The 3-choosability of plane graphs of girth 4

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Abstract

A set $S$ of vertices of the graph $G$ is called $k$-reducible if the following is true: $G$ is $k$-choosable if and only if $G - S$ is $k$-choosable. A $k$-reduced subgraph $H$ of $G$ is a subgraph of $G$ such that $H$ contains no $k$-reducible set of some specific forms. In this paper, we show that a 3-reduced subgraph of a non-3-choosable plane graph $G$ contains either adjacent 5-faces, or an adjacent 4-face and $k$-face, where $k \leq 6$. Using this result, we obtain some sufficient conditions for a plane graph to be 3-choosable. In particular, if $G$ is of girth 4 and contains no 5- and 6-cycles, then $G$ is 3-choosable.

1. Introduction

In this paper, we consider only finite and simple graphs. Undefined terms may be found in [2]. Suppose $k$ is an integer. Then $k^+$ and $k^-$ denote integers $\geq k$ and $\leq k$, respectively. A vertex $u$ is called a $k$-vertex if $d_G(u) = k$. If $k = 3$, then $u$ is called a minor vertex. A face $f$ is called a $k$-face if $d_G(f) = k$. If $k = 4$, then $f$ is called a minor face. If no confusion can arise, $d(v)$ and $d(f)$ will be used instead of $d_G(v)$ and $d_G(f)$, respectively. A face of a plane graph is incident with all edges and vertices on its boundary. Two faces are adjacent.
if they have an edge in common. Let \(F_j\) be the set of \(j\)-faces incident with \(j\) minor vertices and \(F_i^1\) be the subset of \(F_j\) consisting only of faces adjacent to \(i\) minor faces.

A \(k\)-cycle is a cycle of length \(k\). The name of a cycle will also be used to denote the set of its vertices. We use \(F(u)\) and \(k(u)\) to denote the set of all faces incident with the vertex \(u\) and \(\min\{d(x)|x \in F(u)\}\), respectively.

A list coloring of \(G\) is an assignment of colors to \(V(G)\) such that each vertex \(v\) receives a color from a prescribed list \(L(v)\) of colors and adjacent vertices receive distinct colors (see [7]). \(L(G) = \{L(v)|v \in V(G)\}\) is called a color list of \(G\). The graph \(G\) is called \(k\)-choosable if \(G\) admits a list coloring for all color lists \(L\) with \(k\) colors in each list.

All 2-choosable graphs have been characterized by Erdős et al. [3]. In [5], Thomassen proved that every plane graph is 2-choosable. From that proof, one can find a simple linear algorithm for finding the corresponding list coloring. Voigt [8] showed that there are planar graphs which are not 4-choosable. It remains to decide whether a given plane graph is 4- or 3-choosable. Gutner [4] proved that these problems are NP-hard. So far, some sufficient conditions have been obtained and some constructions have been found. Alon and Tarsi [1] proved that every plane bipartite graph is 3-choosable. Thomassen [6] proved that every plane graph of girth at least 5 is 3-choosable. Voigt [9] gave an example of a non-3-choosable plane graph of girth 4. The objective of this paper is to study 3-choosability of plane graphs of girth 4. In this paper we give a necessary condition for a plane graph of girth 4 to be 3-choosable. In particular, if \(G\) is of girth 4 and contains no 5- and 6-cycles, then \(G\) is 3-choosable.

2. Some lemmas

Let \(C\) be a cycle in \(G\). A sub-cycle of \(C\) is a cycle in \(G[C]\) with length strictly shorter than \(C\). We shall need some lemmas, the first of which is well known.

**Lemma 1.** Let \(v\) be a vertex of \(G\), not necessarily planar, with \(d(v)\leq k-1\). Then \(G\) is \(k\)-choosable if and only if \(G - \{v\}\) is \(k\)-choosable.

**Lemma 2.** Let \(G\) be a graph having no subgraphs isomorphic to \(K_4\). Suppose \(n \geq 2\) and \(C\) is a \(2n\)-cycle of \(G\) and \(d(u)\leq k\) for all \(u \in C\). Then either (i) there exists an even sub-cycle \(C^*\) of \(C\) and with \(d(u)\leq k\) for all \(u \in C^*\) or (ii) \(G\) is \(k\)-choosable if and only if \(G - C\) is \(k\)-choosable.

**Proof.** Suppose that \(G - C\) is \(k\)-choosable and suppose that \(L = \{L(v)|v \in V(G)\}\) is a color list of \(G\) in which each list contains \(k\) colors. Let \(\phi\) be an \(L\)-list coloring of \(G - C\). For all \(v \in C\), let \(L^0(v) = L(v)\setminus\{\phi(u)\}u \in V(G)\setminus C\) and \(vu \in E(G)\}. Then \(|L^0(v)|\geq 2\).

We have three cases:

Case 1: \(G[C]\) is chordless. Since even cycles are 2-choosable, there exists a 2-list coloring \(\phi^*\) of \(C\). Combining \(\phi\) and \(\phi^*\), we obtain an \(L\)-list coloring of \(G\).

Case 2: \(G[C]\) contains exactly one chord. Without loss of generality, assume that \(C = u_1u_2\cdots u_{2n}\) and \(u_1u_j\) is the chord, where \(3 \leq j \leq 2n - 1\). Then \(|L^0(u_1)|\geq 3\), \(|L^0(u_j)|\geq 3\) and \(|L^0(u_i)|\geq 2\) if \(i \neq 1\) or \(j\). We can then define \(\phi^*\) on \(C\) as follows: choose \(\phi^*(u_j) \in L^0(u_j)\setminus \phi^*(u_{j-1})\), where \(\phi^*(u_{j-1})\) for
Theorem 5. Let $G$ be a non-$3$-choosable plane graph of girth not less than $4$ and let $H$ be any $3$-reduced subgraph of $G$. Then there exists in $H$ adjacent $5$-faces or adjacent $4$-face and $k$-face ($k \leq 6$).

Corollary 6. Let $G$ be a plane graph of girth not less than $4$. If a $3$-reduced subgraph of $G$ contains neither adjacent $5$-faces nor adjacent $4$-face and $k$-face ($k \leq 6$), then $G$ is $3$-choosable.

Corollary 7. Let $G$ be a plane graph of girth not less than $4$. If $G$ contains neither adjacent $5$-cycles nor adjacent $4$-cycle and $k$-cycle ($k \leq 6$), then $G$ is $3$-choosable.
Clearly, graphs satisfying conditions of Corollary 7 may contain 4-, 5- or 6-cycles.

**Corollary 8.** Let $G$ be a plane graph of girth not less than 4. If $G$ contains no 5- and 6-cycles, then $G$ is 3-choosable.

**Proof.** Let $G$ be a plane graph without 5- and 6-cycles. Consider a 3-reduced subgraph $H$ of $G$. Certainly $\delta(H) \geq 3$ and $H$ contains no 5- and 6-cycles. Hence, $H$ contains neither adjacent 5-faces nor adjacent 4-face and $k$-face ($k \leq 6$). It follows from Corollary 6 that $H$ is 3-choosable. Thus, the corollary holds by Lemma 3. □

4. Proof of Theorem 5

Let $G$ be a non-3-choosable plane graph. Without loss of generality, we assume that $G$ is 3-reduced. Suppose the theorem is false. Then there exist neither adjacent 5-faces nor adjacent 4-face and $k$-face with $k \leq 6$. That is, there are no adjacent 5-faces and $d(f) \geq 7$ for every face $f$ adjacent to a 4-face. This also implies that $F_7 = F_7^0 \cup F_7^1 \cup F_7^2 \cup F_7^3$. If we assign a weight of $\sigma(x) = d(x) - 4$ to each $x \in V(H) \cup F(H)$, then by Euler’s formula we have:

$$\sum_{v \in V(H) \cup F(H)} \sigma(x) = -8. \quad (1)$$

If we obtain a new weight $\sigma^*(x)$ for each $x \in V(H) \cup F(H)$ by transferring weights from one element to another, then we also have

$$\sum_{v \in V(H) \cup F(H)} \sigma^*(x) = -8. \quad (2)$$

Moreover, if $\sigma^*(x) \geq 0$ for all $x \in V(H) \cup F(H)$, then the theorem is proved.

Weights will be transferred from $f \in F$ to $v \in V$, where $v$ is incident with $f$, $d(f) \geq 5$ and $d(v) = 3$, according to the following rules:

(R1) Transfer $\frac{1}{7}$ if (a) $d(f) \geq 7$ and $f \notin F_7$ or (b) $f \in F_7$ and $k(v) = 4$.

(R2) Transfer $\frac{5}{7}$ if (a) $d(f) = 6$ or (b) $f \in F_7$ and $k(v) = 5$.

(R3) Transfer $\frac{1}{5}$ if (a) $d(f) = 5$ or (b) $f \in F_7$ and $k(v) = 6$.

Let $v \in V$. If $d(v) \geq 4$, then $\sigma^*(v) = \sigma(v) \geq 0$. Suppose $d(v) = 3$, $v$ is incident with $f_1$, $f_2$ and $f_3$ and $d(f_1) \leq d(f_2) \leq d(f_3)$. By Lemma 4, at most one of these three faces is in $F_7$. If $d(f_1) = 4$, then (R1) is applicable to both $f_2$ and $f_3$. Therefore, $\sigma^*(v) = \sigma(v) + 2 \cdot \frac{1}{7} = 0$.

If $d(f_1) = 5$, then (R3(a)) is applicable to $f_1$ and (R3) is not applicable to $f_2$ and $f_3$. So $\sigma^*(v) = \sigma(v) + \frac{1}{5} + 2 \cdot \frac{2}{5} = 0$. If $d(f_1) = 6$, then (R3(b)) is applicable to at most one of $f_2$ and $f_3$, and (R3(a)) is not applicable. So $\sigma^*(v) = \sigma(v) + \frac{1}{5} + 2 \cdot \frac{2}{5} = 0$. If $d(f_1) \geq 7$, then (R1(a)) is applicable to at least two of $f_1$, $f_2$ and $f_3$. So $\sigma^*(v) = \sigma(v) + 2 \cdot \frac{1}{7} = 0$.

Let $f \in F$. If $d(f) \geq 4$, then $\sigma^*(f) = \sigma(f) = 0$. If $d(f) = 5$, then $\sigma^*(f) = \sigma(f) - 5 \cdot \frac{1}{7} = 0$. If $d(f) = 6$, then at least one 4$^+$-vertex is incident with $f$ by Lemma 4. So $\sigma^*(f) \geq 0$. 


\( \sigma(f) - 5 \cdot \frac{2}{3} = 0 \). If \( d(f) = k \geq 8 \), then \( \sigma^*(f) \geq \sigma(f) - k \cdot \frac{1}{2} \geq k - 4 - (k/2) \geq 0 \). If \( d(f) = 7 \) and \( f \notin F_7 \), then at least one \( 4^+ \)-vertex is incident with \( f \). So \( \sigma^*(f) \geq \sigma(f) - 6 \cdot \frac{1}{2} = 0 \).

If \( f \in F_0^0 \cup F_1^1 \), then weights are transferred from \( f \) to at most two \( 3 \)-vertices according to [R1(b)]. So \( \sigma^*(f) \geq \sigma(f) - 2 \cdot \frac{1}{2} = 0 \).

If \( f \in F_2^2 \) and \( f \) is not adjacent to any \( 6 \)-face, then at least one vertex incident with \( f \), say \( v \), is incident with \( 7^+ \)-faces only, and 0 is transferred from \( f \) to \( v \). So \( \sigma^*(f) \geq \sigma(f) - 4 \cdot \frac{1}{2} = 0 \).

If \( f \in F_2^2 \) and \( f \) is adjacent to a \( 6 \)-face \( f_1 \), then at least one face adjacent to both \( f \) and \( f_1 \) is a \( 7^+ \)-face. So \( \sigma^*(f) \geq \sigma(f) - 4 \cdot \frac{1}{2} = 0 \).

If \( f \in F_3^3 \), then one vertex incident with \( f \) is incident with \( 7^+ \)-faces only, and the weight transferred from \( f \) to \( v \) is 0. So \( \sigma^*(f) \geq \sigma(f) - 6 \cdot \frac{1}{2} = 0 \).

References