Positive solutions for third-order Sturm–Liouville boundary value problems with $p$-Laplacian

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A B S T R A C T

In this paper, by using the fixed point index method, we establish the existence of at least one or at least two positive solutions for the third-order Sturm–Liouville boundary value problem with $p$-Laplacian

$$
\begin{align*}
(\phi_p(u''(t)))' + f(t, u(t)) &= 0, \quad t \in (0, 1), \\
\alpha u(0) - \beta u'(0) &= 0, \\
\gamma u(1) + \delta u'(1) &= 0, \\
u''(0) &= 0,
\end{align*}
$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta, \gamma, \delta \geq 0$.

1. Introduction

The purpose of this paper is to study the existence of positive solutions for the following third-order Sturm–Liouville boundary value problem with $p$-Laplacian

$$
(\phi_p(u''(t)))' + f(t, u(t)) = 0, \quad t \in (0, 1),
$$

$$
\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \quad u''(0) = 0,
$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta, \gamma, \delta \geq 0$.

The equation with $p$-Laplacian operator arises in the modeling of different physical and natural phenomena, non-Newtonian mechanics [1,2], combustion theory [3], population biology [4,5], nonlinear flow laws [6–8] and the system of Monge–Kantorovich partial differential equations [9]. During the past decades, wide attention has been paid to the study the equation with $p$-Laplacian operator, and there exist a very large number of papers devoted to the existence of positive solutions of the $p$-Laplacian operator. The second-order problem

$$
(\phi_p(u'(t)))' + f(t, u(t)) = 0, \quad t \in (0, 1),
$$

with various boundary conditions has been studied by many authors; see [10–17] and the references therein. However, to our knowledge, few papers have been reported on the existence of positive solutions for the third-order Sturm–Liouville boundary value problem (1.1) and (1.2). Motivated by the works [18,19], in this paper we will show the existence of at least one or at least two positive solutions for the problem (1.1) and (1.2), assuming that

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E-mail address: yangchen0809@126.com (C. Yang).
(A1)  \( \rho := \gamma \beta + \alpha \gamma + \alpha \delta > 0, \)  \( 0 < \sigma := \min \left\{ \frac{4\delta + \gamma}{4(\delta + \gamma)}, \frac{\alpha + 4\beta}{4(\alpha + \beta)} \right\} < 1. \)

(A2)  \( f \in C([0, 1] \times [0, \infty); [0, \infty]) \).

By a positive solution of (1.1) and (1.2) we understand a function \( u(t) \) which is positive on \( 0 < t < 1 \) and satisfies the differential equation (1.1) and the boundary conditions (1.2).

In this paper we will work in the Banach space \( C[0, 1] \) and only the sup-norm is used. Set

\[
\min_{t \to 0^+} f(t, u) := \lim_{\substack{t \to 0^+ \\\ m \to -\infty \\ t \in [0,1]}} u^{-\frac{p-1}{m}}, \quad \max_{t \to +\infty} f(t, u) := \lim_{\substack{t \to +\infty \\ t \in [0,1]}} u^{-\frac{p-1}{m}}.
\]

Let \( G(t, s) \) be the Green function of the differential equation \( u''(t) = 0, \ t \in (0, 1) \) with respect to the boundary condition (1.2), then

\[
G(t, s) = \begin{cases} \frac{1}{\rho} \left( (\gamma + \delta - \gamma t)(\beta + \alpha s) \right), & 0 \leq s \leq t \leq 1. \\ \frac{1}{\rho} \left( (\beta + \alpha t)(\gamma + \delta - \gamma s) \right), & 0 \leq t \leq s \leq 1. \end{cases} \tag{1.3}
\]

Evidently, \( G(t, s) \geq 0, 0 \leq t, s \leq 1. \)

The main results of this paper are as follows.

**Theorem 1.1.** Assume that (A1), (A2) hold. In addition, there exist constants \( b, c > 0 \) such that \( b < \min \left\{ \frac{\rho}{\alpha \delta}, \sigma \right\} \cdot c \) implies

(B1)  \( f(t, u) \geq (lb)^{p-1} \) for \( \frac{1}{4} \leq t \leq \frac{3}{4}, \) \( b \leq u \leq \frac{b}{\sigma }; \)

(B2)  \( f(t, u) \leq (mc)^{p-1} \) for \( 0 \leq t \leq 1, \) \( 0 \leq u \leq c, \)

where

\[
l = \frac{2}{\sigma 4^{1-q}} \left( \int_{\frac{1}{4}}^{\frac{3}{4}} G \left( \frac{1}{2}, s \right) \, ds \right)^{-1} = \frac{2}{\sigma 4^{1-q}} \cdot \frac{32\rho}{3\alpha \gamma + 7\alpha \delta + 7\beta \gamma + 16\beta \delta}, \tag{1.4}
\]

\[
m = \left( \int_{0}^{1} G(s, s) \, ds \right)^{-1} = \frac{6\rho}{\alpha \gamma + 3\alpha \delta + 3\beta \gamma + 6\beta \delta}. \tag{1.5}
\]

Then the problem (1.1) and (1.2) has at least one positive solution \( u^* \) with \( \|u^*\| \leq c \) and \( \min_{t \in [\frac{1}{4}, \frac{3}{4}]} u^*(t) > b. \)

**Theorem 2.2.** Assume (A1), (A2) hold. If the following assumptions are satisfied:

(B3)  \( \min f_0 = \max f_\infty = \infty; \)

(B4)  there exists a constant \( \rho_1 > 0 \) such that

\[
f(t, u) < (m\rho_1)^{p-1}, \quad t \in [0, 1], \ u \in [0, \rho_1], \text{ where } m \text{ is given as in (1.5).}
\]

Then the problem (1.1) and (1.2) has at least two positive solutions \( u_1 \) and \( u_2 \) such that \( 0 < \|u_1\| < \rho_1 < \|u_2\|. \)

**Theorem 3.3.** Assume (A1), (A2) hold. If the following assumptions are satisfied:

(B5)  \( \min f_0 = \max f_\infty = 0; \)

(B6)  there exists a constant \( \rho_2 > 0 \) such that

\[
f(t, u) > (l\rho_2)^{p-1}, \quad t \in \left[ \frac{1}{4}, \frac{3}{4} \right], \ u \in [\sigma \rho_2, \rho_2], \text{ where } l \text{ is given as in (1.4).}
\]

Then the problem (1.1) and (1.2) has at least two positive solutions \( u_1 \) and \( u_2 \) such that \( 0 < \|u_1\| < \rho_2 < \|u_2\|. \)

Our approach is based on the properties of fixed point index.

2. Preliminaries

In this section we summarize some basic concepts and results.

**Definition 2.1** (See [20]). Let \( E \) be a real Banach space. Let \( P \) be a nonempty, convex closed set in \( E \). We say that \( P \) is a cone if it satisfies the following properties:

(i)  \( \lambda u \in P \) for \( u \in P, \lambda \geq 0; \)

(ii)  \( -u \in P \) implies \( u = \theta (\theta \) denotes the null element of \( E \)).

If \( P \subset E \) is a cone, we denote the order induced by \( P \) on \( E \) by \( \leq \). For \( u, v \in P \), we write \( u \leq v \) if and only if \( v - u \in P \). Define the convex sets \( P_r, \ P_r(r > 0) \) by \( P_r = \{y \in P| \|y\| < r\}, P_r = \{y \in P| \|y\| \leq r\}. \)
We require some knowledge of the classical fixed point index for compact maps; see for example [20–22] for further information. The index has the following properties.

**Lemma 2.2.** Let $K$ be a closed convex set in a Banach space $E$ and let $D$ be a bounded open set such that $D_k := D \cap K \neq \emptyset$. Let $T : D_k \to K$ be a compact map. Suppose that $x \neq Tx$ for all $x \in \partial D_k$.

1. (Existence) If $i(T, D_k, K) \neq 0$, then $T$ has a fixed point in $D_k$.
2. (Normalization) If $u \in D_k$, then $i(T, D_k, K) = 1$, where $i(x) = u$ for $x \in D_k$.
3. (Homotopy) Let $\mu : [0, 1] \times \bar{D}_k \to K$ be a compact map such that $x \neq \mu(t, x)$ for $x \in \partial D_k$ and $t \in [0, 1]$. Then $i(\mu, 0, D_k, K) = i(\mu, 1, D_k, K)$.

**Lemma 2.3** (See [21,22]). Let $P$ be a cone in a Banach space $E$. For $q > 0$, define $\Omega_q = \{ x \in P \mid \|x\| < q \}$. Assume that $T : \bar{\Omega}_q \rightarrow P$ is a compact map such that $x \neq Tx$ for $x \in \partial \Omega_q$.

1. If $\|x\| \leq \|Tx\|$ for $x \in \partial \Omega_q$, then $i(T, \Omega_q, P) = 0$.
2. If $\|x\| \geq \|Tx\|$ for $x \in \partial \Omega_q$, then $i(T, \Omega_q, P) = 1$.

In the following, we can always assume that $\Omega_q = C[0, 1]$ and $P = \{ x \in E \mid x(t) \geq 0, t \in [0, 1] \}$. Define an operator $T$ by

$$ (Tu)(t) = \int_0^1 G(t, v)\phi_q \left( \int_0^v f(s, u(s))ds \right) dv, \quad \forall u \in P. \quad (2.1) $$

From (A1) and (A2), we can easily get $(Tu)(t) \geq 0$, $t \in [0, 1]$ for $u \in P$.

**Remark 2.4.** Suppose that $u \in P$ is a solution of the operator equation $Tu = u$, then

$$ u'(t) = -\rho \int_0^t (\beta + \gamma v)\phi_q \left( \int_0^v f(s, u(s))ds \right) dv + \frac{\alpha}{\rho} \int_t^1 (\gamma + \delta - \gamma v)\phi_q \left( \int_0^v f(s, u(s))ds \right) dv, $$

$$ u''(t) = (Tu)''(t) = -\phi_q \left( \int_0^t f(s, u(s))ds \right). $$

So $\phi_p(u''(t)) = -\int_0^t f(s, u(s))ds$, and in consequence, $(\phi_p(u''(t)))' = -f(t, u(t))$. Moreover,

$$ \alpha u(0) - \beta u'(0) = \alpha \int_0^1 G(0, v)\phi_q \left( \int_0^v f(s, u(s))ds \right) dv - \beta \cdot \int_0^1 (\gamma + \delta - \gamma v)\phi_q \left( \int_0^v f(s, u(s))ds \right) dv = 0; $$

$$ \gamma u(1) + \delta u'(1) = \gamma \int_0^1 G(1, v)\phi_q \left( \int_0^v f(s, u(s))ds \right) dv + \left( \frac{\gamma}{\rho} \right) \cdot \int_0^1 (\beta + \alpha v)\phi_q \left( \int_0^v f(s, u(s))ds \right) dv = 0. $$

Further, $u''(0) = 0$. That is to say, all the fixed points of operator $T$ are the solutions for the problem (1.1) and (1.2).

**Lemma 2.5** (See [23]). Let $G(t, s)$ be given as in (1.3), then we have the following results:

$$ \frac{G(t, s)}{G(s, s)} \leq 1 \quad \text{for } t \in [0, 1] \text{ and } s \in [0, 1], $$

$$ \frac{G(t, s)}{G(s, s)} \geq \sigma \quad \text{for } t \in \left[ \frac{1}{4}, \frac{3}{4} \right] \text{ and } s \in [0, 1]. $$

### 3. Proofs of main theorems

We are now in a position to prove the existence Theorems 1.1–1.3 stated in the introduction. Some ideas in the proofs are taken from our paper [19]. Also let $E = C[0, 1]$ and $P = \{ x \in E \mid x(t) \geq 0, t \in [0, 1] \}$. We define an operator $T : P \to P$ by (2.1), then from Remark 2.4 we know that $T : P \to P$.

**Proof of Theorem 1.1.** We divide the proof into several steps.

**Step 1.** We will show that operator $T : P \to P$ is completely continuous. Let $\Omega \subset P$ be a bounded set. Then there exists $K > 0$ such that $\Omega \subset \{ u \in P \mid \|u\| \leq K \}$. Set $M = \max\{f(t, u) \mid t \in [0, 1], u \in \Omega \}$. For any $u \in \Omega$,
\[(Tu)(t) = \int_0^1 G(t, \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \leq \int_0^1 G(v, \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \]

\[
\leq \int_0^1 G(v, \upsilon) \phi_\upsilon(M) \, d\upsilon = \phi_\upsilon(M) \int_0^1 G(v, \upsilon) \, d\upsilon,
\]

which implies that \(T(\Omega)\) is uniformly bounded. Further, for any \(u \in \Omega\) and \(t \in [0, 1]\) we have

\[
|\{(Tu)'(t)\}| = \left| \frac{\gamma}{\rho} \int_0^t (\beta + \alpha \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon + \frac{\alpha}{\rho} \int_t^1 (\gamma + \delta - \gamma \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \right|
\]

\[
\leq \frac{\gamma}{\rho} \int_0^t (\beta + \alpha \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon + \frac{\alpha}{\rho} \int_t^1 (\gamma + \delta - \gamma \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon
\]

\[
\leq \phi_\upsilon(M) \left[ \frac{\gamma}{\rho} \int_0^t (\beta + \alpha \upsilon) \phi_\upsilon \, d\upsilon + \frac{\alpha}{\rho} \int_t^1 (\gamma + \delta - \gamma \upsilon) \phi_\upsilon \right] = \phi_\upsilon(M) t \leq \phi_\upsilon(M).
\]

Hence \(\|\{(Tu)'\}\| \leq \phi_\upsilon(M)\). For any \(0 \leq t_1 \leq t_2 \leq 1\) and \(u \in \Omega\),

\[
|Tu(t_1) - Tu(t_2)| = \int_{t_1}^{t_2} (Tu)'(t) \, dt \leq \int_{t_1}^{t_2} \|\{(Tu)'\}\| \, dt \leq \phi_\upsilon(M) |t_2 - t_1|.
\]

So we can easily prove that \(T(\Omega)\) is equi-continuous. In view of the continuity of \(f\) and the Lebesgue dominated convergence theorem, we know that \(T\) is continuous on \(\Omega\). Thus, the Arzela–Ascoli theorem implies that \(T : P \rightarrow P\) is completely continuous.

Step 2. For any \(u \in P\) and \(t \in [0, 1]\), from Lemma 2.5 we have

\[
Tu(t) = \int_0^1 G(t, \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \leq \int_0^1 G(v, \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon.
\]

Consequently,

\[
\|Tu\| \leq \int_0^1 G(v, \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon.
\]

Further, for \(u \in P\) and \(t \in [\frac{1}{4}, \frac{3}{4}]\), from Lemma 2.5 we obtain

\[
\min_{t \in [\frac{1}{4}, \frac{3}{4}]} Tu(t) = \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_0^1 G(t, \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \\
\geq \sigma \int_0^1 G(v, \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \geq \sigma \|Tu\|.
\]

If \(u \in \tilde{P}_\sigma = \{u \in P \mid \|u\| \leq c\}\), then \(0 \leq u(t) \leq c\) for \(t \in [0, 1]\). From (B2), we have

\[
\|Tu\| = \max_{t \in [0, 1]} |Tu(t)| = \max_{t \in [0, 1]} \int_0^1 G(t, \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \\
\leq \max_{t \in [0, 1]} \int_0^1 G(t, \upsilon) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \leq \max_{t \in [0, 1]} \int_0^1 G(t, \upsilon) \phi_\upsilon (mc)^{p-1} \, d\upsilon \\
= mc \cdot \max_{t \in [0, 1]} \int_0^1 G(t, \upsilon) \, d\upsilon = mc \int_0^1 G(v, \upsilon) \, d\upsilon = c.
\]

Hence, \(T : \tilde{P}_\sigma \rightarrow \tilde{P}_\sigma\) and

\[
\min_{t \in [\frac{1}{4}, \frac{3}{4}]} Tu(t) \geq \sigma \|Tu\|, \quad \forall u \in \tilde{P}_\sigma.
\] (3.1)

Next, for \(b \leq u(t) \leq \frac{b}{\sigma}, t \in [\frac{1}{4}, \frac{3}{4}]\), by (B1) we know that \(f(t, u(t)) \geq (lb)^{p-1}\) for \(t \in [\frac{1}{4}, \frac{3}{4}]\) and then we have

\[
Tu \left( \frac{1}{2} \right) = \int_0^1 G \left( \frac{1}{2}, \upsilon \right) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \geq \int_{\frac{1}{4}}^{\frac{3}{4}} G \left( \frac{1}{2}, \upsilon \right) \phi_\upsilon \left( \int_0^\upsilon f(s, u(s)) \, ds \right) \, d\upsilon \\
\geq \int_{\frac{1}{4}}^{\frac{3}{4}} G \left( \frac{1}{2}, \upsilon \right) lb \upsilon^{q-1} \, d\upsilon \geq \left( \frac{1}{4} \right)^{q-1} lb \int_{\frac{1}{4}}^{\frac{3}{4}} G \left( \frac{1}{2}, \upsilon \right) \, d\upsilon = \frac{2b}{\sigma} > \frac{b}{\sigma}.
\]
Consequently, \[
\min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} Tu(t) \geq \sigma \|u\| > \sigma \times \frac{b}{\sigma} = b \quad \text{for } b \leq u(t) \leq \frac{b}{\sigma}, \, t \in \left[\frac{1}{4}, \frac{3}{4}\right].
\] (3.2)

Step 3. Now we can conclude from Lemma 2.2 and (3.1), (3.2) that \( T \) has at least one nonzero fixed point. In fact, let \( U = \{x \in \bar{P}_{c} \mid \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x(t) > b\} \). Evidently, \( U \) is a nonempty bounded, convex open set for \( \frac{b}{\sigma} \in U \) in \( \bar{P}_{c} \).

Firstly, we prove \( Tx \neq x \) for \( x \in \partial U \). Suppose that there is \( x_0 \in \partial U \) such that \( Tx_0 = x_0 \), then we have \( \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x_0(t) = b \) and either (i) \( x_0 \in \{x \in \bar{P}_{c} \mid \|x\| \leq \frac{b}{\sigma}, \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x(t) \geq b\} \) or (ii) \( \|x_0\| > \frac{b}{\sigma} \).

For case (i), we know that \( b \leq \|x_0\| \leq \frac{b}{\sigma} \), thus, we have by (3.2)
\[
b = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x_0(t) = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} Tx_0(t) > b.
\]

This is a contradiction. For case (ii), we have by (3.1)
\[
b = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x_0(t) = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} Tx_0(t) \geq \sigma \|Tx_0\| = \sigma \|x_0\| > \sigma \times \frac{b}{\sigma} > b.
\]

This is a contradiction. Hence, \( Tx \neq x \) for \( x \in \partial U \). So \( i(T, U, \bar{P}_{c}) \) is meaningful.

Secondly, we take \( u_0 \in P \) such that \( \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u_0(t) > b \), \( \|u_0\| \leq \frac{b}{\sigma} \). Set \( \mu(t, x) = Tu_0 + (1 - t)Tx \), then \( \mu : [0, 1] \times \bar{U} \rightarrow \bar{P}_{c} \) is completely continuous. Suppose that there is \( (t_0, x_0) \in [0, 1] \times \partial U \) such that \( \mu(t_0, x_0) = x_0 \), then \( \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x_0(t) = b \).

We distinguish two cases: (1) If \( \|x_0\| > \frac{b}{\sigma} \), then by (3.1), we have \( \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} Tx_0(t) \geq \sigma \|Tx_0\| > b \). Consequently, \( b = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x_0(t) = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} [t_0u_0(t) + (1 - t_0)Tx_0(t)] \geq \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} t_0u_0(t) + \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} (1 - t_0)Tx_0(t) > t_0b + (1 - t_0)b = b \).

This is a contradiction.

(2) If \( \|x_0\| \leq \frac{b}{\sigma} \), then we obtain
\[
\|x_0\| = \|t_0u_0 + (1 - t_0)Tx_0\| \leq t_0\|u_0\| + (1 - t_0)\|Tx_0\| \\
\leq t_0 \frac{b}{\sigma} + (1 - t_0) \frac{b}{\sigma} = \frac{b}{\sigma}.
\]

That is, \( b = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x_0(t) \leq \|x_0\| \leq \frac{b}{\sigma} \), thus, we have by (3.2)
\[
b = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x_0(t) = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} [t_0u_0(t) + (1 - t_0)Tx_0(t)] \\
\geq \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} t_0u_0(t) + \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} (1 - t_0)Tx_0(t) > t_0b + (1 - t_0)b = b.
\]

This is a contradiction. So we have \( \mu(t, x) \neq x \) for \( (t, x) \in [0, 1] \times \partial U \). Finally, applying (D2) and (D3) in Lemma 2.2, we get \( i(T, U, \bar{P}_{c}) = i(u_0, U, \bar{P}_{c}) = 1 \).

Thus, (D4) in Lemma 2.2 shows that \( T \) has a fixed point \( x^* \in U \). Further, \( \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} x^*(t) > b \), \( \|x^*\| \leq c \). That is, \( x^* \) is a nonzero fixed point of \( T \) in \( \bar{P}_{c} \). That is to say, the third-order Sturm–Liouville boundary value problem (1.1) and (1.2) has at least one positive solution. \( \square \)

**Proof of Theorem 1.2.** Define \( P_1 = \{u|u \in P, \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u(t) \geq \sigma \|u\|\} \), where \( \sigma \) is given as in (A4). It is clear that \( P_1 \) is also a cone. From the proof of Theorem 1.1, we get \( T : P_1 \rightarrow P_1 \) is completely continuous.

To begin with, in view of \( \min_{t \in [0, 1]} f(t, u) \geq (\delta_1 u)^{p-1} \) for \( t \in [0, 1] \) and \( 0 < u \leq r_1 \), where \( \delta_1 \in (\frac{1}{4}, +\infty) \).

Let \( \Omega_{r_1} = \{u|u \in P_1, \|u\| < r_1\} \). Then for any \( u \in \partial \Omega_{r_1} \), by using the same calculation in the proof of Theorem 1.1, we have
\[
Tu \left( \frac{1}{2} \right) = \int_{0}^{1} G \left( \frac{1}{2}, v \right) \phi_1 \left( \int_{0}^{v} f(s, u(s))ds \right) dv \geq \int_{0}^{\frac{1}{4}} G \left( \frac{1}{2}, v \right) \phi_1 \left( \int_{0}^{v} f(s, u(s))ds \right) dv \\
\geq \int_{0}^{\frac{1}{4}} G \left( \frac{1}{2}, v \right) \phi_1 \left( \int_{0}^{v} (\delta_1 u(s))^{p-1}ds \right) dv \geq \int_{0}^{\frac{1}{4}} G \left( \frac{1}{2}, v \right) \phi_1 \left( \int_{0}^{\frac{1}{4}} (\delta_1 u)^{p-1}ds \right) dv
\]
\[ \geq \left( \frac{1}{4} \right)^{q-1} \delta_1 \sigma \int_0^{1/4} G \left( \frac{1}{2}, v \right) dv \| u \| = \frac{2\delta_1}{l} \| u \| > \| u \| \]

which implies \( \| Tu \| > \| u \| \) for \( u \in \partial \Omega_{r_1} \). Thus, Lemma 2.3 implies
\[ i(T, \Omega_{r_1}, P_1) = 0. \tag{3.3} \]

Next, since \( \max f_\infty = \infty \), there exists \( R > \rho_1 \) such that \( f(t, u) \geq (\delta_2 u)^{p-1} \) for \( u \geq R \), where \( \delta_2 \in \left( \frac{1}{2}, +\infty \right) \).

Let \( r_2 > \max \{ \frac{K}{\sigma}, \rho_1 \} \) and set
\[ \Omega_{r_2} = \{ u \in P_1 \mid \| u \| < r_2 \}. \]

If \( u \in \partial \Omega_{r_2} \), then \( \min_{t \in [1/4, 1]} u(t) \geq \sigma \| u \| > R \). Therefore, for any \( u \in \partial \Omega_{r_2} \), by using the method to get (3.3), we then get
\[ Tu \left( \frac{1}{2} \right) > \frac{2\delta_2}{l} \| u \| > \| u \| , \]

which implies \( \| Tu \| > \| u \| \) for \( u \in \partial \Omega_{r_2} \). Hence, Lemma 2.3 shows that
\[ i(T, \Omega_{r_2}, P_1) = 0. \tag{3.4} \]

Finally, let \( \Omega_{\rho_1} = \{ u \in P_1 \mid \| u \| < \rho_1 \} \). Then for \( u \in \partial \Omega_{\rho_1} \), from (B4) we have
\[
Tu(t) = \int_0^1 G(t, s) \phi_q \left( \int_0^s f(s, u(s)) ds \right) dv \leq \int_0^1 G(v, v) \phi_q \left( \int_0^v f(s, u(s)) ds \right) dv \\
< m \rho_1 \int_0^1 G(v, v) dv = \rho_1 = \| u \|.
\]

which implies \( \| Tu \| < \| u \| \) for \( u \in \partial \Omega_{\rho_1} \). An application of Lemma 2.3 again shows that

\[ i(T, \Omega_{\rho_1}, P_1) = 1. \tag{3.5} \]

Note that \( r_1 < \rho_1 < r_2 \): it follows from the additivity of fixed point index and (3.3)-(3.5) that

\[ i(T, \Omega_{\rho_1} \setminus \Omega_{r_1}, P_1) = i(T, \Omega_{\rho_1}, P_1) - i(T, \Omega_{r_1}, P_1) = 1 \]

and

\[ i(T, \Omega_{r_2} \setminus \Omega_{\rho_1}, P_1) = i(T, \Omega_{r_2}, P_1) - i(T, \Omega_{\rho_1}, P_1) = -1. \]

Consequently, \( T \) has a fixed point \( u_1 \) in \( \Omega_{\rho_1} \setminus \Omega_{r_1} \), and a fixed point \( u_2 \) in \( \Omega_{r_2} \setminus \Omega_{\rho_1} \). Both are positive solutions of the problem (1.1) and (1.2) and \( 0 < \| u_1 \| < \rho_1 < \| u_2 \| \). The proof is therefore complete. \( \square \)

**Proof of Theorem 1.3.** Firstly, since \( \min f_0 = 0 \), there exists \( r_1 \in (0, \rho_2) \) such that \( f(t, u) \leq (\tau_1 u)^{p-1} \) for \( t \in [0, 1] \) and \( 0 < u < r_1 \), where \( \tau_1 \in (0, m) \).

Let \( \Omega_{r_1} = \{ u \in P_1 \mid \| u \| < r_1 \} \). Then for any \( u \in \partial \Omega_{r_1} \), by using the same calculation in the proof of Theorem 1.2, we have
\[
Tu(t) = \int_0^1 G(t, s) \phi_q \left( \int_0^s f(s, u(s)) ds \right) dv \leq \int_0^1 G(v, v) \phi_q \left( \int_0^v f(s, u(s)) ds \right) dv \\
\leq \tau_1 \| u \| \int_0^1 G(v, v) dv = \frac{\tau_1}{m} \| u \| < \| u \| ,
\]

which implies \( \| Tu \| < \| u \| \) for \( u \in \partial \Omega_{r_1} \). Therefore, by Lemma 2.3,

\[ i(T, \Omega_{r_1}, P_1) = 1. \tag{3.6} \]

Secondly, in view of \( \max f_\infty = 0 \), there exists \( R > \rho_2 \) such that \( f(t, u) \leq (\tau_2 u)^{p-1} \) for \( t \in [0, 1] \) and \( u \geq R \), where \( \tau_2 \in (0, m) \).

We divide the proof into two cases: \( f \) is bounded and \( f \) is unbounded.

**Case (i).** Suppose that \( f \) is bounded, which implies that there exists \( M > 0 \) such that \( f(t, u) \leq M^{p-1} \) for all \( t \in [0, 1] \) and \( u \in (0, +\infty) \).

Now choose \( r_2 > \max \{ \frac{M}{m}, R \} \) so that for \( u \in P_1 \) with \( \| u \| = r_2 \), we have
\[
Tu(t) = \int_0^1 G(t, s) \phi_q \left( \int_0^s f(s, u(s)) ds \right) dv \leq \int_0^1 G(v, v) \phi_q \left( \int_0^v f(s, u(s)) ds \right) dv \\
\leq M \int_0^1 G(v, v) dv = \frac{M}{m} < r_2 = \| u \| .
\]
Case (ii), suppose that $f$ is unbounded. Then because $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous, we know that there exist $t_0 \in [0, 1]$ and $r_2 > \max\{\rho_2, \frac{\rho_2}{\tau}\}$ such that $f(t, u) \leq f(t_0, r_2)$ for $t \in [0, 1]$ and $0 < u \leq r_2$. Then for $u \in P_1$ with $\|u\| = r_2$, we have

$$
Tu(t) = \int_0^1 G(t, v)\phi_q\left(\int_0^v f(s, u(s))\,ds\right)\,dv \leq \int_0^1 G(v, v)\phi_q\left(\int_0^v f(s, u(s))\,ds\right)\,dv
$$

$$
\leq \int_0^1 G(v, v)\phi_q\left(\int_0^v f(t_0, r_2)\,ds\right)\,dv
$$

$$
\leq \tau_2 r_2 \int_0^1 G(v, v)\,dv = \frac{\tau_2 r_2}{m} < r_2 = \|u\|.
$$

Hence, in either case, we may always set $\Omega_{r_2} = \{u \in P_1 \|u\| < r_2\}$ such that

$$
\|Tu\| < \|u\| \quad \text{for} \quad u \in \partial\Omega_{r_2}.
$$

Thus, Lemma 2.3 implies

$$
i(T, \Omega_{r_2}, P_1) = 1.
$$

(3.7)

Finally, let $\Omega_{p_2} = \{u \in P_1 \|u\| < p_2\}$. Since $u \in \partial\Omega_{p_2} \subset P_1$, $\min_{t \in \{1, 1\}} u(t) \geq \sigma \|u\| = \sigma p_2$. Hence, for any $u \in \partial\Omega_{p_2}$, by (B_6) we have

$$
Tu\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2}, v\right)\phi_q\left(\int_0^v f(s, u(s))\,ds\right)\,dv \geq \int_0^{1/4} G\left(\frac{1}{2}, v\right)\phi_q\left(\int_0^{1/4} f(s, u(s))\,ds\right)\,dv
$$

$$
\geq \frac{1}{4} p_2 \int_0^{1/4} G\left(\frac{1}{2}, v\right)\,dv = \frac{2p_2}{\sigma} > p_2 = \|u\|,
$$

which yields

$$
\|Tu\| > \|u\| \quad \text{for} \quad u \in \partial\Omega_{p_2}.
$$

An application of Lemma 2.3 again shows that

$$
i(T, \Omega_{p_2}, P_1) = 0.
$$

(3.8)

Note that $r_1 < p_2 < r_2$. As before, from (3.6)–(3.8), we have

$$
i(T, \Omega_{p_2} \setminus \Omega_{r_2}, P_1) = -1, \quad i(T, \Omega_{r_2} \setminus \Omega_{p_2}, P_1) = 1,
$$

which shows that $T$ has two positive fixed points, and consequently, the problem (1.1) and (1.2) has two positive solutions. This completes the proof.  \[\square\]

Remark 3.1. When $p = 2$, the problem (1.1) and (1.2) is the usual form of third-order Sturm–Liouville boundary value problem

$$
u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1),
$$

$$
\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \quad u''(0) = 0.
$$

By using the same method we can also present some sufficient conditions which guarantee the existence of at least one or at least two positive solutions for this class of problem. Moreover, the results are also new and different from the previous ones.

Example 3.2. Now we consider an example to illustrate our main result—Theorem 1.1. Consider the following third-order Sturm–Liouville problem with $p$-Laplacian

$$
(\phi_p(u''(t)))' + [\varphi(t)h(u(t))]^2 = 0, \quad t \in (0, 1),
$$

$$
u(0) - u'(0) = 0, \quad u(1) = 0, \quad u''(0) = 0.
$$

(3.9)  \quad  (3.10)
where \( \varphi(t) = 4t, \ t \in [0, 1] \) and

\[
h(u) = \begin{cases} 
\frac{512}{5} & 0 \leq u \leq \frac{5}{512}; \\
1 & \frac{5}{512} \leq u \leq \frac{5}{128}; \\
\frac{32}{251} & \frac{999}{1004}; \\
\frac{5}{128} & \frac{5}{128} \leq u \leq 2; \\
\frac{5}{8} & u \geq 2.
\end{cases}
\]

In this example, we note that \( p = 3, \alpha = \beta = \gamma = 1 \) and \( \delta = 0 \). After a simple calculation, we get \( q = \frac{3}{5}, \rho = 2, \sigma = \frac{1}{4} < 1 \). \( G(s, s) = \frac{1}{2}(1 - s^2) \) and

\[
m = \frac{6\rho}{\alpha\gamma + 3\beta\delta + 3\beta\gamma + 6\beta\delta} = 3, \quad l = \frac{2}{\sigma^{2-1-q}} \cdot \frac{32\rho}{3\alpha\gamma + 7\alpha\delta + 7\beta\gamma + 16\beta\delta} = \frac{512}{5}.
\]

We choose \( b = \frac{5}{512} \) and \( c = 2 \). Evidently, \( b < \min\{\frac{m}{l}, \sigma\} \cdot c \) and

(i) for \( t \in \left[\frac{1}{4}, \frac{3}{4}\right] \), \( \frac{5}{312} \leq u \leq \frac{b}{\sigma} = \frac{5}{128} \), we have

\[
f(t, u) = \left(\varphi(t)h(u)\right)^2 \geq \left[4 \times \frac{1}{4} \times 1\right]^2 = (lb)^2.
\]

(ii) for \( t \in [0, 1], \ 0 \leq u \leq 2 \), we have

\[
f(t, u) = \left(\varphi(t)h(u)\right)^2 \leq \left[4 \times 1 \times \left(\frac{32}{251} \times 2 + \frac{999}{1004}\right)\right]^2 \leq (mc)^2.
\]

Hence, all the conditions of Theorem 1.1 are satisfied; then the problem (3.9) and (3.10) has at least one positive solution \( u^* \) with \( \|u^*\| \leq 2, \ \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u^*(t) > \frac{5}{512}. \)

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References

[13] O’Regan, Some general existence principles results for \( (\psi_{\nu}(y')) = q(t)f(t, y, y'), \ 0 < t < 1, \ SIAM J. Math. Anal. 24(3) (1993) 648–668.