqslant 3$, we first define more useful sets. Let $R_n = \{\phi^{v_s}(a)\cdots\phi^{v_1}(a) \in \pi\colon\ v_1 = 2,\ v_s = n+1, \text{ and } v_i - v_{i-1} \geqslant 2, \forall i\}$, and let $Q_n = R_n \cup T_n$. Note that $R_n \cap T_n = \emptyset$. We remind the reader that with these sets we will switch freely between elements in $S_n \subseteq \pi$ and their representations in vector notation.

Next we prove that $|Q_n| + 1 = F_{n+2}$. (Recall that $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$.) For $n = 3$, $Q_3 = \{(2),(3),(4),(4,2)\}$, so $|Q_3| + 1 = 5 = F_5$. With $n = 4$ we get $Q_4 = Q_3 \cup \{(5),(5,2),(5,3)\}$ and $Q_2 = T_2 = \{(2),(3)\}$. Let $\circ$ be used to denote the concatenation of vectors that occur by themselves or in sets. This gives $Q_4 = Q_3 \cup [(5) \circ Q_2] \cup \{(5)\}$, so $|Q_4| + 1 = |Q_3| + 4 = 8 = F_6$. By induction, for all $n \geqslant 5$, $Q_n = Q_{n-1} \cup [(n+1) \circ Q_{n-2}] \cup \{(n+1)\}$. Thus, $|Q_n| = |Q_{n-1}| + |Q_{n-2}| + 1 = F_{n+1} - 1 + F_n - 1 + 1 = F_{n+2} - 1$. Because

$$R_n = \bigl[(n+1) \circ (Q_{n-4} + 2) \circ (2)\bigr] \cup \bigl\{(n+1, 2)\bigr\},$$

we note that $|R_n| = |Q_{n-4}| + 1 = F_{n-2}$. Thus $|T_n| = |Q_n| - |R_n| = F_{n+2} - 1 - F_{n-2} = L_n - 1$. For n even we have that $|S_n| = |T_n| - |U_n| = |T_n| - 2$ so that $|S_n| + 1 = |T_n| - 1 = L_n - 2$. For n odd, we have $|S_n| + 1 = |T_n| + 1 = L_n$. □

Proof of Proposition 3.6. We first consider the calculation of A_n for all n. We observe that $\phi(aba^{-1}b^{-1}) = bab^{-1}a^{-1}$ and $\phi(bab^{-1}a^{-1}) = aba^{-1}b^{-1}$ and similarly that $\phi(cdc^{-1}d^{-1}) = dcd^{-1}c^{-1}$ and $\phi(dcd^{-1}c^{-1}) = cdc^{-1}d^{-1}$. Thus we can calculate $\phi^n_F(R)$ for all n. For odd n we note that, as elements of the free group F,

$$\phi^n_F(R) = bab^{-1}a^{-1}dcd^{-1}c^{-1} = bab^{-1}a^{-1}R^{-1}(bab^{-1}a^{-1})^{-1},$$

and $A_n = -bab^{-1}a^{-1}$ from [4]. Using the relator R we have

$$[bab^{-1}a^{-1}]^n = \bigl[\phi^n(c^{-1}d^{-1})bab^{-1}a^{-1}(c^{-1}d^{-1})^{-1}\bigr]^n = \bigl[(c^{-1}d^{-1})R^{-1}(dc)\bigr] = [1]^n.$$

Thus $[A_n]^n = -[1]^n$ when n is odd. When n is even, to find A_n we note that $\phi_F^n(R) = R$, so $[A_n]^n = [1]^n$. Thus for all n we have $[A_n]^n = (-1)^n[1]^n$.

Next we prove the entire proposition for $n = 1$ and $n = 2$. Because $\phi(a) = b^{-1}a^{-1}$, $\phi(b) = ab^2$, $\phi(c) = d$, and $\phi(d) = c$, we see that $\phi^2(a) = b^{-1}$, $\phi^2(b) = bab^2$, $\phi^2(c) = c$, and $\phi^2(d) = d$. So

$$R(f, \tilde{f}) = \tau_1(b^{-1}a^{-1} - a - ab) \quad \text{and} \quad R(f^2, \tilde{f}^2) = \tau_2(-1 - ba - bab).$$

We note that $[a]^1 = [\phi(a)]^1 = [b^{-1}a^{-1}]^1$, and that $[ab]^1 = [\phi(b)b^{-1}]^1 = [1]^1$. Thus $R(f, \tilde{f}) = -[1]^1$ which means that $L(f) = -1 = -L_1$ and $N(f) = 1$. Clearly the length and depth of $[1]^1$ are 1, so this proves Proposition 3.6 for $n = 1$. For $n = 2$ we observe that $[1]^2 = [\phi^2(b)b^{-1}]^2 = [bab]^2$, and $[ba]^2 = [\phi^2(a)baa^{-1}]^2 = [1]^2$. We are left with $R(f^2, \tilde{f}^2) = -3[1]^2$. Hence $L(f^2) = -3 = -L_2$ and $N(f^2) = 1$. Because the length and depth of $[1]^2$ are both one, all of the statements in Proposition 3.6 are true for $n = 2$.

We continue with the proof of Proposition 3.6 for $n \geqslant 3$.

(1) We prove that $R(f^n, \tilde{f}^n)$ can be expressed in the given form.

Step 1. We show that

$$R(f^n; \tilde{f}^n) = \tau_n\left(-\frac{\partial\phi^n(a)}{\partial a} + \frac{\partial\phi^{n+2}(a)}{\partial b}b\right).$$

Using the method of [5] and [4] for odd n, by applying (4) of Claim 3.3 we have that

$$\frac{\partial\phi^n(c)}{\partial c} = \frac{\partial\phi^n(d)}{\partial d} = 0.$$

Thus, for odd n,

$$\begin{aligned} R(f^n; \tilde{f}^n) &= \tau_n\left(1 - \frac{\partial\phi^n(a)}{\partial a} - \frac{\partial\phi^n(b)}{\partial b} - \frac{\partial\phi^n(c)}{\partial c} - \frac{\partial\phi^n(d)}{\partial d} + A_n\right) \\ &= \tau_n\left(-\frac{\partial\phi^n(a)}{\partial a} + \frac{\partial\phi^{n+2}(a)}{\partial b}b\right). \end{aligned}$$

This completes Step 1 for n odd. For even n, we have that

$$\frac{\partial\phi^n(c)}{\partial c} = \frac{\partial\phi^n(d)}{\partial d} = 1,$$

so (again from (4) of Claim 3.3)

$$\begin{aligned} R(f^n; \tilde{f}^n) &= \tau_n\left(1 - \frac{\partial\phi^n(a)}{\partial a} - \frac{\partial\phi^n(b)}{\partial b} - 2 + A_n\right) \\ &= \tau_n\left(-1 - \frac{\partial\phi^n(a)}{\partial a} + \frac{\partial\phi^{n+2}(a)}{\partial b}b + A_n\right) \\ &= \tau_n\left(-\frac{\partial\phi^n(a)}{\partial a} + \frac{\partial\phi^{n+2}(a)}{\partial b}b\right). \end{aligned}$$

Thus for all values of n Step 1 is complete.

Step 2. Let Q_n be defined as in the proof of Proposition 3.5. Note that Q_n is then all vectors $(v_s, \ldots, v_1)$ with $v_s \leqslant n+1$, $v_1 \geqslant 2$, and the difference between successive entries at least 2. We next prove that

$$\frac{\partial\phi^{n+2}(a)}{\partial b}b = -1 - \sigma_n, \quad \text{where } \sigma_n = \sum_{q \in Q_n} q.$$

For $n = 1$, because $\phi^3(a) = b^{-2}a^{-1}$, we have $\partial\phi^3(a)/\partial b = -b^{-1} - b^{-2}$. Thus $(\partial\phi^3(a)/\partial b) \cdot b = -1 - b^{-1} = -1 - \phi^2(a)$.

For $n = 2$, we have

$$\begin{aligned}\frac{\partial\phi^4(a)}{\partial b}b &= \frac{\partial b^{-2}a^{-1}b^{-1}}{\partial b}b = (-b^{-1} - b^{-2} - b^{-2}a^{-1}b^{-1})b \\ &= -1 - b^{-1} - b^{-2}a^{-1} = -1 - \phi^2(a) - \phi^3(a) = -1 - \sigma_2.\end{aligned}$$

By induction on n, with base steps $n = 1$ and $n = 2$, we use (2) of Claim 3.3 and the product rule to deduce that

$$\begin{aligned}\frac{\partial\phi^{n+2}(a)}{\partial b}b &= \frac{\partial\phi^{n+1}(a)\phi^n(a)}{\partial b}b = \frac{\partial\phi^{n+1}(a)}{\partial b}b + \phi^{n+1}(a)\frac{\partial\phi^n(a)}{\partial b}b \\ &= -1 - \sigma_{n-1} + \phi^{n+1}(a)(-1 - \sigma_{n-2}).\end{aligned}$$

As in the proof of Proposition 3.5, we have $Q_n = Q_{n-1} \cup [(n+1) \circ Q_{n-2}] \cup \{(n+1)\}$. Thus $\sigma_n = \sigma_{n-1} + \phi^{n+1}(a)\sigma_{n-2} + \phi^{n+1}(a)$. Finally, we see that $-1 - \sigma_n = -1 - (\sigma_{n-1} + \phi^{n+1}(a)\sigma_{n-2} + \phi^{n+1}(a))$, which we have proven is equal to $(\partial\phi^{n+2}(a)/\partial b) \cdot b$. Step 2 is complete.

Step 3. Recall that, as in the proof of Proposition 3.5,

$$R_n = \left\{\phi^{v_s}(a) \cdots \phi^{v_1}(a) \in \pi\colon\ v_1 = 2,\ v_s = n+1 \text{ and } v_i - v_{i-1} \geqslant 2,\ \forall i\right\}.$$

We next prove that for $n \geqslant 3$

$$\tau_n\left(-\frac{\partial\phi^n(a)}{\partial a}\right) = \tau_n\left(\sum_{r \in R_n} r\right).$$

It is helpful to define two more sets. Let

$$V_n = \left\{\phi^{v_s}(a)\phi^{v_{s-1}}(a) \cdots \phi^{v_2}(a)\phi^{v_1}(a) \in Q_n\colon\ v_1 \text{ is odd, } v_s \leqslant n-1\right\}.$$

For odd n we define $W_n = V_n \cup \{\phi^n(a)\}$. For even n let $W_n = V_n$.

We will first prove that

$$-\frac{\partial\phi^n(a)}{\partial a} = \sum_{w \in W_n} w.$$

For $n = 3$,

$$W_3 = \left\{\phi^3(a)\right\}, \quad \text{and} \quad -\frac{\partial\phi^3(a)}{\partial a} = \phi^3(a) = \sum_{w \in W_3} w.$$

Similarly, for $n=4$,

$$W_4=\{\phi^3(a)\} \quad \text{and} \quad -\frac{\partial\phi^4(a)}{\partial a}=\phi^3(a)=\sum_{w\in W_4} w.$$

By induction on n, using (2) of Claim 3.3, we have

$$\begin{aligned}-\frac{\partial\phi^n(a)}{\partial a}&=-\frac{\partial\phi^{n-1}(a)\phi^{n-2}(a)}{\partial a}=-\frac{\partial\phi^{n-1}(a)}{\partial a}-\phi^{n-1}(a)\frac{\partial\phi^{n-2}(a)}{\partial a}\\&=\sum_{w\in W_{n-1}} w+\sum_{w\in W_{n-2}}\phi^{n-1}(a)w=\sum_{w\in W_n} w,\end{aligned}$$

where the fact that $\phi^{n-1}(a)\phi^{n-2}(a)=\phi^n(a)\in W_n$ proves the last equality when n is odd.

To complete Step 3 we will prove that

$$\tau_n\left(\sum_{w\in W_n} w\right)=\tau_n\left(\sum_{r\in R_n} r\right).$$

We will do this by defining a bijection $\psi: R_n\to W_n$ with $[\psi(r)]^n=[r]^n$. For any $r\in R_n$, let $\psi(r)=\phi^n(\phi(a)^{-1})r\phi(a)$. Suppose $r=(n+1,r_{s-1},\ldots,r_2,2)$. Then $\psi(r)$ can be represented by the vector $(r_{s-1},\ldots,r_2,2,1)$. Using (2) of Claim 3.3 repeatedly, the vector collapses on the right until all remaining coordinates differ by at least 2. Note that, after the collapsing, the right-most coordinate will be odd. Also, the left-most coordinate will be at most n when n is odd and will be at most $n-1$ when n is even. It follows that $\psi(r)\in W_n$. To see that ψ is a bijection we note that $\psi^{-1}(w)=\phi^n(\phi(a))w\phi(a)^{-1}$ for all $w\in W_n$. Step 3 is complete.

To complete the proof of Proposition 3.6(1), we note that $T_n\cup R_n=Q_n$, and $T_n\cap R_n=\emptyset$. Thus

$$\sum_{t\in T_n} t=\sum_{q\in Q_n} q-\sum_{r\in R_n} r\in\mathbb{Z}[\pi].$$

Combining the results from Steps 2 and 3 we have

$$\tau_n\left(\sum_{t\in T_n} t\right)=\tau_n\left(-1-\frac{\partial\phi^{n+2}(a)}{\partial b}b\right)+\tau_n\left(\frac{\partial\phi^n(a)}{\partial a}\right).$$

Thus, using the result of Step 1,

$$R(f^n;\tilde f^n)=\tau_n\left(-\frac{\partial\phi^n(a)}{\partial a}+\frac{\partial\phi^{n+2}(a)}{\partial b}b\right)=\tau_n\left(-1-\sum_{t\in T_n} t\right).$$

For n odd, $S_n=T_n$, and the proof of Proposition 3.6(1) is completed. For n even, we must observe that the two elements of U_n are both Reidemeister equivalent at level n to the identity of π. The two elements of U_n are x and $\phi(x)$, where $x=(n,n-2,\ldots,4,2)$. Then, as above at the end of Step 3, by repeated applications of (2) from Claim 3.3, we have $x\phi(a)=\phi^{n+1}(a)$. Thus $[x]^n=[\phi^n(\phi(a)^{-1})x\phi(a)]^n=[1]^n$, and $[\phi(x)]^n=[\phi(1)]^n=[1]^n$. For n even we have $\tau_n\left(-1-\sum_{t\in T_n} t\right)=\tau_n\left(-3-\sum_{s\in S_n} s\right)$, and the proof of Proposition 3.6(1) is complete.

(2) The Lefschetz number $L(f^n)$ is the sum of the coefficients of $R(f^n, \tilde{f}^n)$, so we have

$$L(f^n) = \begin{cases} -|S_n| - 3 & \text{for } n \text{ even,} \\ -|S_n| - 1 & \text{for } n \text{ odd,} \end{cases}$$

which in either case is $-(|S_n| + 1) - (1 + (-1)^n)$. By Proposition 3.5, this means that $L(f^n) = -L_n$.

(3) We prove that the terms in the given expression for $R(f^n, \tilde{f}^n)$ are distinct.

For $n = 1, 2$, we are already done since the only essential class is that of 1.

For $x \in S_n$ let $\nu = \nu(x)$ denote the visible length of x (as defined on p. 19, just before Claim 3.8) and let $\tau = \tau(x)$ be the true length of x. Our goal is to prove that $\nu = \tau$ for all elements of S_n.

The set S_n is partitioned by the action of γ_n into what we will call visible orbits. The visible orbit of x contains exactly ν elements. These elements correspond to ν terms in $R(f^n, \tilde{f}^n)$ as expressed in Proposition 3.6(1). The (true) orbit of $[x]^n$ contains exactly τ elements. Thus the reduced form of $R(f^n, \tilde{f}^n)$ contains $\nu - \tau$ fewer terms than the original.

Jiang and Guo in [15] prove that for any self-homeomorphism g of a compact surface X with negative Euler characteristic $\chi(X)$ and with Lefschetz number $L(g)$, the inequality $|L(g) - \chi(X)| \leqslant N(g) - \chi(X)$ holds. Applying this to $g = f^n$ we have $|L(f^n) + 2| \leqslant N(f^n) + 2$. Thus for $n \geqslant 3$ we have $L_n - 4 \leqslant N(f^n) \leqslant |S_n| + 1$, where the upper bound is the number of terms in the (possibly) unreduced version of $R(f^n, \tilde{f}^n)$. So for n even we have $L_n - 4 \leqslant N(f^n) \leqslant L_n - 2$, and for n odd we have $L_n - 4 \leqslant N(f^n) \leqslant L_n$. Thus for n even, when $R(f^n, \tilde{f}^n)$ is written in reduced form, at most two classes disappear when they are combined with others. Similarly for n odd, when $R(f^n, \tilde{f}^n)$ is written in reduced form at most four classes disappear. We will use these bounds and the fact that reducibility is a property of orbits to prove that in fact no classes disappear.

Because $[x]^n = [\phi^\nu(x)]^n = [\phi^\tau(x)]^n = [\phi^n(x)]^n$, and because τ is minimal, the division algorithm implies that $\tau|\nu$ and $\nu|n$.

Assume $\tau < \nu$, then $\tau \leqslant \nu/2$. If n is even then the number of classes that disappear is $\nu - \tau \leqslant 2$. Thus $\nu \leqslant 4$. (Otherwise, $\nu = \tau$.) Similarly, for n odd, if $\tau < \nu$ then $\nu \leqslant 8$.

For n odd we have shown that for $\nu \geqslant 9$ we must have $\nu = \tau$. Also ν cannot be even because $\nu|n$. Similarly, for n even, we conclude that for $\nu \geqslant 5$ we have $\nu = \tau$.

The remaining values of ν which must be checked are 2, 3, 4 for n even and 3, 5, 7 for n odd. For each ν we must check all divisors α of ν and prove that $\tau \neq \alpha$. We consider only pairs (ν, α) for which the appropriate upper bound for $\nu - \alpha$ (2 for n even and 4 for n odd) is not violated. This eliminates the pairs $\nu = 7$, $\alpha = 1$ and $\nu = 4$, $\alpha = 1$. The remaining pairs (ν, α) to be checked are, for n even, $n_e = \{(2,1), (3,1), (4,2)\}$ and, for n odd, $n_o = \{(3,1), (5,1)\}$.

We must prove that for these remaining pairs (ν, α) an essential class $[x]^n$ with visible length ν does not have length α. This means we must prove that $[x]^n$ is not Reidemeister equivalent to $[\phi^\alpha(x)]^n$. We accomplish this by abelianizing. Let $\xi, \beta \in \mathbb{Z}$, with $\overline{x} = \overline{a}^\xi \overline{b}^\beta$. As in the discussion before Example 2, it is sufficient to prove that when

$$\begin{pmatrix} i \\ j \end{pmatrix} = (\bar{\phi}^n - I)^{-1}(\bar{\phi}^\alpha - I)\begin{pmatrix} \xi \\ \beta \end{pmatrix},$$

then either $i \notin \mathbb{Z}$ or $j \notin \mathbb{Z}$. (It is sufficient to distinguish Reidemeister classes only for elements from $\langle a, b\rangle$. Thus, for this example, we can view $\bar{\phi}$, $\bar{\iota}_{m,n}$ and I as 2×2 rather than 4×4 matrices. This is because all of these matrices are block diagonal so that exponents for $\bar{c}, \bar{d} \in \pi$ do not contribute to the exponents for $\bar{a}, \bar{b} \in \pi$. We observe also here that since the eigenvalues of $\bar{\phi}$ are $(1 \pm \sqrt{5})/2$ which do not have modulus 1, we know that $\bar{\phi}^n - I$ will be invertible for every n.)

The abelianization of $\iota_{\nu,n}$ is the matrix given by

$$\bar{\iota}_{\nu,n} = I + \sum_{i=1}^{n/\nu-1} \bar{\phi}^{i\nu} = (\bar{\phi}^n - I)(\bar{\phi}^\nu - I)^{-1}.$$

By Claim 3.8 we know that there is some $y \in S_\nu$ with $\iota_{\nu,n}(y) = x$. Let $\overline{y} = \overline{a}^{\delta}\overline{b}^{\gamma}$.

Then, from the previous formulation for $\bar{\iota}_{\nu,n}$, we have that there exist $i, j \in \mathbb{Z}$ such that

$$\begin{aligned}\begin{pmatrix} i \\ j \end{pmatrix} &= (\bar{\phi}^n - I)^{-1}(\bar{\phi}^\alpha - I)(\bar{\phi}^n - I)(\bar{\phi}^\nu - I)^{-1}\begin{pmatrix} \delta \\ \gamma \end{pmatrix} \\ &= (\bar{\phi}^\alpha - I)(\bar{\phi}^\nu - I)^{-1}\begin{pmatrix} \delta \\ \gamma \end{pmatrix}.\end{aligned}$$

Because $\alpha|\nu$, we have

$$(\bar{\phi}^\nu - I) = \left(I + \sum_{i=1}^{\nu/\alpha-1} \bar{\phi}^{i\alpha}\right)(\bar{\phi}^\alpha - I).$$

Thus

$$\begin{pmatrix} i \\ j \end{pmatrix} = \left(I + \sum_{i=1}^{\nu/\alpha-1} \bar{\phi}^{i\alpha}\right)^{-1}\begin{pmatrix} \delta \\ \gamma \end{pmatrix}.$$

Note that the last line above is independent of n. For the pairs $(\nu, \alpha) = (2, 1)$, $(3, 1)$, $(4, 2)$, and $(5, 1)$, we check each element y of S_ν (see the discussion before Proposition 3.5 for the representation of y as a column vector) to be sure that either i or j is not an integer. Because $S_2 = \emptyset$, there is nothing to check for the pair $(2, 1)$. The number of elements to be checked is so small that calculating i and j for each case is easy. For each of the 17 cases (one for each element of S_3, S_4, and S_5) we obtained at least one of i and j not in $\mathbb{Z}$.

Therefore the above analysis shows that the actual length of an essential class at level n is equal to its visible length. This implies that within a visible orbit at any level n, all terms represent elements of π which are in distinct classes of $\mathcal{R}(\phi^n)$. The only thing needed then to complete the proof of Proposition 3.6(3) is to show that two elements of S_n from *different* visible orbits are in distinct Reidemeister classes. However, if elements from two different essential visible orbits belong to the same Reidemeister class at level n, then the two orbits are of the same length and every element from one orbit is Reidemeister equivalent to exactly one element from the other orbit. Thus the number of classes that disappear when we reduce $R(f^n, \tilde{f}^n)$ is greater than or equal to the length of those orbits. Using our analysis of the Jiang–Guo inequality made above and the fact that visible length equals ordinary length for the elements of S_n, we thus have that the only orbits for which this can occur have length 1, 2, 3, or 4. From Claim 3.8 such orbits must occur as boosts from levels 1, 2, 3 or 4, respectively, to level n. The only essential irreducible orbits of lengths 1, 3, or 4 are $\langle [1]^1 \rangle$ (length 1), $\langle [\phi^2(a)]^3 \rangle = S_3$ (length 3), and $\langle [\phi^2(a)]^4 \rangle = S_4$ (length 4), respectively. Since $S_2 = \emptyset$ there are no orbits of length 2. Thus since there is only one orbit of each of these types, no two such orbits can combine. The images of $\iota_{k,n}$ for $k = 1, 2, 3, 4$ each contain at most one orbit. Thus all elements of S_n are in distinct Reidemeister classes for each n, and none of these classes are Reidemeister equivalent to 1. Hence the Reidemeister traces are reduced as stated in the proposition.

(4) We prove that $N(f^n) = L_n - ((-1)^n + 1)$.

We know that the given expression of $R(f^n, \tilde{f}^n)$is reduced. So $N(f^n)$ is the number of terms, which is $|S_n| + 1$. From Proposition 3.5 we have that this equals $L_n - ((-1)^n + 1)$. The fact that $L_n = q^n + \bar{q}^n$ is well known in number theory and also follows from the facts that q and $\bar{q}$ are the eigenvalues of $\bar{\phi}$ and $L_n = \text{trace}(\bar{\phi}^n)$ (from the discussion just before Proposition 3.5).

This completes the proof of Proposition 3.6. □

Proof of Proposition 3.7. Recall that τ_n takes $x \in \pi$ to $[x]^n$. Along with the class of 1, which always goes to itself under boosting, our vector representation of the other essential classes at any level $m|n$ shows us that $\iota_{m,n}$ is always injective on essential classes. In other words, since (from Proposition 3.6 parts (1) and (3)) $\iota_{m,n} : \tau_m(S_m) \cup \{[1]^m\} \to \tau_n(S_n) \cup \{[1]^n\}$ is injective and all classes in $\tau_n(S_n) \cup \{[1]^n\}$ are essential and distinct, we have that $\iota_{m,n}$ is injective on essential classes and always sends essential classes to essential classes.

We prove next that length equals depth for all essential classes at every level n. Because no two elements of S_n are Reidemeister equivalent, the length of $[x]^n$ for any $x \in S_n$ equals the visible length of $[x]^n$. Combining this with Claim 3.8 and Lemma 2.4, we see that $[x]^n$ has length ℓ with $1 \leqslant \ell \leqslant n$ iff $[x]^n$ reduces essentially to level ℓ and does not reduce essentially to any level $k < \ell$. The fact that $[x]^n$ then cannot reduce to any class (essential or not) below level ℓ follows since length is always less than or equal to depth. For the only case not covered in Claim 3.8 we note that $[1]^n$ has length 1 and reduces to level 1.

From the fact that length equals depth on essential classes and from the proof of Proposition 3.6(3), the visible length of any $[x]^n$ for $x \in S_n$ equals the depth d of $[x]^n$. From Claim 3.8 this implies that $x \in \iota_{d,n}(S_d)$ as is needed to imply that $[x]^n$ reduces essentially to level d. Since $[x]^n$ cannot reduce to any level below its depth, the existence of a unique essential root is proven as follows. There are no other essential roots at any level because such roots would have depth and length equal to the length of $[x]^n$ and thus would have to reduce essentially to level d. There is exactly one essential root at level d since $\iota_{d,n}$ is injective on essential classes.

Suppose that $[y]^n$ is essential and reduces to both $[\alpha]^m$ and $[\beta]^k$, with $[\alpha]^m$ and $[\beta]^k$ both essential classes. Let $[\delta]^d$ be the unique essential root of $[y]^n$. Then the length of $[y]^n$ is d. The injectivity of the boosting functions on essential classes tells us by Lemma 2.4 that length is preserved under boosting of essential classes and thus $[\delta]^d$, $[\beta]^k$, $[\alpha]^m$, and $[y]^n$ all have length d. Thus d divides each of m, k and n. Hence d will also divide

$\ell = \gcd(m, k)$. The facts that $\iota_{k,n}\iota_{d,k} = \iota_{d,n}$ and $\iota_{m,n}\iota_{d,m} = \iota_{d,n}$ imply that $[\beta]^k$ and $[\alpha]^m$ also have $[\delta]^d$ as their unique essential root. Let $[\gamma]^\ell = \iota_{d,\ell}([\delta]^d)$. Since $[\delta]^d$ is essential so is $[\gamma]^\ell$. As above the facts that $\iota_{\ell,m}\iota_{d,\ell} = \iota_{d,m}$ and $\iota_{\ell,k}\iota_{d,\ell} = \iota_{d,k}$ imply that $[\gamma]^\ell$ is a common reduction of both $[\beta]^k$ and $[\alpha]^m$ as desired.

To see that no f^n is weakly Jiang we note first that $N(f^n)$ is never zero. Also, every essential class contains an $x \in \langle a, b\rangle$. For any such x abelianization quickly shows us that $[x]^n \neq [c]^n$. Thus $[c]^n$ is an inessential class and every $\mathcal{R}(\phi^n)$ contains both essential and inessential classes.

As in [8], the ability to reduce all essential classes to their essential roots and the fact that length equals depth for essential classes implies that the orbits of all essential irreducible classes at any level $m|n$ provides a minimal set of n-representatives with $N\Phi_n(f) = \sum_{m|n} NP_m(f)$. That $N\Phi_n(f) = N(f^n)$ follows from the property that essential classes boost to essential classes, which can be called essential boostability. In [8] this was a consequence of the fact that whenever f^n is weakly Jiang and $N(f^n) \neq 0$ and we have essential reducibility, then all classes at level n are essential and hence so is anything that boosts to level n. Here we do not prove essential reducibility but are able to prove essential boostability by different means. The various ι give a natural bijection between the unique roots of essential classes at level n and the essential classes at level n themselves so that $N(f^n) = N\Phi_n(f)$ as needed. (End of Example 4.) □

Corrections to [4]. We take this opportunity to note that, in Example 4 of [4], we can now determine the Nielsen number of the given homeomorphisms that induce $\phi(a) = c^{-n+1}d^{-1}$, $\phi(b) = dc^n$, $\phi(c) = a$, $\phi(d) = b$, where $n \geqslant 2$. Here n does not indicate a number of iterations. The Reidemeister trace at level 1 is $[1] + [bab^{-1}a^{-1}]$, so we knew that $N(f) = 1$ or 2. Now we can use the inequality from [15] as above ($|L(f) + 2| \leqslant N(f) + 2$) to see that there cannot be exactly one essential class if that class has index two. Thus the Reidemeister trace is reduced as stated and $N(f) = 2$. Jiang and Guo also prove in [15] that for self-homeomorphisms on surfaces of negative Euler characteristic there are no essential classes with index greater than 1. See Kelly's work in [16] for related results. In Example 3 of [4], the Reidemeister trace is printed incorrectly. It should be

$$\rho\left(-1 - \frac{\partial a^{-n+1}}{\partial a}\right) = \rho\left(-1 + \sum_{i=1}^{n-1} a^{-i}\right).$$

4. Conclusion

We have exhibited a homeomorphism (Example 4), in which not all of the properties listed in Section 2.3 can be proven. Despite this, the basic equalities $N\Phi_n(f) = N(f^n) = \sum_{m|n} NP_m(f)$ were shown to hold. The analysis used to prove this capitalized on a rich combinatorial structure of essential algebraic orbits that may be present in a similar form for other self mappings. While we have shown that the equalities above will not be true for maps in general on the double torus, there exists the possibility that the following conjecture is true.

Conjecture 4.1. If $f : M \to M$ is a homeomorphism on a surface M of nonpositive Euler characteristic, then $N\Phi_n(f) = \sum_{n|m} NP_n(f)$ and when $N(f^n) \neq 0$ then $N\Phi_n(f) = N(f^n)$. Furthermore, each essential class $[x]^n$ has exactly one essential root $r([x]^n)$ (and perhaps also inessential roots). For a fixed n, the set $\{r([x]^n)\colon [x]^n \text{is essential}\}$ is a minimal set of n-representatives for computing $N\Phi_n(f)$.

If the conjecture is not true in general, then one can ask for conditions on M and f that do make it valid. For example, the case of a pseudo-Anosov homeomorphism on a surface, discussed in [15], is an easy example where the conjecture holds. [6] This is the case because for a pseudo-Anosov homeomorphism f one has that $N(f^n) = |\operatorname{fix}(f^n)|$ for all n. Thus every fixed point of f^n is essential and every geometric Nielsen class of f^n is a singleton. Each periodic point with minimal period k is an *essential* fixed point for f^{mk} for all $m \in \mathbb{N}$. It is immediate that f has essential reducibility, injective boost functions, length = depth, and that every essential class has a unique essential root. Furthermore, for such homeomorphisms, $N\Phi_n(f)$ and $NP_n(f)$ are equal to their respective minimum numbers rather than being merely lower bounds. Using Thurston's classification of surface homeomorphisms and the Jiang–Guo [15] representatives in each isotopy class, it may be possible to do this in general on surfaces. However, given an essential periodic class $[\alpha]^n$, knowing that the length of $[\alpha]^n$ equals the depth of $[\alpha]^n$ does not allow us to complete the computation until we know what these values are. Thus the computation of the ordinary Nielsen numbers $N(f^n)$ will always remain important in the computation of $N\Phi_n(f)$ and $NP_n(f)$. For this reason, while the theory of train tracks for surface homeomorphisms should certainly be a part of such calculations, we feel that our algebraic approach also warrants consideration, not only for homeomorphisms but also when the map may not induce an automorphism. As we have seen, therein lie some very interesting combinatorics which we hope, someday, can be more completely understood.

There are many opportunities for new tools to be applied to the calculation of the Nielsen periodic numbers. Geometric facts, like the Jiang–Guo inequalities (in Example 4) (see [15]) or various results that provide bounds for the total index of a Nielsen class (see [16]), are extremely useful. We have shown that the techniques of abelianization are very powerful. Additionally, subgroups other than the commutator should be considered in this regard. For fundamental groups whose abelianizations have torsion, results could also be developed. For a given f, the graph whose nodes are the Nielsen classes and whose edges are determined by the boosting functions is an important object of study. Understanding the structure of this graph, including the action of f on the graph, is crucial when the properties and formulae of Section 2 do not hold.

[6] We note that the map in Example 4, since it is reducible, is not of this type.

Acknowledgments

We would like to thank Robert F. Brown and Michael R. Kelly for useful discussions. Helpful conversations with John Guaschi demonstrated the use of train tracks to analyze our Example 4 and its generalizations to other homeomorphisms from [4]. This has made us keenly aware that such techniques should play a major role in any further development of the ideas presented here when the self map is a homeomorphism. See the conclusion for further comments on this. The referee's detailed comments contributed to the exposition and were very much appreciated. Both authors would like to thank both spouses (Adger Williams and Jackie Keppelmann) for their generous patience, hospitality, and fine cooking.

References

[1] W. Bosma, J.J. Cannon and G. Mathews, Programming with algebraic structures: Design of the Magma language, in: M. Giesbrecht, ed., Proceedings of the 1994 International Symposium on Symbolic and Algebraic Computation, Oxford, July 20–22, 1994 (ACM, New York, 1994) 52–57.

[2] R.F. Brown, The Lefschetz Fixed Point Theorem (Scott, Foresman and Co., 1971).

[3] R. Crowell and R. Fox, An Introduction to Knot Theory (Ginn and Co., 1963).

[4] O. Davey, E. Hart and K. Trapp, Computation of Nielsen numbers for maps of closed surfaces, Trans. Amer. Math. Soc. 348 (1996) 3245–3266.

[5] E. Fadell and S. Husseini, The Nielsen number on surfaces, Contemp. Math. 21 (1983) 59–98.

[6] R. Geoghegan, Fixed points in finitely dominated compacta: the geometric meaning of a conjecture of H. Bass, in: Shape Theory and Geometric Topology (Dubrovnik, 1981) 6–22; Lecture Notes in Math. 870 (Springer, Berlin, 1981).

[7] R. Graham, D. Knuth and O. Patashnik, Concrete Mathematics: A Foundation for Computer Science (Addison-Wesley, Reading, MA, 1989).

[8] P. Heath and E. Keppelmann, Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds I, Topology Appl. 76 (1997) 217–247.

[9] P. Heath and E. Keppelmann, Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds II, Topology Appl., to appear.

[10] P. Heath, R. Piccinini and C. You, Nielsen type numbers for periodic points I, in: B. Jiang, ed., Topological Fixed Point Theory and Applications, Lecture Notes in Math. 1411 (Springer, Berlin, 1989) 88–106.

[11] P. Heath and C. You, Nielsen type numbers for periodic points II, Topology Appl. 43 (1992) 219–236.

[12] V.E. Hoggatt, Fibonacci and Lucas Numbers (Houghton Mifflin Co., 1969).

[13] S. Husseini, Generalized Lefschetz numbers, Trans. Amer. Math. Soc. 272 (1982) 247–274.

[14] B. Jiang, Lectures on Nielsen Fixed Point Theory, Contemp. Math. 14 (Amer. Math Soc., Providence, RI, 1983).

[15] B. Jiang and J. Guo, Fixed points of surface diffeomorphisms, Pacific J. Math. 160 (1993) 67–89.

[16] M. Kelly, A bound on the fixed-point index for surface mappings, Preprint.

[17] E. Keppelmann and C. McCord, The Anosov theorem for exponential solvmanifolds, Pacific J. Math. 170 (1995) 143–159.

[18] T. Kiang, The Theory of Fixed Point Classes (Springer, Berlin, 1980).

[19] C. Lekkerkerker, Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci, Simon Stevin 29 (1952) 190–195.
[20] R. Lyndon and P. Schupp, Combinatorial Group Theory (Springer, Berlin, 1977).
[21] C. McCord, Nielsen numbers and Lefschetz numbers on solvmanifolds, Pacific J. Math. 147 (1991) 153–164.
[22] A.H. Rarivoson, Calculation of the Nielsen numbers on nilmanifolds and solvmanifolds, Ph.D. dissertation, University of Cincinnati (1993).
[23] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, Bull. de la Société Royale des Sciences de Liège 41 (1972) 179–182.