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# Mean field propagation of Wigner measures and BBGKY hierarchies for general bosonic states

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**Abstract**

Contrary to the finite dimensional case, Weyl and Wick quantizations are no more asymptotically equivalent in the infinite dimensional bosonic second quantization. Moreover neither the Weyl calculus defined for cylindrical symbols nor the Wick calculus defined for polynomials are preserved by the action of a nonlinear flow. Nevertheless taking advantage carefully of the information brought by these two calculuses in the mean field asymptotics, the propagation of Wigner measures for general states can be proved, extending to the infinite dimensional case a standard result of semiclassical analysis.


**Résumé**

Contrairement au cas de la dimension finie, les quantifications de Weyl et de Wick ne sont pas asymptotiquement équivalentes en dimension infinie. De plus, ni le calcul de Weyl, défini pour des symboles cylindriques, ni le calcul de Wick, défini pour des polynômes, ne sont préservés par un flot non linéaire. Néanmoins une utilisation attentive de l'information apportée par ces deux calculs, permet d'établir la propagation des mesures de Wigner pour des données initiales très générales, ce qui étend à la dimension infinie un résultat bien connu de l'analyse semi-classique.


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## 1. Introduction

Our main result is briefly presented in this introduction. Accurate definitions will be found in Section 2.

Let $\mathcal{H} = \Gamma_s(\mathcal{Z})$ be the bosonic Fock space constructed over the complex separable Hilbert-space $\mathcal{Z}$, $\Gamma_s(\mathcal{Z}) = \bigoplus_{n=0}^{\infty} \bigvee^n \mathcal{Z}$ where $\bigvee^n \mathcal{Z}$ is the symmetric $n$-th Hilbertian tensor power of $\mathcal{Z}$. Consider the Hamiltonian

$$H_\varepsilon = \mathrm{d}\Gamma(A) + \left(\sum_{j=2}^{r} \langle z^{\otimes j}, \tilde{Q}_j z^{\otimes j}\rangle\right)^{Wick}$$

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defined for the self-adjoint operator $(A, \mathcal{D}(A))$ on $\mathcal{Z}$ and $\tilde{Q}_j = \tilde{Q}_j^* \in \mathcal{L}(\bigvee^j \mathcal{Z})$. It is the Wick-quantized version of the classical Hamiltonian:

$$h(z, \bar{z}) = \langle z, Az\rangle + \sum_{j=2}^{r} \langle z^{\otimes j}, \tilde{Q}_j z^{\otimes j}\rangle, \quad z \in \mathcal{D}(A) \subset \mathcal{Z}.$$

When $\mathcal{Z} = L^2(\mathbb{R}^d)$, the operator $H_\varepsilon$ is formally written

$$\begin{aligned} H_\varepsilon = & \int_{\mathbb{R}^{2d}} A(x, y)a^*(x)a(y)\,dx\,dy \\ & + \sum_{j=2}^{r} \int_{\mathbb{R}^{2dj}} \tilde{Q}_j(x_1, \ldots, x_j, y_1, \ldots, y_j)a^*(x_1)\ldots a^*(x_j)a(y_1)\ldots a(y_j)\,dx\,dy, \end{aligned}$$

with the $\varepsilon$-dependent canonical commutation relations $[a(x), a^*(y)] = \varepsilon\delta(x - y)$. Here $A(.,.)$ and $\tilde{Q}_j(.,.)$ denote the kernels of the operators $A$ and $\tilde{Q}_j$. The mean field asymptotics is concerned with the limit as $\varepsilon \to 0$, where $\frac{1}{\varepsilon} = N_\varepsilon$ represents a large number of particles and where $\varepsilon$ enters in the CCR-relations by:

$$\forall f, g \in \mathcal{Z}, \quad \left[a(f), a^*(g)\right] = \varepsilon\langle f, g\rangle I.$$

The number operator is $\mathbf{N} = \mathrm{d}\Gamma(I_{\mathcal{Z}})$, with $\mathbf{N}z^{\otimes n} = \varepsilon n z^{\otimes n}$. For a normal state $\varrho_\varepsilon \in \mathcal{L}^1(\bigvee^{N_\varepsilon} \mathcal{Z}) \subset \mathcal{L}^1(\mathcal{H})$ with $\mathcal{Z} = L^2(\mathbb{R}^d)$, a standard tool considered in the mean field limit is the BBGKY hierarchy of reduced density matrices:

$$\gamma_\varepsilon^{(p)}(x, y) = \int_{\mathbb{R}^{2d(N_\varepsilon - p)}} \varrho_\varepsilon(x, X, y, X)\,dX, \quad p \in \mathbb{N},$$

and such a definition will be extended to general $\mathcal{Z}$ and normal states $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{H})$ fulfilling the condition $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^k] < +\infty$ for all $k \in \mathbb{N}$.

For a cylindrical function, $b(z) = b(\wp z)$ for some finite rank projection $\wp$ and $b$ belonging to the Schwartz class $\mathcal{S}(\wp\mathcal{Z})$, the Weyl quantization can be given by

$$b^{Weyl} = \int_{\wp\mathcal{Z}} \mathcal{F}[b](z)\, W(\sqrt{2}\pi z)\, L_p(dz),$$

where $W(\sqrt{2\pi} z) = e^{i\pi(a(z)+a^*(z))}$ and where $L_p$ and $\mathcal{F}$ are respectively the Lebesgue measure on $\wp\mathcal{Z}$ and the ($\varepsilon$-independent) Fourier transform on $\mathcal{S}(\wp\mathcal{Z})$. Associated with a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$, Wigner measures can be defined by:

$$\lim_{k\to\infty} \mathrm{Tr}\left[\varrho_{\varepsilon_k} b^{Weyl}\right] = \int_{\mathcal{Z}} b(z)\,d\mu(z)$$

after extracting subsequences under the sole uniform estimate $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\delta] \leqslant C_\delta$ for some $\delta > 0$.

The problem of the mean field dynamics questions whether the asymptotic quantities as $\varepsilon \to 0$ associated with

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon} \varrho_\varepsilon e^{-i\frac{t}{\varepsilon}H_\varepsilon}, \quad t \in \mathbb{R},$$

are transported by the flow $\mathbf{F}_t$ generated by the classical Hamiltonian $h(z, \bar{z})$ and given, after writing $z_t = \mathbf{F}_{t-s}(z_s)$, by

$$i\partial_t z_t = (\partial_{\bar{z}} h)(z_t, \bar{z}_t) = Az_t + \sum_{j=2}^{r} j\langle z_t^{\otimes j-1}, \tilde{Q}_j z_t^{\otimes j}\rangle. \tag{1}$$

The finite dimensional case enters in the standard framework of semiclassical analysis and has been studied extensively in the 80's and 90's by various authors and with various methods ([48,35,29,42,18,43,24] and references therein).

It was first considered by Hepp in [36] and extended by Ginibre and Velo in [30,31] by the squeezed coherent states method well-known as the Hepp method (see also [49,6]). More recently the question of the mean field dynamics has been tackled with the so-called BBGKY-hierarchy approach inspired by the BBGKY method of classical kinetic theory (see [52,13,22,14,32,1,3,23] and also the related works [37,17]). In [25–27] a specific use of the structure of the Wick calculus in the bosonic Fock space was used to make work truncated Dyson expansions for the mean field dynamics of specific states. The aim of our work started in [7] was to restore the phase-space geometric nature of the problem in the spirit of [11,33,38,39] and to extend as much as possible to the infinite dimensional case, the methods well understood for the semiclassical finite dimensional problem. In this first article, we explained the construction of Wigner measures, analyzed accurately the gap of information carried by Weyl observables and Wick observables and use these Wigner (or semiclassical) measures to reformulate known propagation results. In [8], we reconsidered the truncated Dyson expansion method of [25–27] in order to prove the propagation of Wigner measures for some specific families of states. We are now able to state the following general result (still with a regular interaction term contrary to many other works cited above).

**Theorem 1.1.** *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$ with a single Wigner measure $\mu_0$ and such that*

$$\forall\alpha\in\mathbb{N},\quad \lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon\mathbf{N}^\alpha\big]=\int_{\mathcal{Z}}|z|^{2\alpha}\,d\mu_0(z)<+\infty. \tag{2}$$

*Then for all $t\in\mathbb{R}$, the family $(\varrho_\varepsilon(t)=e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon})_{\varepsilon\in(0,\bar\varepsilon)}$ has a unique Wigner measure $\mu_t=(\mathbf{F}_t)_*\mu_0$, which is the initial measure $\mu_0$ pushed forward by the flow associated with* (1).

*Moreover the convergence,*

$$\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon(t)b^{Wick}\big]=\int_{\mathcal{Z}}b\circ\mathbf{F}_t(z)\,d\mu_0(z),$$

*holds for any $b\in\mathcal{P}_{alg}(\mathcal{Z})=\bigoplus^{alg}_{p,q\in\mathbb{N}}\mathcal{P}_{p,q}(\mathcal{Z})$.*

*Finally, the convergence of the reduced density matrices,*

$$\lim_{\varepsilon\to 0}\gamma_\varepsilon^{(p)}(t)=\frac{1}{\int_{\mathcal{Z}}|z|^{2p}\,d\mu_t(z)}\int_{\mathcal{Z}}\big|z^{\otimes p}\big\rangle\big\langle z^{\otimes p}\big|\,d\mu_t(z)=:\gamma_0^{(p)}(t),$$

*holds in the $\mathcal{L}^1(\bigvee^p\mathcal{Z})$-norm for all $p\in\mathbb{N}$.*

**Comments.** The existence of Wigner measures as Borel probability measures requires a uniform estimate $\mathrm{Tr}[\varrho_\varepsilon\mathbf{N}^\delta]\leqslant C_\delta$ for some $\delta>0$, but such an assumption would be redundant with the existence of bounded limits stated in (2).

The uniqueness of the Wigner measure $\mu_0$ is not really a strong assumption since it suffices to replace the whole family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ by a suitable extracted sequence $(\varrho_{\varepsilon_k})_{k\in\mathbb{N}}$, $\lim_{k\to\infty}\varepsilon_k=0$, in order to fulfill this requirement. Such a reduction argument after extraction will often be used.

The fact that the quantities $\mathrm{Tr}[\varrho_\varepsilon\mathbf{N}^\alpha]$ are uniformly bounded w.r.t. $\varepsilon\in(0,\bar\varepsilon)$ is also very natural within the mean field framework and satisfied by all known physical examples.

Actually the strong assumption which is not satisfied in all cases is that the limit in (2) equals $\int_{\mathcal{Z}}|z|^{2\alpha}\,d\mu_0$. This condition prevents from the phenomenon of "infinite dimensional defect of compactness" identified in [7] and which was shown to appear in the physical example of the Bose–Einstein free gas (the non-condensated phase is responsible for a discrepancy between the left- and right-hand sides of (2)). The analysis of this phenomenon is improved in Section 2.

Finally our proof no more uses truncated Dyson expansions of the quantum flow and relies only on the good properties of the classical flow, after exploiting all the a priori information given by the Weyl and Wick calculus.

**Outline.** Section 2 introduces the various objects used for our analysis, Wick and Weyl calculuses, Wigner measures, reduced density matrices. The conditions presented in [8] are reduced to the simple equivalent form (2) in Section 2.7. After this Section 2.8 is devoted to the notion of states localized in a ball.

The dynamics is studied in Section 3. First a simple condition is proved to ensure, via some equicontinuity argument, the possibility of a common extraction process $(\varepsilon_k)_{k\in\mathbb{N}}$ for all times $t\in\mathbb{R}$. Then the propagation of Wigner measures is proved for states localized in a ball. Then the truncation is removed and all the arguments are gathered for the proof of Theorem 1.1 in Section 3.4. Finally, additional simple consequences are listed in Section 3.5.

Examples are presented in Section 4. It is recalled that the regular interactions are physically relevant within the modeling of the rapidly rotating Bose condensates in the Lowest Landau Level approximation. Details are given about the propagation of non-trivial Wigner measures supported on a torus, which shows the advantage of this formulation compared to the BBGKY hierarchy method. Finally, the propagation of Wigner measures provides a nice formulation of the Hartree–von Neumann limit.

## 2. Information carried by Wigner measures

After introducing the symmetric Fock space with $\varepsilon$-dependent CCR's and recalling some properties of the Wick quantization, the connection between infinite dimensional Wigner measures and the BBGKY presentation of the many body problem is explicitly specified. This section ends with the notion of states localized in a ball, which will be useful in the proof of Theorem 1.1.

### *2.1. Fock space*

Consider a separable Hilbert space $\mathcal{Z}$ endowed with a scalar product $\langle .,. \rangle$ which is anti-linear in the left argument and linear in the right one and with the associated norm $|z| = \sqrt{\langle z, z\rangle}$. Let $\sigma = \operatorname{Im}\langle .,. \rangle$ and $S = \operatorname{Re}\langle .,. \rangle$ respectively denote the canonical symplectic form and the real scalar product over $\mathcal{Z}$. The symmetric Fock space on $\mathcal{Z}$ is the Hilbert space,

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \bigvee^n \mathcal{Z} = \Gamma_s(\mathcal{Z}),$$

where $\bigvee^n \mathcal{Z}$ is the $n$-fold symmetric tensor product. Almost all the direct sums and tensor products are completed within the Hilbert framework. This is omitted in the notation. On the contrary, a specific ${}^{alg}$ superscript will be used for the algebraic direct sums or tensor products.

For any $n \in \mathbb{N}$, the orthogonal projection of $\bigotimes^n \mathcal{Z}$ onto the closed subspace $\bigvee^n \mathcal{Z}$ will be denoted by $\mathcal{S}_n$. For any $(\xi_1, \xi_2, \dots, \xi_n) \in \mathcal{Z}^n$, the vector $\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \in \bigvee^n \mathcal{Z}$ will be:

$$\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n = \mathcal{S}_n(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \xi_{\pi(1)} \otimes \xi_{\pi(2)} \otimes \cdots \otimes \xi_{\pi(n)}, \tag{3}$$

where $\mathfrak{S}_n$ is the symmetric group of degree $n$. The family of vectors $(\xi_1 \vee \cdots \vee \xi_n)_{\xi_i \in \mathcal{Z}}$ is a total family of $\bigvee^n \mathcal{Z}$ and thanks to the polarization identity,

$$\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_n \left( \sum_{j=1}^n \varepsilon_j \xi_j \right)^{\otimes n}, \tag{4}$$

the same property holds for $(\xi^{\otimes n})_{n \in \mathbb{N}, \xi \in \mathcal{Z}}$.

For two operators $A_k : \bigvee^{i_k} \mathcal{Z} \to \bigvee^{j_k} \mathcal{Z}$, $k = 1, 2$, the notation $A_1 \vee A_2$ stands for:

$$A_1 \vee A_2 = \mathcal{S}_{j_1+j_2} \circ (A_1 \otimes A_2) \circ \mathcal{S}_{i_1+i_2} \in \mathcal{L}\Big(\bigvee^{i_1+i_2} \mathcal{Z}, \bigvee^{j_1+j_2} \mathcal{Z}\Big).$$

Any $z \in \mathcal{Z}$ is identified with the operator $|z\rangle : \bigvee^0 \mathcal{Z} = \mathbb{C} \ni \lambda \mapsto \lambda z \in \mathcal{Z} = \bigvee^1 \mathcal{Z}$ while $\langle z|$ denotes the linear form $\mathcal{Z} \ni \xi \mapsto \langle z, \xi\rangle \in \mathbb{C}$. The creation and annihilation operators $a^*(\xi)$ and $a(\xi)$, parametrized by $\varepsilon > 0$, are then defined by:

$$a(\xi)|_{\bigvee^n \mathcal{Z}} = \sqrt{\varepsilon n} \langle \xi | \otimes I_{\bigvee^{n-1} \mathcal{Z}},$$
$$a^*(\xi)|_{\bigvee^n \mathcal{Z}} = \sqrt{\varepsilon(n+1)} \mathcal{S}_{n+1} \circ \big(|\xi\rangle \otimes I_{\bigvee^n \mathcal{Z}}\big) = \sqrt{\varepsilon(n+1)} \xi \vee I_{\bigvee^n \mathcal{Z}}$$

and satisfy the canonical commutation relations (CCR):

$$\left[a(\xi_1), a(\xi_2)\right] = \left[a^*(\xi_1), a^*(\xi_2)\right] = 0, \qquad \left[a(\xi_1), a^*(\xi_2)\right] = \varepsilon\langle\xi_1, \xi_2\rangle I. \tag{5}$$

We also consider the canonical quantization of the real variables $\Phi(\xi) = \frac{1}{\sqrt{2}}(a^*(\xi) + a(\xi))$ and $\Pi(\xi) = \Phi(i\xi) = \frac{1}{i\sqrt{2}}(a(\xi) - a^*(\xi))$. They are self-adjoint operators on $\mathcal{H}$ and satisfy the identities:

$$\left[\Phi(\xi_1), \Phi(\xi_2)\right] = i\varepsilon\sigma(\xi_1, \xi_2)I, \qquad \left[\Phi(\xi_1), \Pi(\xi_2)\right] = i\varepsilon S(\xi_1, \xi_2)I.$$

The representation of the Weyl commutation relations in the Fock space,

$$W(\xi_1)W(\xi_2) = e^{-\frac{i\varepsilon}{2}\sigma(\xi_1,\xi_2)}W(\xi_1 + \xi_2) = e^{-i\varepsilon\sigma(\xi_1,\xi_2)}W(\xi_2)W(\xi_1), \tag{6}$$

is obtained by setting $W(\xi) = e^{i\Phi(\xi)}$. The number operator is also parametrized by $\varepsilon > 0$,

$$\mathbf{N}|_{\bigvee^n \mathcal{Z}} = \varepsilon n I|_{\bigvee^n \mathcal{Z}}.$$

It is convenient to introduce the subspace,

$$\mathcal{H}_{fin} = \bigoplus_{n\in\mathbb{N}}^{alg} \bigvee^n \mathcal{Z},$$

of $\mathcal{H}$, which is a set of analytic vectors for $\mathbf{N}$.

For any contraction $S \in \mathcal{L}(\mathcal{Z})$, $|S|_{\mathcal{L}(\mathcal{H})} \leqslant 1$, $\Gamma(S)$ is the contraction in $\mathcal{H}$ defined by:

$$\Gamma(S)|_{\bigvee^n \mathcal{Z}} = S \otimes S \otimes \cdots \otimes S.$$

More generally $\Gamma(B)$ can be defined by the same formula as an operator on $\mathcal{H}_{fin}$ for any $B \in \mathcal{L}(\mathcal{Z})$. Meanwhile, for any self-adjoint operator $A : \mathcal{Z} \supset \mathcal{D}(A) \to \mathcal{Z}$, the operator $\mathrm{d}\Gamma(A)$ is the self-adjoint operator given by:

$$e^{\frac{it}{\varepsilon}\mathrm{d}\Gamma(A)} = \Gamma\left(e^{itA}\right),$$

$$\mathrm{d}\Gamma(A)|_{\bigvee^{n,alg}\mathcal{D}(A)} = \varepsilon\left[\sum_{k=1}^{n} I \otimes \cdots \otimes \underbrace{A}_{k} \otimes \cdots \otimes I\right].$$

For example $\mathbf{N} = \mathrm{d}\Gamma(I)$.

### 2.2. Wick operators

The Wick symbolic calculus on (homogenous) polynomials as introduced in [7] is recalled with its basic properties.

**Definition 2.1.** For $p, q \in \mathbb{N}$, $\mathcal{P}_{p,q}(\mathcal{Z})$ denotes the set of $(p,q)$-homogeneous polynomial functions on $\mathcal{Z}$ which fulfill:

$$b(z) = \left\langle z^{\otimes q}, \tilde{b} z^{\otimes p}\right\rangle \quad \text{with } \tilde{b} \in \mathcal{L}\left(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z}\right).$$

The subspace of $\mathcal{P}_{p,q}(\mathcal{Z})$ made of polynomials $b$ such that $\tilde{b}$ is a compact operator $\tilde{b} \in \mathcal{L}^\infty(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})$ (resp. $b \in \mathcal{L}^r(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})$) is denoted by $\mathcal{P}^\infty_{p,q}(\mathcal{Z})$ (resp. $\mathcal{P}^r_{p,q}(\mathcal{Z})$).

On those spaces, the natural norms are:

$$|b|_{\mathcal{P}_{p,q}} = |\tilde{b}|_{\mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})} \quad \text{and} \quad |b|_{\mathcal{P}^r_{p,q}} = |\tilde{b}|_{\mathcal{L}^r(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})}, \quad 1 \leqslant r.$$

The set of non-homogeneous polynomials, the algebraic direct sum $\bigoplus^{alg}_{p,q\in\mathbb{N}} \mathcal{P}_{p,q}(\mathcal{Z})$ (resp. $\bigoplus^{alg}_{p,q\in\mathbb{N}} \mathcal{P}^r_{p,q}(\mathcal{Z})$ with $1 \leqslant r \leqslant \infty$), will be denoted by $\mathcal{P}_{alg}(\mathcal{Z})$ (resp. $\mathcal{P}^r_{alg}(\mathcal{Z})$).

Owing to the condition $\tilde{b} \in \mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})$ for $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, this definition implies that any Gâteaux differential $\partial^j_{\bar{z}}\partial^k_z b(z)$ at the point $z \in \mathcal{Z}$ belongs to $\mathcal{L}(\bigvee^k \mathcal{Z}, \bigvee^j \mathcal{Z})$ with,

$$\left\langle \varphi, \partial^j_{\bar{z}}\partial^k_z b(z)\psi\right\rangle = \frac{p!}{(p-k)!}\frac{q!}{(q-j)!}\left\langle z^{\otimes q-j} \vee \varphi, \tilde{b} z^{\otimes p-k} \vee \psi\right\rangle.$$

In particular, we recover the operator $\tilde{b}$ from $b(z)$ via the relation,

$$\tilde{b} = \frac{1}{p!}\frac{1}{q!}\partial_z^p \partial_{\bar{z}}^q b(z) \in \mathcal{L}\Big(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z}\Big).$$

With any "symbol" $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, a linear operator $b^{Wick}$ called Wick monomial can be associated according to:

$$b^{Wick} : \mathcal{H}_{fin} \to \mathcal{H}_{fin},$$

$$b|^{Wick}_{\bigvee^n \mathcal{Z}} = 1_{[p,+\infty)}(n)\frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\varepsilon^{\frac{p+q}{2}}\big(\tilde{b} \vee I_{\bigvee^{n-p}\mathcal{Z}}\big) \in \mathcal{L}\Big(\bigvee^n \mathcal{Z}, \bigvee^{n+q-p} \mathcal{Z}\Big), \tag{7}$$

with $\tilde{b} = (p!)^{-1}(q!)^{-1}\partial_z^p \partial_{\bar{z}}^q b(z)$. The basic symbol–operator correspondence:

$$\begin{array}{llllll} \langle z, \xi\rangle \longleftrightarrow a^*(\xi) & \quad \sqrt{2}S(\xi, z) \longleftrightarrow \Phi(\xi) & \quad \langle z, Az\rangle \longleftrightarrow \mathrm{d}\Gamma(A) \\ \langle \xi, z\rangle \longleftrightarrow a(\xi) & \quad \sqrt{2}\sigma(\xi, z) \longleftrightarrow \Pi(\xi) & \quad |z|^2 \longleftrightarrow \mathbf{N}, \end{array}$$

and more generally,

$$\left(\prod_{i=1}^p \langle z, \eta_i\rangle \times \prod_{j=1}^q \langle \xi_j, z\rangle\right)^{Wick} = a^*(\eta_1)\cdots a^*(\eta_p)a(\xi_1)\cdots a(\xi_q).$$

We have the following properties.

**Proposition 2.2.** *The following identities hold true on $\mathcal{H}_{fin}$ for every $b \in \mathcal{P}_{p,q}(\mathcal{Z})$:*

(i) $(b^{Wick})^* = \bar{b}^{Wick}$.
(ii) $(C(z)b(z)A(z))^{Wick} = C^{Wick}b^{Wick}A^{Wick}$, *if $A \in \mathcal{P}_{\alpha,0}(\mathcal{Z})$, $C \in \mathcal{P}_{0,\beta}(\mathcal{Z})$.*
(iii) $e^{i\frac{t}{\varepsilon}\mathrm{d}\Gamma(A)} b^{Wick} e^{-i\frac{t}{\varepsilon}\mathrm{d}\Gamma(A)} = (b(e^{-itA}z))^{Wick}$, *if $A$ is a self-adjoint operator on $\mathcal{Z}$.*

A consequence of (i) says that $b^{Wick}$ is symmetric when $q = p$ and $\tilde{b}^* = \tilde{b}$. Moreover the definition (7) gives:

$$(q = p \text{ and } \tilde{b} \geqslant 0) \quad \Rightarrow \quad \big(b^{Wick} \geqslant 0 \text{ on } \mathcal{H}_{fin}\big), \tag{8}$$

which is false for general non-negative polynomial symbols.[1] For an increasing net of non-negative operators $(\tilde{b}_\alpha)_\alpha$, $\tilde{b}_\alpha \in \mathcal{L}(\bigvee^p \mathcal{Z})$ (again $q = p$), it also gives

$$\Big(\tilde{b} = \sup_\alpha \tilde{b}_\alpha \text{ in } \mathcal{L}\Big(\bigvee^p \mathcal{Z}\Big)\Big) \quad \Rightarrow \quad \big(\forall \varphi \in \mathcal{H}_{fin},\ \langle \varphi, b^{Wick}\varphi\rangle = \sup_\alpha \langle \varphi, b_\alpha^{Wick}\varphi\rangle\big). \tag{9}$$

When $\mathcal{Z} = L^2(\mathbb{R}^d, dx)$, the general formula for $b^{Wick}$ with $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ is simply:

$$b^{Wick} = \int_{\mathbb{R}^{d(p+q)}} \tilde{b}(y_1, \ldots, y_q, x_1, \ldots, x_p)a^*(y_1)\ldots a^*(y_q)a(x_1)\ldots a(x_p)\, dx_1 \cdots dx_p\, dy_1 \cdots dy_q,$$

where $\tilde{b}(y, x)$ is the Schwartz kernel of $\tilde{b}$ and where $a(x_k) = a(\delta_{x_k})$ according to the usual convention.

**Proposition 2.3.** *For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the following number estimate holds*:

$$\big|\langle \mathbf{N}\rangle^{-\frac{q}{2}} b^{Wick} \langle \mathbf{N}\rangle^{-\frac{p}{2}}\big|_{\mathcal{L}(\mathcal{H})} \leqslant |b|_{\mathcal{P}_{p,q}}. \tag{10}$$

The relations (8) and (9) now become for $b \in \mathcal{P}_{p,p}(\mathcal{Z})$ or $b_\alpha \in \mathcal{P}_{p,p}(\mathcal{Z})$,

[1] This property should not be confused with the positivity of the finite dimensional *Anti-Wick* quantization which associates a non-negative operator to any non-negative symbol.

$$(q = p \text{ and } \tilde{b} \geqslant 0) \quad \Rightarrow \quad \big(\langle \mathbf{N}\rangle^{-p/2} b^{Wick} \langle \mathbf{N}\rangle^{-p/2} \geqslant 0 \text{ in } \mathcal{L}(\mathcal{H})\big), \tag{11}$$

$$\Big(\tilde{b} = \sup_{\alpha} \tilde{b}_\alpha \text{ in } \mathcal{L}\Big(\bigvee^p \mathcal{Z}\Big)\Big) \quad \Rightarrow \quad \big(\langle \mathbf{N}\rangle^{-p/2} b^{Wick} \langle \mathbf{N}\rangle^{-p/2} = \sup_{\alpha} \langle \mathbf{N}\rangle^{-p/2} b_\alpha^{Wick} \langle \mathbf{N}\rangle^{-p/2} \text{ in } \mathcal{L}(\mathcal{H})\big). \tag{12}$$

An important property of our class of Wick polynomials is that a composition of $b_1^{Wick} \circ b_2^{Wick}$ with $b_1, b_2 \in \mathcal{P}_{alg}(\mathcal{Z})$ is a Wick polynomial with symbol in $\mathcal{P}_{alg}(\mathcal{Z})$. For $b_1 \in \mathcal{P}_{p_1,q_1}(\mathcal{Z})$, $b_2 \in \mathcal{P}_{p_2,q_2}(\mathcal{Z})$, $k \in \mathbb{N}$ and any fixed $z \in \mathcal{Z}$, $\partial_z^k b_1(z) \in \mathcal{L}(\bigvee^k \mathcal{Z}; \mathbb{C})$ while $\partial_{\bar{z}}^k b_2(z) \in \bigvee^k \mathcal{Z}$. The $\mathbb{C}$-bilinear duality product $\partial_z^k b_1(z).\partial_{\bar{z}}^k b_2(z)$ defines a function of $z \in \mathcal{Z}$ simply denoted by $\partial_z^k b_1.\partial_{\bar{z}}^k b_2$. We also use the following notation for multiple Poisson brackets:

$$\{b_1, b_2\}^{(k)} = \partial_z^k b_1.\partial_{\bar{z}}^k b_2 - \partial_z^k b_2.\partial_{\bar{z}}^k b_1, \quad k \in \mathbb{N},$$
$$\{b_1, b_2\} = \{b_1, b_2\}^{(1)}.$$

**Proposition 2.4.** *Let $b_1 \in \mathcal{P}_{p_1,q_1}(\mathcal{Z})$ and $b_2 \in \mathcal{P}_{p_2,q_2}(\mathcal{Z})$. For any $k \in \{0, \dots, \min\{p_1, q_2\}\}$, $\partial_z^k b_1.\partial_{\bar{z}}^k b_2$ belongs to $\mathcal{P}_{p_2-k,q_1-k}(\mathcal{Z})$ with the estimate*:

$$\big|\partial_z^k b_1.\partial_{\bar{z}}^k b_2\big|_{\mathcal{P}_{p_2,q_1}} \leqslant \frac{p_1!}{(p_1-k)!}\frac{q_2!}{(q_2-k)!}|b_1|_{\mathcal{P}_{p_1,q_1}}|b_2|_{\mathcal{P}_{p_2,q_2}}.$$

*The formulas,*

$$\text{(i)} \quad b_1^{Wick} \circ b_2^{Wick} = \left(\sum_{k=0}^{\min\{p_1,q_2\}} \frac{\varepsilon^k}{k!}\partial_z^k b_1.\partial_{\bar{z}}^k b_2\right)^{Wick} = \big(e^{\varepsilon\langle\partial_z,\partial_{\bar{\omega}}\rangle} b_1(z) b_2(\omega)|_{z=\omega}\big)^{Wick},$$

$$\text{(ii)} \quad \big[b_1^{Wick}, b_2^{Wick}\big] = \left(\sum_{k=1}^{\max\{\min\{p_1,q_2\},\min\{p_2,q_1\}\}} \frac{\varepsilon^k}{k!}\{b_1, b_2\}^{(k)}\right)^{Wick},$$

*hold as identities on $\mathcal{H}_{fin}$.*

## *2.3. Cylindrical functions and Weyl quantization*

Let $\mathbb{P}$ denote the set of all finite rank orthogonal projections on $\mathcal{Z}$ and for a given $p \in \mathbb{P}$ let $L_p(dz)$ denote the Lebesgue measure on the finite dimensional subspace $p\mathcal{Z}$. A function $f : \mathcal{Z} \to \mathbb{C}$ is said cylindrical if there exists $p \in \mathbb{P}$ and a function $g$ on $p\mathcal{Z}$ such that $f(z) = g(pz)$, for all $z \in \mathcal{Z}$. In this case we say that $f$ is based on the subspace $p\mathcal{Z}$. We set $\mathcal{S}_{cyl}(\mathcal{Z})$ to be the cylindrical Schwartz space:

$$\big(f \in \mathcal{S}_{cyl}(\mathcal{Z})\big) \quad \Leftrightarrow \quad \big(\exists p \in \mathbb{P},\ \exists g \in \mathcal{S}(p\mathcal{Z}),\ f(z) = g(pz)\big).$$

The Fourier transform of a function $f \in \mathcal{S}_{cyl}(\mathcal{Z})$ based on the subspace $p\mathcal{Z}$ is defined as

$$\mathcal{F}[f](z) = \int_{p\mathcal{Z}} f(\xi) e^{-2\pi i S(z,\xi)}\, L_p(d\xi)$$

and its inverse Fourier transform is

$$f(z) = \int_{p\mathcal{Z}} \mathcal{F}[f](z) e^{2\pi i S(z,\xi)}\, L_p(dz).$$

With any symbol $b \in \mathcal{S}_{cyl}(\mathcal{Z})$ based on $p\mathcal{Z}$, a *Weyl observable* can be associated according to

$$b^{Weyl} = \int_{p\mathcal{Z}} \mathcal{F}[b](z) W(\sqrt{2}\pi z)\, L_p(dz). \tag{13}$$

After the tensor decompositions

$$\mathcal{H} = \Gamma_s(\mathcal{Z}) = \Gamma_s(p\mathcal{Z}) \otimes \Gamma_s\big((1-p)\mathcal{Z}\big) \quad \text{due to } \mathcal{Z} = p\mathcal{Z} \overset{\perp}{\oplus} (1-p)\mathcal{Z} \quad \forall z \in p\mathcal{Z},$$
$$W(\sqrt{2}\pi z) = W_{p\mathcal{Z}}(\sqrt{2}\pi z) \otimes I_{\Gamma_s(1-p)\mathcal{Z}}$$

where $W_{p\mathcal{Z}}$ denotes the reduced representation in $\Gamma_s(p\mathcal{Z})$, one sees that the Weyl quantization of cylindrical observables based on $p\mathcal{Z}$ amounts to the usual finite dimensional Weyl quantization. Hence more general classes of symbols can be considered.

For $p \in \mathbb{P}$, the symbol classes defined for $0 \leqslant \nu \leqslant 1$ on the finite dimensional phase space $p\mathcal{Z}$,

$$S^{\nu}_{p\mathcal{Z}} = \bigoplus_{n\in\mathbb{Z}}^{alg} S\left(\langle z\rangle^{n}_{p\mathcal{Z}}, \frac{dz^2}{\langle z\rangle^{2\nu}_{p\mathcal{Z}}}\right), \tag{14}$$

where $\langle z\rangle_p^2 = 1 + |z|^2_{p\mathcal{Z}}$, are natural Weyl–Hörmander algebras associated with the finite dimensional harmonic oscillator Hamiltonian, $\mathbf{N}_p = (|z|^2_{p\mathcal{Z}})^{Wick} = (|z|^2_{p\mathcal{Z}})^{Weyl} - \frac{\dim p\mathcal{Z}}{2}\varepsilon$. They contain the polynomial functions on $p\mathcal{Z}$. The associated class of Weyl-quantized operators after tensorization with $I_{\Gamma_s((1-p))\mathcal{Z}}$ is denoted by $\mathrm{Op}\, S^{\nu}_{p\mathcal{Z}}$. For a cylindrical polynomial $b \in \mathcal{P}_{alg}(\mathcal{Z})$ based on $p\mathcal{Z}$, $b(z) = b(pz)$, the asymptotic equivalence of the Weyl and Wick quantization in finite dimension says for any $\nu \in [0, 1]$,

$$b^{Wick} = b^{Weyl} + \mathcal{O}_b(\varepsilon) \quad \text{in } \mathrm{Op}\, S^{\nu}_{p\mathcal{Z}}. \tag{15}$$

Such polynomials have finite rank kernels and make a dense set in $\mathcal{P}^{\infty}_{alg}(\mathcal{Z})$ *but not in* $\mathcal{P}_{alg}(\mathcal{Z})$.

### *2.4. Wick observables and BBGKY hierarchy*

When $\mathcal{Z} = L^2(\mathbb{R}^d)$, mean field results are often presented or even analyzed in terms of reduced density matrices or more precisely in terms of a sequence $(\gamma_\varepsilon^{(p)})_{p\in\mathbb{N}}$ with $\gamma_\varepsilon^{(p)} \in \mathcal{L}^1(\bigvee^p \mathcal{Z})$. This follows the general BBGKY approach of the kinetic theory and the $\gamma_\varepsilon^p$ correspond in the classical case to the empirical distributions.

The basic example is when $\varrho_\varepsilon \in \mathcal{L}^1(\bigvee^n \mathcal{Z})$, $n = [\frac{1}{\varepsilon}]$: For any $p \in \mathbb{N}$, $p \leqslant n$, $\gamma_\varepsilon^{(p)} \in \mathcal{L}^1(\bigvee^p \mathcal{Z})$ is defined as the partially traced operator with the kernel

$$\gamma_\varepsilon^{(p)}(x_1, \dots, x_p; y_1, \dots, y_p) := \int_{\mathbb{R}^{d(n-p)}} \varrho_\varepsilon(x_1, \dots, x_p, X, y_1, \dots, y_p, X)\, L_{\mathbb{R}^{d(n-p)}}(dX).$$

With the polarization identity (4), the family $(|\psi^{\otimes n}\rangle\langle\psi^{\otimes n}|)_{\psi\in\mathcal{Z}}$ forms a total set of $\mathcal{L}^1(\bigvee^n \mathcal{Z})$. Hence the formal identity

$$\begin{aligned}
&\varepsilon^p \frac{n!}{(n-p)!} |\psi|^{2(n-p)} \psi(x_1)\dots\psi(x_p)\overline{\psi(y_1)}\dots\overline{\psi(y_p)} \\
&\quad = \langle a(y_1)\dots a(y_p)\psi^{\otimes n}, a(x_1)\dots a(x_p)\psi^{\otimes n}\rangle \\
&\quad = \mathrm{Tr}\big[a^*(y_1)\dots a^*(y_p)a(x_1)\dots a(x_p)\big|\psi^{\otimes n}\big\rangle\big\langle\psi^{\otimes n}\big|\big]
\end{aligned}$$

carries over to $\varrho_\varepsilon \in \mathcal{L}^1(\bigvee^n \mathcal{Z})$:

$$\forall p \in \{1, \dots, n\}, \quad \varepsilon^p \frac{n!}{(n-p)!} \gamma_\varepsilon^{(p)}(x_1, \dots, x_p, y_1, \dots, y_p) = \mathrm{Tr}\big[a^*(y_1)\dots a^*(y_p)a(x_1)\dots a(x_p)\varrho_\varepsilon\big].$$

The correct meaning of this definition is:

$$\mathrm{Tr}\big[\gamma_\varepsilon^{(p)}\tilde{b}\big] = \frac{1_{[p,+\infty)}(n)}{\varepsilon^p n(n-1)\dots(n-p+1)} \mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big], \quad \forall b \in \mathcal{P}_{p,p}(\mathcal{Z}).$$

Moreover after noticing that the factor $\varepsilon^p n(n-1)\dots(n-p+1)$ is nothing but $\mathrm{Tr}[\varrho_\varepsilon(|z|^{2p})^{Wick}]$ when $\mathrm{Tr}[\varrho_\varepsilon] = 1$ and $\varrho_\varepsilon \in \mathcal{L}^1(\bigvee^n \mathcal{Z})$, it becomes:

$$\mathrm{Tr}\big[\gamma_\varepsilon^{(p)}\tilde{b}\big] = \frac{\mathrm{Tr}[\varrho_\varepsilon]}{\mathrm{Tr}[\varrho_\varepsilon(|z|^{2p})^{Wick}]} \mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big], \quad \forall b \in \mathcal{P}_{p,p}(\mathcal{Z}), \tag{16}$$

with the convention that the right-hand side is 0 when $\mathrm{Tr}[\varrho_\varepsilon(|z|^{2p})^{Wick}] = 0$. The extension to general $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{H})$ requires an assumption. Moreover it works for a general separable Hilbert space $\mathcal{Z}$.

**Proposition 2.5.** *Assume that $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{H})$ satisfies $\varrho_\varepsilon \geqslant 0$ and $\mathbf{N}^{k/2}\varrho_\varepsilon \mathbf{N}^{k/2} \in \mathcal{L}^1(\mathcal{H})$ for all $k \in \mathbb{N}$. Then for any $p \in \mathbb{N}$, the relation* (16) *defines a unique element $\gamma_\varepsilon^{(p)} \geqslant 0$ of $\mathcal{L}^1(\bigvee^p \mathcal{Z})$.*

**Proof.** Suppose $\mathrm{Tr}[\varrho_\varepsilon(|z|^{2p})^{Wick}] > 0$. Writing,

$$\mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big] = \mathrm{Tr}\big[(1+\mathbf{N})^{p/2}\varrho_\varepsilon(1+\mathbf{N})^{p/2}(1+\mathbf{N})^{-p/2}b^{Wick}(1+\mathbf{N})^{-p/2}\big],$$

with our assumptions and the estimates (10) ensures that $\tilde{b} \to \mathrm{Tr}[\varrho_\varepsilon b^{Wick}]$ defines a continuous linear form on $\mathcal{L}(\bigvee^p \mathcal{Z})$. The positivity comes from (11) and the normality of the associated state after normalization, which says $\gamma_\varepsilon^{(p)} \in \mathcal{L}^1(\bigvee^p \mathcal{Z})$, is a consequence of (12). □

We end with this discussion with a natural definition:

**Definition 2.6.** When $\varrho_\varepsilon \in \mathcal{L}^1(\mathcal{Z})$ satisfies $\varrho_\varepsilon \geqslant 0$ and $\mathbf{N}^{k/2}\varrho_\varepsilon\mathbf{N}^{k/2} \in \mathcal{L}^1(\mathcal{H})$ for all $k \in \mathbb{N}$, the reduced density matrix $\gamma_\varepsilon^{(p)}$, $p \in \mathbb{N}$, associated with $\varrho_\varepsilon$ is the element of $\mathcal{L}^1(\bigvee^p \mathcal{Z})$ defined by:

$$\mathrm{Tr}\big[\gamma_\varepsilon^{(p)}\tilde{b}\big] = \frac{\mathrm{Tr}[\varrho_\varepsilon]}{\mathrm{Tr}[\varrho_\varepsilon(|z|^{2p})^{Wick}]}\,\mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big], \quad \forall b \in \mathcal{P}_{p,p}(\mathcal{Z}), \tag{17}$$

with $\gamma_\varepsilon^{(p)} = 0$ in the case when $\mathrm{Tr}[\varrho_\varepsilon(|z|^{2p})^{Wick}] = 0$.

### 2.5. Wigner measures

The Wigner measures are defined after the next result proved in [7, Theorem 6.2].

**Theorem 2.7.** *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states on $\mathcal{H}$ parametrized by $\varepsilon$. Assume $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\delta] \leqslant C_\delta$ uniformly w.r.t. $\varepsilon \in (0,\bar{\varepsilon})$ for some fixed $\delta > 0$ and $C_\delta \in (0,+\infty)$. Then for every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\varepsilon_n = 0$ there exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a Borel probability measure $\mu$ on $\mathcal{Z}$ such that*

$$\lim_{k\to\infty}\mathrm{Tr}\big[\varrho_{\varepsilon_{n_k}} b^{Weyl}\big] = \int_{\mathcal{Z}} b(z)\,d\mu(z),$$

*for all $b \in \bigcup_{p\in\mathbb{P}}\mathcal{F}^{-1}(\mathcal{M}_b(p\mathcal{Z}))$.*

*Moreover this probability measure $\mu$ satisfies $\int_{\mathcal{Z}}|z|^{2\delta}\,d\mu(z) < \infty$.*

**Definition 2.8.** The set of Wigner measures associated with a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ (resp. a sequence $(\varrho_{\varepsilon_n})_{n\in\mathbb{N}}$) which satisfies the assumptions of Theorem 2.7 is denoted by

$$\mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big) \quad \big(\text{resp. } \mathcal{M}(\varrho_{\varepsilon_n}, n \in \mathbb{N})\big).$$

Wigner measures are in practice identified via their characteristic functions according to the relation

$$\mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big) = \{\mu\} \quad \Leftrightarrow \quad \lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon W(\sqrt{2}\pi\xi)\big] = \mathcal{F}(\mu)(\xi).$$

The expression $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) = \{\mu\}$ simply means that the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ is "pure" in the sense,

$$\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon b^{Weyl}\big] = \int_{\mathcal{Z}} b(z)\,d\mu,$$

for all cylindrical symbol $b$ without extracting a subsequence. Actually the general case can be reduced to this after reducing the range of parameter to $\varepsilon \in \{\varepsilon_{n_k}, k \in \mathbb{N}\}$.

A simple a priori estimate argument allows to extend the convergence to symbols which have a polynomial growth and to test to Wick-quantized symbols with compact kernels belonging to $\mathcal{P}_{alg}^\infty(\mathcal{Z})$ (see [7, Corollary 6.14]).

**Proposition 2.9.** *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{L}(\mathcal{H})$ parametrized by $\varepsilon$ such that $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\alpha] \leqslant C_\alpha$ holds uniformly with respect to $\varepsilon \in (0,\bar\varepsilon)$ for all $\alpha \in \mathbb{N}$ and such that $\mathcal{M}(\varrho_\varepsilon, \varepsilon\in(0,\bar\varepsilon)) = \{\mu\}$. Then the convergence,*

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\big[\varrho_\varepsilon b^{quantized}\big] = \int_{\mathcal{Z}} b(z)\,d\mu(z), \tag{18}$$

*holds for the Weyl quantization of any $b \in S^\nu_{p\mathcal{Z}}$ with $p \in \mathbb{P}$ and $\nu \in [0,1]$, and for the Wick quantization of any $b \in \mathcal{P}^\infty_{alg}(\mathcal{Z})$.*

Wigner measures are completely identified by testing with Weyl-quantized observable and possibly by restricting to some countable subset $\bigcup_{n\in\mathbb{N}} D_{n,p_n}$ where $D_{n,p_n}$ is a countable dense subset of $\mathcal{F}^{-1}(\mathcal{M}_b(p_n\mathcal{Z}))$, and $(p_n)_{n\in\mathbb{N}}$ is a sequence of $\mathbb{P}$ such that $\sup_{n\in\mathbb{N}} p_n = I_{\mathcal{Z}}$ (see [7]). One may question whether testing on all the $b^{Wick}$ with $b \in \mathcal{P}^\infty_{alg}(\mathcal{Z})$ also identifies the Wigner measures. When $\mathcal{Z}$ is finite dimensional, this amounts to the well-known Hambürger moment problem of identifying a probability measure $\nu$ on $\mathbb{R}$ from its moments $a_n = \int_{\mathbb{R}} x^n\,d\nu(x)$, $n\in\mathbb{N}$, for which uniqueness fails without growth conditions on the sequence $(a_n)_{n\in\mathbb{N}}$ [47,5], which can be translated in our case to growth conditions of $(\sup_{\varepsilon\in(0,\bar\varepsilon)} \mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\alpha])_{\alpha\in\mathbb{N}}$. We shall circumvent this difficulty, by identifying the Wigner measures in two steps by approximating the states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ by states $(\varrho_\varepsilon^{app})_{\varepsilon\in(0,\bar\varepsilon)}$ for which the growth condition is satisfied. We shall reconsider the moment problem later, but the comparison argument is given below.

**Proposition 2.10.** *Let $(\varrho^j_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$, $j=1,2$, be two families (or sequences) of normal states on $\mathcal{H}$ such that $\mathrm{Tr}[\varrho^j_\varepsilon \mathbf{N}^\delta] \leqslant C_\delta$ uniformly w.r.t. $\varepsilon\in(0,\bar\varepsilon)$ for some $\delta>0$ and $C_\delta\in(0,+\infty)$. Assume further $\mathcal{M}(\varrho^j_\varepsilon, \varepsilon\in(0,\bar\varepsilon)) = \{\mu_j\}$ for $j=1,2$. Then*

$$\int |\mu_1-\mu_2| \leqslant \liminf_{\varepsilon\to 0} \big|\varrho^1_\varepsilon - \varrho^2_\varepsilon\big|_{\mathcal{L}^1(\mathcal{H})}.$$

**Proof.** For a symbol $b\in\mathcal{S}_{cyl}(\mathcal{Z})$, the finite dimensional Weyl semiclassical calculus says $|b^{Weyl}|_{\mathcal{L}(\mathcal{H})} \leqslant \|b\|_\infty + \mathcal{O}_b(\varepsilon)$ with $\|b\|_\infty = \|b\|_{L^\infty(p\mathcal{Z})}$. This implies for a given $b\in\mathcal{S}_{cyl}(\mathcal{Z})$,

$$\left|\int_{\mathcal{Z}} b(z)\,d(\mu_1-\mu_2)(z)\right| = \lim_{\varepsilon\to 0}\big|\mathrm{Tr}\big[\big(\varrho^1_\varepsilon - \varrho^2_\varepsilon\big)b^{Weyl}\big]\big| \leqslant \|b\|_\infty \liminf_{\varepsilon\to 0}\big|\varrho^1_\varepsilon-\varrho^2_\varepsilon\big|_{\mathcal{L}^1(\mathcal{H})}.$$

The measure $\mu_1-\mu_2$ is absolutely continuous with respect to the Borel probability measure $\frac{\mu_1+\mu_2}{2}$. Hence there exists a Borel function $\lambda$ on $\mathcal{Z}$ such that $\mu_1-\mu_2 = \lambda(z)\frac{\mu_1+\mu_2}{2}$ with the additional property $|\lambda(z)|\leqslant 2$ $\frac{\mu_1+\mu_2}{2}$-almost everywhere. But for any Borel probability measure $\nu$ on $\mathcal{Z}$, it was checked in [7] that $\mathcal{S}_{cyl}(\mathcal{Z})$ is dense in $L^p(\mathcal{Z},\nu)$ for $p\in[1,\infty)$ on the basis of a general measurable version of Stone–Weierstrass theorem (see for instance [19]). Hence there exists a sequence $(\beta_n)_{n\in\mathbb{N}}$ in $\mathcal{S}_{cyl}(\mathcal{Z})$ such that

$$\lim_{n\to\infty}\left\|\beta_n - \frac{|\lambda|}{\lambda}1_{\{\lambda\neq 0\}}\right\|_{L^1(\mathcal{Z},\frac{\mu_1+\mu_2}{2})} = 0$$

and after extraction $\lim_{k\to\infty}\beta_{n_k}(z) = \frac{|\lambda|}{\lambda}(z)1_{\{\lambda\neq0\}}(z)$, $\frac{\mu_1+\mu_2}{2}$-almost everywhere. By setting $b_k = 2\frac{\beta_{n_k}}{1+|\beta_{n_k}|^2}$, we get a sequence $(b_k)_{k\in\mathbb{N}}$ such that

$$\forall k\in\mathbb{N},\ b_k\in\mathcal{S}_{cyl} \text{ and } \|b_k\|_\infty\leqslant 1,\qquad \lim_{k\to\infty} b_k(z) = \frac{|\lambda|(z)}{\lambda(z)}1_{\{\lambda\neq0\}}(z)\frac{\mu_1+\mu_2}{2}\quad \text{a.e.}$$

We conclude with

$$\begin{aligned}\int|\mu_1-\mu_2| &= \int_{\mathcal{Z}}\big|\lambda(z)\big|\,d\frac{\mu_1+\mu_2}{2}(z)\\ &= \left|\lim_{k\to\infty}\int_{\mathcal{Z}} b_k(z)\,d\big(\mu^1-\mu^2\big)(z)\right|\\ &\leqslant 1\times\liminf_{\varepsilon\to0}\big|\varrho^1_\varepsilon-\varrho^2_\varepsilon\big|_{\mathcal{L}^1(\mathcal{H})}. \qquad \square\end{aligned}$$

When the two sets $\mathcal{M}(\varrho_\varepsilon^j, \varepsilon \in (0, \bar{\varepsilon}))$ have more than one element, the extraction of subsequences, $(\varepsilon_n)_{n\in\mathbb{N}}$, can be made simultaneously and the result has to be modified into:

$$\inf_{(\mu_1,\mu_2)\in\mathcal{M}(\varrho_\varepsilon^1,\varepsilon\in(0,\bar{\varepsilon}))\times\mathcal{M}(\varrho_\varepsilon^2,\varepsilon\in(0,\bar{\varepsilon}))} \int |\mu_1 - \mu_2| \leqslant \limsup_{\varepsilon\to 0} |\varrho_\varepsilon^1 - \varrho_\varepsilon^2|_{\mathcal{L}^1(\mathcal{H})}. \tag{19}$$

*2.6. Wigner measures and the BBGKY hierarchy*

The compactness condition $b \in \mathcal{P}_{alg}^\infty(\mathcal{Z})$ for the Wick quantization in Proposition 2.9 is not a technical restriction and the convergence is no more true for a general $b \in \mathcal{P}_{alg}(\mathcal{Z})$. It was identified in [7] as a "dimensional defect of compactness" and illustrated with examples, one of them being related with the Bose–Einstein condensation of the free Bose gas.

This terminology comes from the idea that this defect of compactness does not come from the infinity in the phase space like in the finite dimensional case (see [53,28]) but from the non-compactness in the norm topology of balls in infinite dimension. Actually this was made more accurate in [8]: under the assumptions $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0, \bar{\varepsilon})) = \{\mu\}$ and $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^k] \leqslant \lambda^k$, we proved (T) $\Rightarrow$ (P) with,

$$\text{(P)} \quad \forall b \in \mathcal{P}_{alg}(\mathcal{Z}), \quad \lim_{\varepsilon\to 0} \mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big] = \int_{\mathcal{Z}} b(z)\, d\mu(z),$$

$$\text{(T)} \quad \forall \eta > 0,\ \exists P_\eta \in \mathbb{P}, \quad \mathrm{Tr}\big[\big(1 - \Gamma(P_\eta)\big)\varrho_\varepsilon\big] < \eta,$$

where (T) appears as a quantum Prokhorov condition (or tightness condition in the strong topology).

The condition (P) which will be simplified in the next subsection, actually contains, for all $\alpha \in \mathbb{N}$, the uniform bound w.r.t. $\varepsilon$ of $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\alpha]$ since $\mathbf{N}^\alpha = [(|z|^2)^{Wick}]^\alpha$. It implies actually a strong relationship between the Wigner measure formulation and the convergence of reduced density matrices.

**Proposition 2.11.** *Assume that* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ *is a family of* $\mathcal{L}^1(\mathcal{H})$ *with* $\varrho_\varepsilon \geqslant 0$, $\mathrm{Tr}[\varrho_\varepsilon] = 1$, $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0, \bar{\varepsilon})) = \{\mu\}$ *with the condition* (P) *and assume* $\mu \neq \delta_0$. *Define for* $p \in \mathbb{N}$,

$$\gamma_0^{(p)} := \frac{1}{\int_{\mathcal{Z}} |z|^{2p}\, d\mu(z)} \int_{\mathcal{Z}} \big|z^{\otimes p}\big\rangle\big\langle z^{\otimes p}\big|\, d\mu(z). \tag{20}$$

*Then for all* $p \in \mathbb{N}$, *the reduced density matrix* $\gamma_\varepsilon^{(p)}$ *converges to* $\gamma_0^{(p)}$ *in the* $\mathcal{L}^1$*-norm.*

**Proof.** For $p = 0$, the result is nothing but $1 = \int \mu = \lim_{\varepsilon\to 0} \mathrm{Tr}[\varrho_\varepsilon] = 1$.

For $p \in \mathbb{N}^*$, the condition (P) with $\mu \neq \delta_0$ says first,

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\big[\varrho_\varepsilon \big(|z|^{2p}\big)^{Wick}\big] = \int_{\mathcal{Z}} |z|^{2p}\, d\mu(z) > 0.$$

Hence, the reduced density matrix $\gamma_\varepsilon^{(p)}$ is well defined according to Definition 2.6 for $\varepsilon < \bar{\varepsilon}_p$ small enough (with $\mathrm{Tr}[\varrho_\varepsilon] = 1$). The condition (P) gives the general convergence:

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\big[\gamma_\varepsilon^{(p)} \tilde{b}\big] = \lim_{\varepsilon\to 0} \frac{\mathrm{Tr}[\varrho_\varepsilon b^{Wick}]}{\mathrm{Tr}[\varrho_\varepsilon (|z|^{2p})^{Wick}]} = \frac{\int_{\mathcal{Z}} b(z)\, d\mu(z)}{\int_{\mathcal{Z}} |z|^{2p}\, d\mu(z)} = \mathrm{Tr}\big[\gamma_0^{(p)} \tilde{b}\big],$$

for all $b \in \mathcal{P}_{p,p}(\mathcal{Z})$, where the last equality is a $\mu$-integration of the equality of continuous functions,

$$b(z) = \big\langle z^{\otimes p}, \tilde{b} z^{\otimes p}\big\rangle = \mathrm{Tr}\big[\big|z^{\otimes p}\big\rangle\big\langle z^{\otimes p}\big| \tilde{b}\big].$$

This proves the weak convergence of $\gamma_\varepsilon^{(p)}$ to $\gamma_0^{(p)}$ in $\mathcal{L}^1(\bigvee^p \mathcal{Z})$. But since $\gamma^{(p)_\varepsilon}$ and $\gamma_0^{(p)}$ are non-negative with $\mathrm{Tr}[\gamma_\varepsilon^{(p)}] = 1 = \mathrm{Tr}[\gamma_0^{(p)}]$, this implies the norm convergence according to [50,4,20].[2] □

[2] In a more general framework, it is said that $\mathcal{L}^1(\bigvee^p \mathcal{Z})$ has a uniform Kadec–Klee property (see [40] and references therein).

### 2.7. A simple criterion for the reliability of Wick observables

The proof of Proposition 2.11 can be adapted in order to make an equivalent condition to (P) with a weaker and easier to handle formulation:

$$\text{(PI)}\quad \forall\alpha\in\mathbb{N},\quad \lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon \mathbf{N}^\alpha\big]=\int_{\mathcal{Z}}|z|^{2\alpha}\,d\mu(z)<+\infty.$$

**Proposition 2.12.** *For a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ in $\mathcal{L}^1(\mathcal{H})$ such that $\varrho_\varepsilon\geqslant 0$, $\mathrm{Tr}[\varrho_\varepsilon]=1$, $\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon))=\{\mu\}$, the conditions* (P) *and* (PI) *are equivalent*:

$$\left(\forall\alpha\in\mathbb{N},\ \lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon\mathbf{N}^\alpha\big]=\int_{\mathcal{Z}}|z|^{2\alpha}\,d\mu(z)\right)\quad\Leftrightarrow\quad\left(\forall b\in\mathcal{P}_{alg}(\mathcal{Z}),\ \lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big]=\int_{\mathcal{Z}}b\,d\mu\right).$$

**Proof.** The condition (PI) is a particular case of (P). Let us prove (PI) $\Rightarrow$ (P).
We start with two remarks:

- For $k\in\mathbb{N}^*$, $(|z|^{2k})^{Wick}=\mathbf{N}(\mathbf{N}-\varepsilon)\dots(\mathbf{N}-(k-1)\varepsilon)$. Hence the condition (PI) is equivalent to:
$$\forall\alpha\in\mathbb{N},\quad\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon\big(|z|^{2\alpha}\big)^{Wick}\big]=\int_{\mathcal{Z}}|z|^{2\alpha}\,d\mu(z).$$
- For $p=0$ (resp. $q=0$) the operators in $\mathcal{L}(\mathbb{C},\bigvee^q\mathcal{Z})$ (resp. in $\mathcal{L}(\bigvee^p\mathcal{Z},\mathbb{C})$) are compact and $\mathcal{P}_{0,q}(\mathcal{Z})=\mathcal{P}^\infty_{0,q}(\mathcal{Z})$ (resp. $\mathcal{P}_{p,0}(\mathcal{Z})=\mathcal{P}^\infty_{p,0}(\mathcal{Z})$). Hence the convergence $\lim_{\varepsilon\to 0}\mathrm{Tr}[\varrho_\varepsilon b^{Wick}]=\int b\,d\mu$, is consequence of Proposition 2.9 when $p=0$ or $q=0$.

According to Proposition 2.11, there are two cases.

**If $\mu=\delta_0$:** Then for $b\in\mathcal{P}_{p,p}(\mathcal{Z})$, $p\in\mathbb{N}^*$, such that $\tilde b\geqslant 0$, the inequality $0\leqslant\tilde b\leqslant|b|_{\mathcal{P}_{p,p}}I_{\bigvee^p\mathcal{Z}}$ and the positivity (11) says:

$$0\leqslant\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big]\leqslant\lim_{\varepsilon\to 0}|b|_{p,p}\,\mathrm{Tr}\big[\varrho_\varepsilon\big(|z|^{2p}\big)^{Wick}\big]=\int_{\mathcal{Z}}|z|^{2p}\delta_0(z)=0.$$

For a general $b\in\mathcal{P}_{p,p}(\mathcal{Z})$, $p\in\mathbb{N}^*$, the decomposition $\tilde b=\tilde b_{R,+}-\tilde b_{R,-}+i\tilde b_{I,+}-i\tilde b_{I,-}$ with all the $\tilde b_\bullet\geqslant 0$ now gives:

$$\forall p\in\mathbb{N}^*,\ \forall b\in\mathcal{P}_{p,p}(\mathcal{Z}),\quad\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big]=0.$$

For $p\neq q$, $p,q\in\mathbb{N}^*$, write

$$\big|\mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big]\big|=\big|\mathrm{Tr}\big[\varrho_\varepsilon^{1/2}\big(\varrho_\varepsilon^{1/2}b^{Wick}\big)\big]\big|\leqslant\mathrm{Tr}[\varrho_\varepsilon]^{1/2}\,\mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}b^{Wick,*}\big]^{1/2}.$$

Proposition 2.4 says that $b^{Wick}b^{Wick,*}=\sum_{\ell=0}^p\frac{\varepsilon^\ell}{\ell!}\partial_z^\ell b.\partial_{\bar z}^\ell\bar b$ belongs to $\bigoplus_{k=0}^{p+q}\mathcal{P}_{k,k}(\mathcal{Z})$ with an $\mathcal{O}(\varepsilon)$ term in $\mathcal{P}_{0,0}(\mathcal{Z})$. We have proved,

$$\forall p,q\in\mathbb{N}^*,\ \forall b\in\mathcal{P}_{p,q}(\mathcal{Z}),\quad\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big]=0=\int_{\mathcal{Z}}b(z)\,\delta_0(z),$$

while the cases $(0,q)$ and $(p,0)$ are already known.

**If $\mu\neq\delta_0$:** Then we know by Proposition 2.11 that $\lim_{\varepsilon\to 0}\|\gamma_\varepsilon^{(p)}-\gamma_0^{(p)}\|_{\mathcal{L}^1}=0$, which implies:

$$\forall b\in\mathcal{P}_{p,p}(\mathcal{Z}),\quad\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big]=\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\gamma_\varepsilon^{(p)}\tilde b\big]=\mathrm{Tr}\big[\gamma_0^{(p)}\tilde b\big]=\int_{\mathcal{Z}}b(z)\,d\mu(z).$$

Let us consider the general case $b \in \mathcal{P}_{p,q}(\mathcal{Z})$. The above convergence is still true when the kernel $\tilde{b}$ is compact by Proposition 2.9. Consider now a general $b \in \mathcal{P}_{p,q}(\mathcal{Z})$. Since $\int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes q}|\, d\mu(z)$ is nuclear (or trace-class in $\bigvee^q \mathcal{Z} \oplus \bigvee^p \mathcal{Z}$), for any $n \in \mathbb{N}$ there exists a compact operator $\tilde{b}_n \in \mathcal{L}^\infty(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})$ such that $|b_n|_{\mathcal{P}_{p,q}} = |\tilde{b}_n|_{\mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})} = |\tilde{b}|_{\mathcal{L}(\bigvee^p \mathcal{Z}, \bigvee^q \mathcal{Z})} = |b|_{\mathcal{P}_{p,q}}$, and

$$\left|\int_{\mathcal{Z}} \big(b(z) - b_n(z)\big)\, d\mu(z)\right| = \left|\operatorname{Tr}\left[\int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes q}|\, d\mu(z)[\tilde{b} - \tilde{b}_n]\right]\right| \leqslant \frac{1}{n+1}.$$

The Lebesgue convergence theorem with,

$$\forall n \in \mathbb{N}, \quad \big|b(z) - b_n(z)\big|^r \leqslant \big(2|b|_{\mathcal{P}_{p,q}}\big)^r |z|^{r(p+q)}, \qquad \int_{\mathcal{Z}} |z|^{r(p+q)}\, d\mu(z) < \infty,$$

$$\forall z \in \mathcal{Z}, \quad \lim_{n\to\infty} b_n(z) = \lim_{n\to\infty} \big\langle z^{\otimes q}, \tilde{b}_n z^{\otimes p}\big\rangle = b(z),$$

yields

$$\lim_{n\to\infty} \int_{\mathcal{Z}} \big|b(z) - b_n(z)\big|^r\, d\mu(z) = 0.$$

Set $\eta_r(n) = \int_{\mathcal{Z}} |b(z) - b_n(z)|^r\, d\mu(z)$ and use again the Cauchy–Schwarz inequality,

$$\big|\operatorname{Tr}\big[\varrho_\varepsilon\big(b^{Wick} - b_n^{Wick}\big)\big]\big| \leqslant \operatorname{Tr}\big[\varrho_\varepsilon\big(b^{Wick} - b_n^{Wick}\big)\big(b^{Wick,*} - b_n^{Wick,*}\big)\big]^{1/2}.$$

Owing to the result valid when $p = q$ we deduce:

$$\limsup_{\varepsilon\to 0}\big|\operatorname{Tr}\big[\varrho_\varepsilon\big(b^{Wick} - b_n^{Wick}\big)\big]\big| \leqslant \left[\int_{\mathcal{Z}} \big|b(z) - b_n(z)\big|^2\, d\mu(z)\right]^{1/2} = \eta_2(n)^{1/2}.$$

Since for $n \in \mathbb{N}$ fixed, $\lim_{\varepsilon\to 0} \operatorname{Tr}[\varrho_\varepsilon b_n^{Wick}] = \int_{\mathcal{Z}} b_n(z)\, d\mu(z)$, we deduce:

$$\forall n \in \mathbb{N}, \quad \limsup_{\varepsilon\to 0}\left|\operatorname{Tr}\big[\varrho_\varepsilon b^{Wick}\big] - \int_{\mathcal{Z}} b(z)\, d\mu(z)\right| \leqslant \frac{1}{n+1} + \eta_2(n)^{1/2},$$

while the right-hand side goes to 0 as $n \to \infty$. □

### *2.8. States localized in a ball*

The condition, $\operatorname{Tr}[\varrho_\varepsilon \mathbf{N}^\alpha] \leqslant \lambda^\alpha$ for all $\alpha \in \mathbb{N}$, used in [8] is actually equivalent to

$$\varrho_\varepsilon = 1_{[0,\lambda]}(\mathbf{N})\varrho_\varepsilon 1_{[0,\lambda]}(\mathbf{N})$$

(locate the spectral measure of $\varrho_\varepsilon$ for the self-adjoint operator $\mathbf{N}$). Such an assumption remains an important step in the present analysis, and $\mathbf{N} = (|z|^2)^{Wick}$ suggests that such a state is localized in ball of the phase-space.

**Definition 2.13.** A family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ (or a sequence $(\varrho_{\varepsilon_n})_{n\in\mathbb{N}}$) of normal states on $\mathcal{H}$, is said to be localized in the ball of radius $R > 0$, if $\varrho_\varepsilon = 1_{[0,R^2]}(\mathbf{N})\varrho_\varepsilon 1_{[0,R^2]}(\mathbf{N})$ for all $\varepsilon \in (0, \bar{\varepsilon})$.

The meaning of the geometric intuition contained in the terminology "localized in a ball of radius $R$", can be made more accurate.

**Lemma 2.14.** *For a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ (or a sequence $(\varrho_{\varepsilon_n})_{n\in\mathbb{N}}$) of normal states on $\mathcal{H}$ localized in a ball of radius $R > 0$, all its Wigner measures are supported in the ball $\{|z| \leqslant R\}$.*

**Proof.** A family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ localized in a ball of radius $R$ satisfies $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\delta] \leqslant R^{2\delta}$ for all $\delta > 0$. Therefore the set of Wigner measures $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon))$ is well defined and the convergence after extraction can be tested with Weyl-quantized cylindrical functions in the symbol class $S_p^\nu$ introduced in (14) for any $p \in \mathbb{P}$. Let $\mu \in \mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon))$ be associated with the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. For any finite rank projection $p \in \mathbb{P}$, the Wick-quantized operator $(|pz|^2)^{Wick}$ is $\mathbf{N}_p \otimes I_{\Gamma_s((1-p)\mathcal{Z})}$ where $\mathbf{N}_p$ is the number operator on $\Gamma_s(p\mathcal{Z})$ and equals $(|z|^2_{p\mathcal{Z}} - C_p\varepsilon)^{Weyl}$ in the finite dimensional framework of $p\mathcal{Z}$. For any cut-off function $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $[0, R^2]$, the finite dimensional Weyl semiclassical calculus tells us $(1-\chi)(\mathbf{N}_p) = (1-\chi)(|z|^2_{p\mathcal{Z}})^{Weyl} + \mathcal{O}_p(\varepsilon)$ in $\mathcal{L}(\Gamma_s(p\mathcal{Z}))$. Further the commutative decomposition $\mathbf{N} = \mathbf{N}_p \otimes I_{\Gamma_s((1-p)\mathcal{Z})} + I_{\Gamma_s(p\mathcal{Z})} \otimes \mathbf{N}_{(1-p)} \geqslant \mathbf{N}_p \otimes I_{\Gamma_s((1-p)\mathcal{Z})}$ and choosing $\chi$ decreasing on $[0,+\infty)$ implies:

$$(1-\chi)\big(|pz|^2\big)^{Weyl} + \mathcal{O}_p(\varepsilon) \leqslant (1-\chi)(\mathbf{N}_p \otimes I_{\Gamma((1-p)\mathcal{Z})}) \leqslant (1-\chi)(\mathbf{N}).$$

We deduce

$$0 \leqslant \int_{\mathcal{Z}} \big(1-\chi\big(|pz|\big)^2\big)\, d\mu(z) = \lim_{n\to\infty} \mathrm{Tr}\big[\varrho_\varepsilon \big(1-\chi\big(|pz|^2\big)\big)^{Weyl}\big]$$

$$\leqslant \lim_{n\to\infty} \mathrm{Tr}\big[\varrho_{\varepsilon_n} 1_{[0,R^2]}(\mathbf{N})\big(1-\chi(\mathbf{N})\big)\big] = 0.$$

Hence the measure $\mu$ vanishes outside a cylinder $\{|pz| \geqslant R\}$. This yields the result. □

With such localized states we can solve the moment problem.

**Proposition 2.15.** *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family (or a sequence $(\varrho_{\varepsilon_n})_{n\in\mathbb{N}}$) of normal states on $\mathcal{H}$, localized in the ball of radius $R > 0$. If there exists a Borel measure $\mu$ on $\mathcal{Z}$ such that*

$$\forall b \in \mathcal{P}^\infty_{alg}(\mathcal{Z}), \quad \lim_{\varepsilon\to 0} \mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big] = \int_{\mathcal{Z}} b(z)\, d\mu(z),$$

*then*

$$\mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)\big) = \{\mu\}.$$

**Proof.** Although this is shown in [7, Proposition 6.15], we provide here a different proof.

Let $p \in \mathbb{P}$ and consider the direct image by $p$ of the measure $\mu$:

$$\forall E \in \mathcal{B}(p\mathcal{Z}), \quad \mu_p(E) = \int_{\mathcal{Z}} 1_{p^{-1}(E)}(z)\, d\mu(z),$$

where $\mathcal{B}(p\mathcal{Z})$ denotes the Borel $\sigma$-set on $p\mathcal{Z}$.

For any $b \in \mathcal{P}^\infty_{alg}(\mathcal{Z})$, such that $b(pz) = b(z)$ we have:

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\big[\varrho_\varepsilon b^{Wick}\big] = \int_{p\mathcal{Z}} b(z)\, d\mu_p(z).$$

This holds in particular when $b(z) = |pz|^{2k}$ with $b^{Wick} = \mathbf{N}_p^k + \mathcal{O}(\varepsilon) \leqslant \mathbf{N}^k + \mathcal{O}(\varepsilon)$ with,

$$\int_{p\mathcal{Z}} |z|^{2k}\, d\mu_p(z) \leqslant \lim_{\varepsilon\to 0} \mathrm{Tr}\big[\varrho_\varepsilon \mathbf{N}_p^k\big] \leqslant \lim_{\varepsilon\to 0} \mathrm{Tr}\big[\varrho_\varepsilon \mathbf{N}^k\big] \leqslant R^{2k}.$$

Hence all the moments $\int_{p\mathcal{Z}} |z|^{2k}\, d\mu_p(z)$ are bounded by $R^{2k}$ and the finite dimensional moment problem applies (see [47,5]): $\mu_p$ is completely determined by the set of values $\{\int_{p\mathcal{Z}} b\, d\mu_p,\ b \text{ polynomial}\}$. Let $\mu'$ be a Wigner measure of the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$. It is supported in the ball $\{z \in \mathcal{Z},\ |z| \leqslant R\}$ so that its direct image by $p$, $\mu'_p$ is supported in the ball $\{z \in p\mathcal{Z},\ |z| \leqslant R\}$. Moreover there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, such that

$$\forall b \in S^\nu_{p\mathcal{Z}}, \quad \lim_{n\to\infty} \mathrm{Tr}\big[\varrho_{\varepsilon_n} b^{Weyl}\big] = \int_{p\mathcal{Z}} b(z)\, d\mu'_p(z),$$

where the $b^{Weyl}$ can be replaced by $b^{Wick}$ for any polynomial $b$ such that $b(z) = b(pz)$ according to the finite dimensional comparison of the Weyl and Wick calculus in (15). We deduce $\mu_p = \mu'_p$. Since this holds for all the $p \in \mathbb{P}$, this ends the proof. □

Let $\chi$ be a continuous cut-off function supported in [0, 1], with $0 \leqslant \chi \leqslant 1$ and such that $\chi \equiv 1$ in $[0, \frac{1}{2}]$. Within the assumptions of Theorem 2.7 and especially $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^\delta] \leqslant C_\delta$, the difference between the state $\varrho_\varepsilon$ and the localized state $\varrho_\varepsilon^{\chi,R} = \frac{1}{\mathrm{Tr}[\varrho_\varepsilon \chi^2(\frac{\mathbf{N}}{R^2})]}\chi(\frac{\mathbf{N}}{R^2})\varrho_\varepsilon\chi(\frac{\mathbf{N}}{R^2})$ can be made arbitrarily small according to

$$\forall \varepsilon \in (0, \bar{\varepsilon}), \quad \left|\varrho_\varepsilon - \varrho_\varepsilon^{\chi,R}\right|_{\mathcal{L}^1(\mathcal{H})} \leqslant \frac{K_\delta}{(R/2)^\delta - K_\delta}, \tag{21}$$

where the right-hand side vanishes as $R \to \infty$. Actually the previous estimate comes from $|\langle\mathbf{N}\rangle^{\delta/2}\varrho|_{\mathcal{L}^1} \leqslant |\langle\mathbf{N}\rangle^{\delta/2}\varrho\langle\mathbf{N}\rangle^{\delta/2}|_{\mathcal{L}^1} = \mathrm{Tr}[\varrho\langle\mathbf{N}\rangle^\delta] \leqslant C'_\delta$.

Then the comparison result in Proposition 2.10 or its variant (19) says that the Wigner measures $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ can be identified by its approximation by states localized in balls:

$$\inf_{(\mu,\mu')\in\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar{\varepsilon}))\times\mathcal{M}(\varrho_\varepsilon^{\chi,R},\varepsilon\in(0,\bar{\varepsilon}))} \int \left|\mu - \mu'\right| \leqslant \frac{C_\delta}{(R/2)^\delta - C_\delta}. \tag{22}$$

Then the question arises whether the family $(\varrho_\varepsilon^{\chi,R})_{\varepsilon\in(0,\bar{\varepsilon})}$, or an extracted subsequence, satisfies the condition (PI) (or equivalently (P)) if the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ does.

**Proposition 2.16.** *Assume that the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ of normal states on $\mathcal{H}$ satisfies $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) = \{\mu\}$ and the condition* (PI). *Let the function $f \in \mathcal{C}^0([0,+\infty),\mathbb{R})$ be polynomially bounded such that the quantity $\mathrm{Tr}[\varrho_\varepsilon f^2(\mathbf{N})]$ is uniformly bounded from below for $\varepsilon \in (0,\bar{\varepsilon})$. Then the family $(\varrho_\varepsilon^f)_{\varepsilon\in(0,\bar{\varepsilon})}$ given by $\varrho_\varepsilon^f = \frac{1}{\mathrm{Tr}[\varrho_\varepsilon f^2(\mathbf{N})]} f(\mathbf{N})\varrho_\varepsilon f(\mathbf{N})$ has a unique Wigner measure $\mathcal{M}(\varrho_\varepsilon^f, \varepsilon \in (0,\bar{\varepsilon})) = \{\frac{f^2(|z|^2)\mu}{\int f^2(|z|^2)\,d\mu}\}$ and satisfies the condition* (PI).

We will need the next lemma:

**Lemma 2.17.** *Let the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ (or a sequence $(\varrho_{\varepsilon_n})_{n\in\mathbb{N}}$) of normal states be localized in the ball of radius $R$ and assume the condition* (PI) *with $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) = \{\mu\}$. Then the equality,*

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\left[e^{\alpha_1\mathbf{N}}\varrho_\varepsilon e^{\alpha_2\mathbf{N}} b^{Wick}\right] = \int_{\mathcal{Z}} e^{(\alpha_1+\alpha_2)|z|^2} b(z)\,d\mu(z), \tag{23}$$

*holds for all $\alpha_1, \alpha_2 \in \mathbb{C}$ and all $b \in \mathcal{P}_{alg}(\mathcal{Z})$.*

**Proof.** The right-hand side of (23) is the sum of the double series,

$$\sum_{k_1,k_2\in\mathbb{N}} \frac{(\alpha_1)^{k_1}(\alpha_2)^{k_2}}{k_1!k_2!} \int_{\mathcal{Z}} |z|^{2k_1+2k_2} b(z)\,d\mu(z),$$

for $\mu$ is a Borel probability measure supported in $\{|z| \leqslant R\}$ and $b$ is a polynomial function.

Due to $\varrho_\varepsilon = \varrho_\varepsilon 1_{[0,R^2]}(\mathbf{N})$, the sum

$$S_{K_2,k} = \sum_{k_2=0}^{K_2} \varrho_\varepsilon \frac{(\alpha_2\mathbf{N})^{k_2}}{k_2!}(1+\mathbf{N})^k, \quad K_2, k \in \mathbb{N}$$

and the remainder term,

$$R_{K_2,k} = \varrho_\varepsilon e^{\alpha_2\mathbf{N}}(1+\mathbf{N})^k - S_{K_2,k} = \int_0^1 \frac{(1-t)^{K_2}}{K_2!}\varrho_\varepsilon(\alpha_2\mathbf{N})^{K_2+1}e^{\alpha_2 t\mathbf{N}}(1+\mathbf{N})^k\,dt,$$

satisfy

$$1_{[0,R^2]}(\mathbf{N})S_{K_2,k}=S_{K_2,k}\quad \text{with } |S_{K_2,k}|_{\mathcal{L}^1(\mathcal{H})}\leqslant e^{|\alpha_2|R^2}\big(1+R^2\big)^k,$$

and

$$1_{[0,R^2]}(\mathbf{N})R_{K_2,k}=R_{K_2,k}\quad \text{with } |R_{K_2,k}|_{\mathcal{L}^1(\mathcal{H})}\leqslant e^{|\alpha_2|R^2}\big(1+R^2\big)^k\frac{(|\alpha_2|R^2)^{K_2+1}}{(K_2+1)!}.$$

Repeating the same estimate on the left-hand side with $S_{K_2,k}$ and $R_{K_2,k}$ instead of $\varrho_\varepsilon$ implies that the $\mathcal{L}^1(\mathcal{H})$ norm of,

$$(1+\mathbf{N})^k\left[e^{\alpha_1\mathbf{N}}\varrho_\varepsilon e^{\alpha_2\mathbf{N}}-\sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2}\frac{(\alpha_1\mathbf{N})^{k_1}}{k_1!}\varrho_\varepsilon\frac{(\alpha_2\mathbf{N})^{k_2}}{k_2!}\right](1+\mathbf{N})^k,$$

is bounded by:

$$e^{|\alpha_2|R^2+|\alpha_1|R^2}\big(1+R^2\big)^{2k}\left[\frac{(|\alpha_1|R^2)^{K_1+1}}{(K_1+1)!}+\frac{(|\alpha_2|R^2)^{K_2+1}}{(K_2+1)!}+\frac{(|\alpha_1|R^2)^{K_1+1}(|\alpha_2|R^2)^{K_2+1}}{(K_1+1)!(K_2+1)!}\right],$$

which vanishes as $\min(K_1,K_2)\to\infty$. We conclude with a $\delta/3$-argument after noticing that $(1+\mathbf{N})^{-k}b^{Wick}(1+\mathbf{N})^{-k}$ is bounded for $k\geqslant k_b$ and that the convergence as $\varepsilon\to 0$ holds for $b\in\mathcal{P}_{alg}(\mathcal{Z})$ fixed and for the finite sums $\sum_{k_1=0}^{K_1}\sum_{k_2=0}^{K_2}$ owing to the condition (PI). □

**Proof of Proposition 2.16.** Let $C_f>1$ be a constant such that $\mathrm{Tr}[\varrho_\varepsilon f^2(\mathbf{N})]\geqslant\frac{1}{C_f}$ and $\sup_{s\in[0,+\infty)}f(s)(1+s)^{-\nu}\leqslant C_f$. The inequalities,

$$\mathrm{Tr}\big[\varrho_\varepsilon^f\mathbf{N}^\alpha\big]\leqslant C_f^2\,\mathrm{Tr}\big[\varrho_\varepsilon\mathbf{N}^\alpha(1+\mathbf{N})^{2\nu}\big],\quad \alpha\in\mathbb{N},$$

ensure that the family $(\varrho_\varepsilon^f)_{\varepsilon\in(0,\bar\varepsilon)}$ admits Wigner measures without any way to identify them for the moment. So take a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty}\varepsilon_n=0$ and $\mathcal{M}(\varrho_{\varepsilon_n}^f,n\in\mathbb{N})=\{\mu^f\}$. We first prove that the sequence $(\varrho_{\varepsilon_n}^f)_{n\in\mathbb{N}}$ satisfies the condition (PI), then check that $\mu^f=\frac{f^2(|z|^2)\mu}{\int f^2(|z|^2)\,d\mu}$ in the cases when $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ is localized in a ball and then when $f$ is compactly supported, and finally conclude with approximation arguments.

**1) The condition (PI) for the sequence:** The uniform control of $\mathrm{Tr}[\varrho_{\varepsilon_n}^f\mathbf{N}^\alpha]\leqslant C_\alpha$, $\alpha\in\mathbb{N}$, implies $\int_{\mathcal{Z}}|z|^{2\alpha}\,d\mu^f(z)<+\infty$ and Proposition 2.9 says that the convergence,

$$\lim_{n\to\infty}\mathrm{Tr}\big[\varrho_{\varepsilon_n}^f b^{Wick}\big]=\int_{\mathcal{Z}}b(z)\,d\mu^f(z),$$

holds for any $b\in\mathcal{P}_{alg}^\infty(\mathcal{Z})$ with a compact kernel. In particular for $b(z)=|pz|^{2k}$ with $p\in\mathbb{P}$ and $k\in\mathbb{N}$,

$$\lim_{n\to\infty}\mathrm{Tr}\big[\varrho_{\varepsilon_n}^f\big(\big(|pz|^2\big)^{Wick}\big)^k\big]=\lim_{n\to\infty}\mathrm{Tr}\big[\varrho_{\varepsilon_n}^f\big(|pz|^{2k}\big)^{Wick}\big]=\int_{\mathcal{Z}}|pz|^{2k}\,d\mu^f(z),\tag{24}$$

while we assumed,

$$\forall b\in\mathcal{P}_{alg}(\mathcal{Z}),\quad \lim_{n\to\infty}\mathrm{Tr}\big[\varrho_{\varepsilon_n}b^{Wick}\big]=\int_{\mathcal{Z}}b(z)\,d\mu(z).\tag{25}$$

Fix $\alpha\in\mathbb{N}^*$ and take $\delta>0$. By Lebesgue's convergence, there exists $p\in\mathbb{P}$ such that

$$\int_{\mathcal{Z}}\big||z|^{2\alpha}-|pz|^{2\alpha}\big|\,d\mu^f(z)\leqslant\delta\quad\text{and}\quad\int_{\mathcal{Z}}\big||z|^{2\alpha}-|pz|^{2\alpha}\big|\big(1+|z|^2\big)^{2\nu}\,d\mu(z)\leqslant\delta.$$

Remember that $(|pz|^2)^{Wick}=\mathbf{N}_p\otimes I_{\Gamma_s((1-p)\mathcal{Z})}=\mathbf{N}_p$ with $\mathbf{N}_p^\alpha\leqslant\mathbf{N}^\alpha$, where both sides commute with $f(\mathbf{N})$ and we get:

$$\begin{aligned}
0 &\leqslant \mathrm{Tr}\big[\varrho_{\varepsilon_n}^f\big(\mathbf{N}^\alpha - \mathbf{N}_p^\alpha\big)\big] \\
&\leqslant C_f\,\mathrm{Tr}\big[f(\mathbf{N})\big(\mathbf{N}^\alpha - \mathbf{N}_p^\alpha\big)^{1/2}\varrho_{\varepsilon_n}\big(\mathbf{N}^\alpha - \mathbf{N}_p^\alpha\big)^{1/2} f(\mathbf{N})\big] \\
&\leqslant C_f\big|f(\mathbf{N})(1+\mathbf{N})^{-\nu}\big|_{\mathcal{L}(\mathcal{H})}^2\,\mathrm{Tr}\big[(1+\mathbf{N})^\nu\big(\mathbf{N}^\alpha - \mathbf{N}_p^\alpha\big)^{1/2}\varrho_{\varepsilon_n}\big(\mathbf{N}^\alpha - \mathbf{N}_p^\alpha\big)^{1/2}(1+\mathbf{N})^\nu\big] \\
&\leqslant C_f^3\,\mathrm{Tr}\big[\varrho_{\varepsilon_n}\big(\mathbf{N}^\alpha - \mathbf{N}_p^\alpha\big)(1+\mathbf{N})^{2\nu}\big].
\end{aligned}$$

But we know by (25) that the right-hand side converges as $n \to \infty$ to

$$C_f^3 \int_{\mathcal{Z}} \big(|z|^{2\alpha} - |pz|^{2\alpha}\big)\big(1+|z|^2\big)^{2\nu}\,d\mu(z) \leqslant C_f^3\delta,$$

while (24) with $(|pz|^2)^{Wick} = \mathbf{N}_p$ gives:

$$\lim_{n\to\infty} \mathrm{Tr}\big[\varrho_{\varepsilon_n}^f \mathbf{N}_p^\alpha\big] = \int_{\mathcal{Z}} |pz|^{2\alpha}\,d\mu^f(z).$$

Hence there exist $n_\delta \in \mathbb{N}$ such that

$$\forall n \geqslant n_\delta, \quad \left|\mathrm{Tr}\big[\varrho_{\varepsilon_n}^f \mathbf{N}^\alpha\big] - \int_{\mathcal{Z}} |pz|^{2\alpha}\,d\mu^f(z)\right| \leqslant \big(C_f^3+1\big)\delta.$$

From $\int_{\mathcal{Z}} ||z|^{2\alpha} - |pz|^{2\alpha}|\,d\mu^f(z) \leqslant \delta$, we deduce:

$$\limsup_{n\to\infty}\left|\mathrm{Tr}\big[\varrho_{\varepsilon_n}^f \mathbf{N}^\alpha\big] - \int_{\mathcal{Z}} |z|^{2\alpha}\,d\mu(z)\right| \leqslant \big(C_f^3+2\big)\delta.$$

Letting $\delta \to 0$ ends the proof of this part.

**2) Identification of $\mu^f$ when $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ is localized in a ball:** Assume that $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ is localized in a ball of radius $R > 0$. Lemma 2.17 tells us that for any $t_1, t_2 \in \mathbb{R}$ and any $b \in \mathcal{P}_{alg}(\mathcal{Z})$,

$$\lim_{n\to\infty} \mathrm{Tr}\big[e^{it_2\mathbf{N}}\varrho_{\varepsilon_n} e^{it_1\mathbf{N}} b^{Wick}\big] = \int_{\mathcal{Z}} e^{i(t_1+t_2)|z|^2} b(z)\,\mu(z),$$

while the uniform boundedness of $(1+\mathbf{N})^{-k_b} b^{Wick}(1+\mathbf{N})^{-k_b}$ entails:

$$\big|\mathrm{Tr}\big[e^{it_2\mathbf{N}}\varrho_{\varepsilon_n} e^{it_1\mathbf{N}} b^{Wick}\big]\big| \leqslant C_b\,\mathrm{Tr}\big[\varrho_{\varepsilon_n}(1+\mathbf{N})^{2k_b}\big] \leqslant C_b\big(1+R^2\big)^{2k_b}.$$

Hence for $f \in \mathcal{F}^{-1}(L^1(\mathbb{R}))$, we get:

$$\lim_{n\to\infty} \frac{\mathrm{Tr}[f(\mathbf{N})\varrho_{\varepsilon_n} f(\mathbf{N}) b^{Wick}]}{\mathrm{Tr}[f(\mathbf{N})\varrho_{\varepsilon_n} f(\mathbf{N})]} = \frac{\int_{\mathcal{Z}} f(|z|^2)^2 b(z)\,d\mu(z)}{\int_{\mathcal{Z}} f(|z|^2)^2\,d\mu(z)}.$$

We have proved:

$$\forall b \in \mathcal{P}_{alg}(\mathcal{Z}), \quad \lim_{n\to\infty} \mathrm{Tr}\big[\varrho_{\varepsilon_n}^f b^{Wick}\big] = \frac{\int_{\mathcal{Z}} f(|z|^2)^2 b(z)\,d\mu(z)}{\int_{\mathcal{Z}} f(|z|^2)^2\,d\mu(z)}.$$

Part 1 and $1_{[0,R^2]}(\mathbf{N})\varrho_{\varepsilon_n}^f 1_{[0,R^2]}(\mathbf{N}) = \varrho_{\varepsilon_n}^f$ ensure that $(\varrho_{\varepsilon_n}^f)_{n\in\mathbb{N}}$ satisfies the sufficient conditions for solving the moment problem (Proposition 2.15) and $\mu^f = \frac{f(|z|^2)^2\mu}{\int_{\mathcal{Z}} f(|z|^2)^2\,d\mu}$ in this case.

**3) Identification of $\mu^f$ when $f$ is compactly supported:** Assume that $f \in \mathcal{C}_c^0([0,+\infty))$ is supported in $[0, R_0]$. Consider for $\chi \in \mathcal{C}_c^0([0,+\infty))$, $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $[0, 1/2]$ and for $R > 0$, the truncated states,

$$\varrho_{\varepsilon_n}^R = \frac{1}{\mathrm{Tr}[\varrho_{\varepsilon_n}\chi^2(\frac{\mathbf{N}}{R^2})]}\chi\left(\frac{\mathbf{N}}{R^2}\right)\varrho_{\varepsilon_n}\chi\left(\frac{\mathbf{N}}{R^2}\right), \quad n \in \mathbb{N}.$$

For $R \geqslant 2R_0$, we have:

$$\forall n \in \mathbb{N}^*, \quad \varrho_{\varepsilon_n}^f = \frac{1}{\operatorname{Tr}[\varrho_{\varepsilon_n}^R f^2(\mathbf{N})]} f(\mathbf{N}) \varrho_{\varepsilon_n}^R f(\mathbf{N}).$$

By extracting a subsequence we can assume $\mathcal{M}(\varrho_{\varepsilon_{n_k}}^R, k \in \mathbb{N}) = \{\mu^R\}$, and Part 1 applied to $(\varrho_{\varepsilon_{n_k}}^R)_{k \in \mathbb{N}}$ ensures that the pair $(\varrho_{\varepsilon_{n_k}}^f, \varrho_{\varepsilon_{n_k}}^R)$ fulfills all the assumptions of Part 2 if $f \in \mathcal{C}_c^0([0,+\infty)) \cap \mathcal{F}^{-1}L^1(\mathbb{R})$. Thus the measure $\mu^f$ equals $\frac{|f(|z|^2)|^2 \mu^R}{\int_{\mathcal{Z}} |f(|z|^2)|^2 \, d\mu^R}$. From the comparison (22) we know $\int |\mu^R - \mu| = \mathcal{O}(R^{-1})$ and since $f$ is a bounded function

$$\int \left| \mu^f - \frac{|f(|z|^2)|^2 \mu}{\int |f(|z|^2)|^2 \, d\mu} \right| \leqslant \frac{C}{R}.$$

Taking the limit as $R \to 0$ gives the result when $f \in \mathcal{C}_c^0([0,+\infty)) \cap \mathcal{F}^{-1}L^1(\mathbb{R})$. Removing the condition $f \in \mathcal{F}^{-1}L^1(\mathbb{R})$ is obtained by a comparison argument between $\varrho_{\varepsilon_n}^f$ and $\varrho_{\varepsilon_n}^{f_\ell}$ with $f_\ell \in \mathcal{C}_c^0 \cap \mathcal{F}^{-1}L^1(\mathbb{R})$ and $\sup_{s \in [0,+\infty]} |f(s) - f_\ell(s)| \leqslant \frac{1}{\ell+1}$, for $\ell \in \mathbb{N}$.

**4) Final approximation argument and uniqueness of $\mu^f$:** Consider now the complete problem with the extracted sequence $(\varrho_{\varepsilon_n}^f)_{n \in \mathbb{N}}$. We again use the cut-off $\chi(\frac{\mathbf{N}}{R^2})$ but now to compare $\varrho_{\varepsilon_n}^f$ with $\varrho_{\varepsilon_n}^{f\chi(R^{-2}.)}$. After extracting a subsequence, we can assume $\mathcal{M}(\varrho_{\varepsilon_{n_k}}^{f\chi(R^{-2}.)}, k \in \mathbb{N}) = \{\mu^{f\chi(R^{-2}.)}\}$. The pair $(\varrho_{\varepsilon_n}^{f\chi(R^{-2}.)}, \varrho_{\varepsilon_n})$ fulfills the assumptions of Part 3, and

$$\mu^{f\chi(R^{-2}.)} = \frac{f^2(|z|^2)\chi^2(R^{-2}|z|^2)\mu}{\int f^2(|z|^2)\chi^2(R^{-2}|z|^2)\, d\mu}.$$

But from the inequalities $f(s)(1-\chi(R^{-2}s))(1+s)^{-\nu-1} \leqslant CR^{-2}$ and $\operatorname{Tr}[\varrho_\varepsilon(1+\mathbf{N})^{2\nu+2}] \leqslant \tilde{C}_\nu$ we deduce the uniform estimate:

$$\forall k \in \mathbb{N}, \quad \left| \varrho_{\varepsilon_{n_k}}^f - \varrho_{\varepsilon_{n_k}}^{f\chi(R^{-2}.)} \right|_{\mathcal{L}^1(\mathcal{H})} \leqslant \frac{C'_f}{R}.$$

Again the comparison argument (22) gives:

$$\int \left| \mu^f - \frac{f^2(|z|^2)\chi^2(R^{-2}|z|^2)\mu}{\int f^2(|z|^2)\chi^2(R^{-2}|z|^2)\, d\mu} \right| \leqslant \frac{C'_f}{R},$$

and we take the limit as $R \to \infty$. We have proved $\mu^f = \frac{f^2(|z|^2)\mu}{\int f^2(|z|^2)\, d\mu}$ for any sequence extracted from $(\varrho_\varepsilon^f)_{\varepsilon \in (0,\bar{\varepsilon})}$ with a single Wigner measure. This proves $\mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})) = \{\frac{f^2(|z|^2)\mu}{\int f^2(|z|^2)\, d\mu}\}$ while the condition (PI) was checked in Part 1. □

## 3. Dynamical mean field limit

Let $Q$ be a real-valued polynomial in $\mathcal{P}_{alg}(\mathcal{Z})$ given by:

$$Q = \sum_{j=2}^{r} Q_j, \quad \text{with } Q_j \in \mathcal{P}_{j,j}(\mathcal{Z}).$$

We consider the many-body quantum Hamiltonian for a system of bosons,

$$H_\varepsilon = \mathrm{d}\Gamma(A) + Q^{Wick}, \tag{26}$$

with $A$ a given self-adjoint operator on $\mathcal{Z}$. Here $Q^{Wick}$ is the operator $\sum_{j=2}^{r} Q_j^{Wick}$ with $Q_j^{Wick}$ given by (7). Clearly, $H_\varepsilon$ acts as a self-adjoint operator on the symmetric Fock space $\mathcal{H}$. When $\mathcal{Z} = L^2(\mathbb{R}^d)$, the Schrödinger Hamiltonian $A = -\Delta + V(x)$ and the semi-relativistic Hamiltonian $A = \sqrt{-\Delta + m^2} + V(x)$ are among the typical examples (e.g. [21]).

*3.1. Existence of Wigner measure for all times*

The first step to prove Theorem 1.1 is to show the existence of Wigner measures for all times. This is accomplished in Proposition 3.3 by following the same lines as in the proof of Theorem 2.7. For this task two useful lemmas are stated below with the first one being proved in [7, Proposition 2.10].

**Lemma 3.1.** *For any $b \in \mathcal{P}_{alg}(\mathcal{Z})$ we have*:

(i) *$b^{Wick}$ is a closable operator with the domain of its closure containing,*

$$\mathcal{H}_0 = \text{vect}\{W(\varphi)\psi,\ \psi \in \mathcal{H}_{fin},\ \varphi \in \mathcal{Z}\}.$$

(ii) *For any $\varphi \in \mathcal{Z}$ the identity,*

$$W(\xi)^* b^{Wick} W(\xi) = \left(b\left(z + \frac{i\varepsilon}{\sqrt{2}}\xi\right)\right)^{Wick},$$

*holds on $\mathcal{H}_0$ with $b(\cdot + \frac{i\varepsilon}{\sqrt{2}}\xi) \in \mathcal{P}_{alg}(\mathcal{Z})$.*

**Lemma 3.2.** *For any $k \in \mathbb{N}$ there exists an $\varepsilon$-independent constant $C_k > 0$ such that*

$$W(\xi)^* \langle \mathbf{N}\rangle^k W(\xi) \leqslant C_k \langle \bar{\varepsilon}\rangle^k \langle \xi\rangle^k \langle \mathbf{N}\rangle^k, \tag{27}$$

*for any $\xi \in \mathcal{Z}$ and uniformly in $\varepsilon \in (0, \bar{\varepsilon})$.*

**Proof.** Since $\mathbf{N}$ is a self-adjoint operator, the functional calculus provides the inequality:

$$\langle \mathbf{N}\rangle^k \leqslant (1 + \mathbf{N})^k.$$

Therefore, it is enough to prove (27) with $\langle \mathbf{N}\rangle$ in the l.h.s. replaced by $(1 + \mathbf{N})$. The Wick calculus in Proposition 2.4 tell us that $(1 + \mathbf{N})^k$ is a Wick operator with symbol $b_k(z)$ in $\bigoplus_{j=0}^k \mathcal{P}_{j,j}(\mathcal{Z})$, i.e.:

$$b_k(z) = \sum_{j=0}^k \langle z^{\otimes j}, \tilde{b}_k^{(j)} z^{\otimes j}\rangle \quad \text{with } b_k^{(j)} \in \mathcal{P}_{j,j}(\mathcal{Z}).$$

Now, applying Lemma 3.1 yields:

$$W(\xi)^*(1 + \mathbf{N})^k W(\xi) = W(\xi)^* b_k^{Wick} W(\xi) = \left(b_k\left(z + \frac{i\varepsilon}{\sqrt{2}}\xi\right)\right)^{Wick}.$$

A Taylor expansion of the symbol gives us,

$$b_k\left(z + \frac{i\varepsilon}{\sqrt{2}}\xi\right) = \sum_{j=0}^k \frac{(i\varepsilon)^j}{j!\sqrt{2^j}} D^{(j)} b_k(z)[\xi],$$

with $D^{(j)}$ is the $j$th derivatives and $D^{(j)} b_k(z)[\xi] \in \bigoplus_{m,n=0}^{k-j} \mathcal{P}_{m,n}(\mathcal{Z})$. So, by the number estimate (2.3) we can derive the following bound,

$$\left|\langle \mathbf{N}\rangle^{-k/2} \left(D^{(j)} b_k(z)[\xi]\right)^{Wick} \langle \mathbf{N}\rangle^{-k/2}\right| \leqslant \tilde{C}_k \langle \xi\rangle^j$$

with $\tilde{C}_k$ only depending on $k \in \mathbb{N}$. Hence, we obtain:

$$\left|\langle \mathbf{N}\rangle^{-k/2} \sum_{j=0}^k \frac{(i\varepsilon)^j}{j!\sqrt{2^j}} \left(D^{(j)} b_k(z)[\xi]\right)^{Wick} \langle \mathbf{N}\rangle^{-k/2}\right| \leqslant C_k \langle \bar{\varepsilon}\rangle^k \langle \xi\rangle^k,$$

with $C_k$ only depending on $k \in \mathbb{N}$. Thus, we conclude that $W(\xi)^*(1+\mathbf{N})^k W(\xi)$ as a positive quadratic form is bounded by $C_k \langle \bar{\varepsilon}\rangle^k \langle \xi\rangle^k \langle \mathbf{N}\rangle^k$. □

**Proposition 3.3.** *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$ satisfying the uniform estimate $\mathrm{Tr}[\varrho_\varepsilon \mathbf{N}^r] \leqslant C_r$ for some $r > 0$.*

*Then for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\bar\varepsilon)$ such that $\lim_{n\to\infty}\varepsilon_n = 0$ there exist a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a family of Borel probability measures $(\mu_t)_{t\in\mathbb{R}}$ satisfying,*

$$\mathcal{M}\big(e^{-i\frac{t}{\varepsilon_n}H_{\varepsilon_n}}\varrho_{\varepsilon_n}e^{i\frac{t}{\varepsilon_n}H_{\varepsilon_n}},\ n\in\mathbb{N}\big) = \{\mu_t\},$$

*for any $t\in\mathbb{R}$. Moreover, we have*:

$$\int_{\mathcal{Z}} |z|^{2r}\, d\mu_t(z) \leqslant C_r.$$

**Proof.** We set:

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon} \quad \text{and} \quad \tilde\varrho_\varepsilon(t) = e^{i\frac{t}{\varepsilon}\mathrm{d}\Gamma(A)}e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{-i\frac{t}{\varepsilon}\mathrm{d}\Gamma(A)}.$$

(i) Consider for $\varepsilon > 0$ the function:

$$G_\varepsilon(t,\xi) = \mathrm{Tr}\big[\tilde\varrho_\varepsilon(t)W(\sqrt{2}\pi\xi)\big].$$

Write for any $(s,\xi),(t,\eta)\in\mathbb{R}\times\mathcal{Z}$,

$$\big|G_\varepsilon(t,\eta) - G_\varepsilon(s,\xi)\big| \leqslant \big|\mathrm{Tr}\big[\big(\tilde\varrho_\varepsilon(t) - \tilde\varrho_\varepsilon(s)\big)W(\sqrt{2}\pi\eta)\big]\big| + \big|\mathrm{Tr}\big[\tilde\varrho_\varepsilon(s)\big(W(\sqrt{2}\pi\eta) - W(\sqrt{2}\pi\xi)\big)\big]\big|.$$

By differentiation, we get:

$$\big|\mathrm{Tr}\big[\big[\tilde\varrho_\varepsilon(t) - \tilde\varrho_\varepsilon(s)\big]W(\sqrt{2}\pi\eta)\big]\big| \leqslant \frac{1}{\varepsilon}\left|\int_s^t \mathrm{Tr}\big[\tilde\varrho_\varepsilon\big(t'\big)\big[Q_{t'}^{Wick}, W(\sqrt{2}\pi\eta)\big]\big]\,dt'\right| \tag{28}$$

with $Q_{t'}(z) = Q(e^{-it'A}z)$, while the second term is estimated by:

$$\big|\mathrm{Tr}\big[\tilde\varrho_\varepsilon(s)\big(W(\sqrt{2}\pi\eta) - W(\sqrt{2}\pi\xi)\big)\big]\big| \leqslant (1+C_r)\big|\big[W(\sqrt{2}\pi\eta) - W(\sqrt{2}\pi\xi)\big](\mathbf{N}+1)^{-1}\big|_{\mathcal{L}(\mathcal{H})}. \tag{29}$$

Now, we claim that there exists a constant $c > 0$ such that the r.h.s. of (28) is bounded by:

$$c|t-s|\left(\sum_{j=2}^{r}\|\tilde Q_j\|\right)\sum_{i=1}^{2r}\varepsilon^{i-1}|\eta|^i. \tag{30}$$

This can be proved by first writing,

$$\begin{aligned}
&\mathrm{Tr}\big[\tilde\varrho_\varepsilon(t')\big[Q_{t'}^{Wick}, W(\sqrt{2}\pi\eta)\big]\big] \\
&\quad = \mathrm{Tr}\big[\langle\mathbf{N}\rangle^r\tilde\varrho_\varepsilon\big(t'\big)\langle\mathbf{N}\rangle^r\big(\langle\mathbf{N}\rangle^{-r}W(\sqrt{2}\pi\eta)\langle\mathbf{N}\rangle^r\big)\langle\mathbf{N}\rangle^{-r}\big[W(\sqrt{2}\pi\eta)^*Q_{t'}^{Wick}W(\sqrt{2}\pi\eta) - Q_{t'}^{Wick}\big]\langle\mathbf{N}\rangle^{-r}\big]
\end{aligned} \tag{31}$$

and second estimating the r.h.s. of (31) using Lemmas 3.2 and 3.1(ii) so that

$$\begin{aligned}
&\left|\int_s^t \mathrm{Tr}\big[\tilde\varrho_\varepsilon\big(t'\big)\big[Q_{t'}^{Wick}, W(\sqrt{2}\pi\eta)\big]\big]\,dt'\right| \\
&\quad \leqslant c|t-s|\sup_{t'\in[s,t]}\left|\langle\mathbf{N}\rangle^{-r}\left[Q_{t'}\left(.+\frac{i\varepsilon}{\sqrt{2}}\eta\right)^{Wick} - Q_{t'}^{Wick}\right]\langle\mathbf{N}\rangle^{-r}\right|_{\mathcal{L}(\mathcal{H})}.
\end{aligned}$$

Thus, the bound (30) follows from the number estimate in Proposition 2.3.

We recall the inequality proved in [7, Lemma 3.1],

$$\big|\big[W(\sqrt{2}\pi\eta) - W(\sqrt{2}\pi\xi)\big](\mathbf{N}+1)^{-1/2}\big| \leqslant \tilde C|\eta-\xi|\big[\min\big(\varepsilon|\eta|,\varepsilon|\xi|\big) + \max(1,\varepsilon)\big].$$

This leads to the following bound on the r.h.s. of (29),

$$\tilde C\langle\bar\varepsilon\rangle|\eta-\xi|\big(1+\sqrt{|\eta|^2+|\xi|^2}\big).$$

Thus, we conclude that $\forall(s,\xi),(t,\eta)\in\mathbb{R}\times\mathcal{Z}$,

$$\big|G_\varepsilon(t,\eta)-G_\varepsilon(s,\xi)\big|\leqslant\tilde{c}\big(|t-s|\big(|\eta|+1\big)^{2r}+|\eta-\xi|\sqrt{|\eta|^2+|\xi|^2}\big),\tag{32}$$

uniformly w.r.t. $\varepsilon\in(0,\bar{\varepsilon})$. Recall also that we have the uniform estimate $|G_\varepsilon(s,\xi)|\leqslant 1$.

Now, we apply an Ascoli type argument:

- Since $\mathbb{R}\times\mathcal{Z}$ is separable, it admits a countable dense set $\mathcal{N}=\{(t_\ell,\xi_\ell)\ \ \ell\in\mathbb{N}\}$. For any $\ell\in\mathbb{N}$ the set $\{G_\varepsilon(t_\ell,\xi_\ell)\}_{\varepsilon\in(0,\bar{\varepsilon})}$ remains in $\{\sigma\in\mathbb{C},\ |\sigma|\leqslant 1\}$. Hence for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n\to 0$ there exists by a diagonal extraction process a subsequence, still denoted by $(\varepsilon_n)_{n\in\mathbb{N}}$, such that for all $\ell\in\mathbb{N}$, $G_{\varepsilon_n}(t_\ell,\xi_\ell)$ converges in $\{\sigma\in\mathbb{C},\ |\sigma|\leqslant 1\}$ as $n\to\infty$. Set
$$G(t_\ell,\xi_\ell):=\lim_{n\to\infty}G_{\varepsilon_n}(t_\ell,\xi_\ell)$$
for all $\ell\in\mathbb{N}$.
- The uniform estimate (32) implies that the limit $G$ is uniformly continuous on any set
$$\mathcal{N}\cap\big\{(t,z)\in\mathbb{R}\times\mathcal{Z}\colon\ |t|+|z|\leqslant R\big\}.$$
Hence it admits a continuous extension still denoted $G$ in $(\mathbb{R}\times\mathcal{Z},||_{\mathbb{R}\times\mathcal{Z}})$. An "epsilon/3"-argument shows that for any $(t,\xi)\in\mathbb{R}\times\mathcal{Z}$, $\lim_{n\to\infty}G_{\varepsilon_n}(t,\xi)$ exists and equals $G(t,\xi)$.

Finally for any $t\in\mathbb{R}$, $G(t,.)$ is a norm continuous normalized function of positive type, since

$$G(t,0)=\lim_{n\to\infty}\operatorname{Tr}\big[\tilde{\varrho}_\varepsilon(t)\big]=1,$$
$$\sum_{i,j=1}^N\lambda_i\overline{\lambda_j}G(t,\xi_i-\xi_j)=\lim_{n\to\infty}\sum_{i,j=1}^N\lambda_i\overline{\lambda_j}\operatorname{Tr}\big[\tilde{\varrho}_{\varepsilon_n}(t)W\big(\sqrt{2}\pi(\xi_i-\xi_j)\big)\big]e^{i\varepsilon_n\pi^2\sigma(\xi_i,\xi_j)}\geqslant 0.$$

The positivity in the last statement follows by Weyl commutation relations (6). Therefore, according to the Bochner theorem (e.g. [12, Corollary 1.4.2]) for any $t\in\mathbb{R}$, $G(t,.)$ is a characteristic function of a weak distribution or equivalently a cylindrical measure $\tilde{\mu}_t$ on $\mathcal{Z}$ (see [51] and also [7, Section 6] for specific information).

(ii) The fact that $\tilde{\mu}_t$ are Borel probability measures satisfying,

$$\tilde{\mu}_t\big(|z|^{2r}\big)\leqslant C_r<\infty,\tag{33}$$

follows directly by [46, Theorem 2.5, Chapter VI] or by part (iv) in the proof of [7, Theorem 6.2].

(iii) Using (13) we see that for any $b\in\mathcal{S}_{cyl}(\mathcal{Z})$ based on a finite dimensional subspace $p\mathcal{Z}$ with $p\in\mathbb{P}$,

$$\begin{aligned}\lim_{n\to\infty}\operatorname{Tr}\big[\tilde{\varrho}_{\varepsilon_n}(t)b^{Weyl}\big]&=\lim_{n\to\infty}\int_{p\mathcal{Z}}G_{\varepsilon_n}(t,\xi)\mathcal{F}[b](\xi)\,L_p(d\xi)\\&=\int_{p\mathcal{Z}}G(t,\xi)\mathcal{F}[b](\xi)\,L_p(d\xi)=\int_{\mathcal{Z}}b(z)\,d\tilde{\mu}_t(z).\end{aligned}$$

Therefore, according to Definition 2.8 of Wigner measures we conclude that

$$\forall t\in\mathbb{R},\quad\mathcal{M}\big(\tilde{\varrho}_{\varepsilon_n}(t),\ n\in\mathbb{N}\big)=\{\tilde{\mu}_t\}.$$

(iv) Finally the family of measures $\mu_t$ which satisfy the claimed statement in the proposition are the push-forward measures,

$$\mu_t=\big(e^{-itA}\big)_*\tilde{\mu}_t.$$

Furthermore, an analogue of (33) can be easily checked for the measures $\mu_t$. $\square$

*3.2. Polynomial approximations of the classical flow*

With the classical Hamiltonian

$$h(z) = \langle z, Az\rangle + Q(z) = \langle z, Az\rangle + \sum_{j=2}^{r} \big\langle z^{\otimes j}, \tilde{Q}_j z^{\otimes j}\big\rangle, \quad z \in \mathcal{D}(A),$$

the related nonlinear field equation is:

$$\begin{cases} i\partial_t z_t = Az_t + \partial_{\bar{z}} Q(z_t), \\ z_{t=0} = z_0. \end{cases}$$

Actually this Cauchy problem is better studied when reformulated as an integral equation,

$$z_t = e^{-itA} z - i \int_0^t e^{-i(t-s)A} \partial_{\bar{z}} Q(z_s)\, ds, \quad \text{for } z \in \mathcal{Z}, \tag{34}$$

which admits a classical $\mathcal{C}^0$-flow $\mathbf{F}_t : \mathbb{R} \times \mathcal{Z} \to \mathcal{Z}$: 1) since the $\tilde{Q}_j$ are bounded a fixed point argument gives the local in time existence and uniqueness; 2) then the conservation $|z_t| = |z_0|$, due to $\langle z, \partial_{\bar{z}} h(z)\rangle \in \mathbb{R}$, ensures the global in time result. As a classical $\mathcal{C}^0$-flow, $\mathbf{F}$ is a $\mathcal{C}^0$-map satisfying $\mathbf{F}_{t+s}(z) = \mathbf{F}_t \circ \mathbf{F}_s(z)$, and $\mathbf{F}_t(z)$ solves (34) for any $z \in \mathcal{Z}$.

Moreover, if $z_t$ solves (34), and $Q_t(z) = Q(e^{-itA}z)$, then $w_t = e^{itA} z_t$ solves the differential equation:

$$\frac{d}{dt} w_t = -i\partial_{\bar{z}} Q_t(w_t).$$

Therefore for any $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the following identity holds:

$$\frac{d}{dt} b(w_t) = \partial_{\bar{z}} b(w_t)\big[-i\partial_{\bar{z}} Q_t(w_t)\big] + \partial_z b(w_t)\big[-i\partial_{\bar{z}} Q_t(w_t)\big] = i\{Q_t, b\}(w_t).$$

Hence, we obtain the Duhamel formula:

$$b(z_t) = b_t(z) + i \int_0^t \{Q_{t_1}, b_t\}\big(e^{it_1 A} z_{t_1}\big)\, dt_1. \tag{35}$$

A simple iteration in (35), using

$$\{Q_{t_1}, b_t\}(w_{t_1}) = \{Q_{t_1}, b_t\}(w_0) + i \int_0^{t_1} \big\{Q_{t_2}, \{Q_{t_1}, b_t\}\big\}(w_{t_2})\, dt_2,$$

yields

$$b(z_t) = b_t(z) + i \int_0^t \{Q_{t_1}, b_t\}(z)\, dt_1 + i^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \big\{Q_{t_2}, \{Q_{t_1}, b_t\}\big\}\big(e^{it_2 A} z_{t_2}\big).$$

Therefore, by induction and after setting $\mathbf{F}_t(z) = z_t$, we obtain for any $K > 1$:

$$\begin{aligned} b \circ \mathbf{F}_t(z) = b_t(z) &+ \sum_{k=1}^{K-1} i^k \int_0^t dt_1 \cdots \int_0^{t_{k-1}} dt_k \big\{Q_{t_k}, \big\{\ldots, \{Q_{t_1}, b_t\}\ldots\big\}\big\}(z) \\ &+ i^K \int_0^t dt_1 \cdots \int_0^{t_{K-1}} dt_K \big\{Q_{t_K}, \big\{\ldots, \{Q_{t_1}, b_t\}\ldots\big\}\big\}\big(e^{it_K A} z_{t_K}\big). \end{aligned}$$

With the polynomial $Q$ we associate the norm,

$$\|Q\| = \max_{j \in \{2,\ldots,r\}} |Q_j|_{\mathcal{P}_{j,j}} = \max_{j \in \{2,\ldots,r\}} |\tilde{Q}_j|_{\mathcal{L}(\bigvee^j \mathcal{Z}, \bigvee^j \mathcal{Z})}, \tag{36}$$

and we note that $\|Q_t\| = \|Q\|$ for all $t \in \mathbb{R}$. Notice that the flow $\mathbf{F}_t$ preserves the norm,

$$\forall z \in \mathcal{Z}, \quad \left|\mathbf{F}_t(z)\right| = |z|,$$

and is gauge invariant,

$$\forall z \in \mathcal{Z},\ \forall \theta \in \mathbb{R}, \quad \mathbf{F}_t\left(e^{i\theta} z\right) = e^{i\theta}\mathbf{F}_t(z).$$

*But* for a given polynomial $b(z)$, the map $z \mapsto b(z_t)$ does not remain a polynomial. Starting from a polynomial $b(z) \in \mathcal{P}_{p,q}(\mathcal{Z})$, we study polynomial approximations of $b(z_t)$.

Consider the expression:

$$b^K(t,z) = b_t(z) + \sum_{k=1}^{K-1} i^k \int_0^t dt_1 \cdots \int_0^{t_{k-1}} dt_k \left\{Q_{t_k}, \left\{\ldots, \{Q_{t_1}, b_t\}\ldots\right\}\right\}(z) = \sum_{k=0}^{K-1} b_k(t,z), \tag{37}$$

$$R^K(t,z) = i^K \int_0^t dt_1 \cdots \int_0^{t_{K-1}} dt_K \left\{Q_{t_K}, \left\{\ldots, \{Q_{t_1}, b_t\}\ldots\right\}\right\}\left(e^{it_K A} z_{t_K}\right). \tag{38}$$

The two approximation results that we will use are given in the two next propositions.

**Proposition 3.4.** *For $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, the polynomial $b^K(t,z) = \sum_{k=0}^{K-1} b_k(t,z)$ defined in* (37) *belongs to $\bigoplus_{j=1}^{K(r-1)} \mathcal{P}_{j+p,j+q}(\mathcal{Z})$ with the estimates*:

$$\left|b_k(t,z)\right| \leqslant 2^{\frac{p+q}{2(r-1)}}(p+q)\left(4r^3\right)^k \|Q\|^k |b|_{\mathcal{P}_{p,q}} |t|^k \langle z\rangle^{2k(r-1)+p+q}. \tag{39}$$

*Moreover, we have for $R^K(t,z)$ the estimates*:

$$\left|R^K(t,z)\right| \leqslant 2^{\frac{p+q}{2(r-1)}}(p+q)\left(4r^3\right)^K \|Q\|^K |b|_{\mathcal{P}_{p,q}} |t|^K \langle z\rangle^{2K(r-1)+p+q}. \tag{40}$$

**Proof.** With $b \in \mathcal{P}_{p,q}(\mathcal{Z})$ and $Q_t = \sum_{j=2}^r Q_{j,t}$, the polynomial,

$$b_k(t) = (i)^k \int_0^t dt_1 \cdots \int_0^{t_{k-1}} dt_k \left\{Q_{t_k}, \left\{\ldots, \{Q_{t_1}, b_t\}\ldots\right\}\right\}(z)$$

is the sum of $(r-1)^k \leqslant r^k$ monomials,

$$b_k(t) = \sum_{\alpha \in \{2,\ldots,r\}^k} b_{k,\alpha}(t),$$

with

$$b_{k,\alpha}(t) = (i)^k \int_0^t dt_1 \cdots \int_0^{t_{k-1}} dt_k \left\{Q_{\alpha_k,t_k}, \left\{\ldots, \{Q_{\alpha_1,t_1}, b_t\}\ldots\right\}\right\} \in \mathcal{P}_{|\alpha|-k+p,|\alpha|-k+q}(\mathcal{Z}).$$

A consequence of Proposition 2.4 says for $c \in \mathcal{P}_{p',q'}(\mathcal{Z})$,

$$\left|\{Q_{\alpha_1,t_1}, c\}(z)\right| \leqslant r\left(p'+q'\right)|Q_{\alpha_1}|_{\mathcal{P}_{\alpha_1,\alpha_1}} |c|_{\mathcal{P}_{p',q'}} \langle z\rangle^{p'+q'+2(\alpha_1-1)}.$$

We deduce:

$$\begin{aligned}\left|b_{k,\alpha}(t,z)\right| &\leqslant \int_0^t dt_1 \cdots \int_0^{t_{k-1}} dt_k\, r^k (p+q)\cdots\left(p+q+2k(r-1)\right)\|Q\|^k |b|_{\mathcal{P}_{p,q}} \langle z\rangle^{p+q+2|\alpha|-2k} \\ &\leqslant (p+q) r^k \left(2(r-1)\right)^{k-1} |t|^k \frac{\Gamma(a+k+2)}{\Gamma(k+1)\Gamma(a+1)} \frac{1}{a+k+1} \|Q\|^k |b|_{\mathcal{P}_{p,q}} \langle z\rangle^{p+q+2k(r-1)},\end{aligned}$$

with $a = \frac{p+q}{2(r-1)}$ and $\Gamma$ denotes the Gamma function. Now, we notice the relation with the Beta function

$$B(k+1, a+1) = \frac{\Gamma(k+1)\Gamma(a+1)}{\Gamma(a+k+2)} = \int_0^1 t^k (1-t)^a \, dt \geqslant \frac{1}{2^{a+k+1}(a+k+1)},$$

which yields (39).

The remainder

$$R^K(t,z) = i\{Q_t, b^K\} = i^K \int_0^t dt_1 \cdots \int_0^{t_{K-1}} dt_K \{Q_{t_K}, \{\ldots, \{Q_{t_1}, b_t\}\ldots\}\}\big(e^{it_K A} z_{t_K}\big)$$

is analyzed like the term $b_k(t)$. □

**Proposition 3.5.** *Let $\mu$ be a positive Borel measure on $\mathcal{Z}$ supported in the ball $\{|z| \leqslant R\}$, $R > 0$, then for any polynomial $b \in \mathcal{P}_{p,q}(\mathcal{Z})$,*

$$\int_{\mathcal{Z}} \big|R^K(t,z)\big| \, d\mu(z) \leqslant \langle R\rangle^{p+q} 2^{\frac{p+q}{2(r-1)}} (p+q)|b|_{\mathcal{P}_{p,q}} \big[4r^3 \|Q\| \langle R\rangle^{2(r-1)} |t|\big]^K.$$

**Proof.** It easily follows from (40). □

### 3.3. Transport for a state localized in a ball

The previous approximation result allows to prove partly Theorem 1.1 for states localized in a ball, introduced according to Definition 2.13 and studied in Section 2.8.

**Proposition 3.6.** *Let $(\varrho_{\varepsilon_n})_{n\in\mathbb{N}}$ be a sequence of normal states on $\mathcal{H}$ localized in a ball with radius $R > 0$ and such that*

$$\forall t \in [-T, T], \quad \mathcal{M}\big(e^{-i\frac{t}{\varepsilon_n}H_{\varepsilon_n}} \varrho_{\varepsilon_n} e^{i\frac{t}{\varepsilon_n}H_{\varepsilon_n}}, \ n \in \mathbb{N}\big) = \{\mu_t\}, \quad \textit{and}$$
$$\forall \alpha \in \mathbb{N}, \quad \lim_{k\to\infty} \mathrm{Tr}\big[\varrho_{\varepsilon_{n_k}} \mathbf{N}^\alpha\big] = \int_{\mathcal{Z}} |z|^{2\alpha} \, d\mu_0(z).$$

*Then for all $t \in [-T, T]$, the probability measure $\mu_t$ is the push-forward by the flow $\mathbf{F}_t$ of the measure $\mu_0$, i.e., $\mu_t = (\mathbf{F}_t)_*\mu_0$. Moreover the identity,*

$$\lim_{n\to\infty} \mathrm{Tr}\big[e^{-i\frac{t}{\varepsilon_n}H_{\varepsilon_n}} \varrho_{\varepsilon_n} e^{i\frac{t}{\varepsilon_n}H_{\varepsilon_n}} b^{quantized}\big] = \int_{\mathcal{Z}} b(z) \, d\mu_t(z) = \int_{\mathcal{Z}} b\big(\mathbf{F}_t(z)\big) \, d\mu_0(z),$$

*holds for Weyl-quantized cylindrical functions $b \in \bigcup_{p\in\mathbb{P}} \mathcal{F}^{-1}(\mathcal{M}_b(p\mathcal{Z}))$ and general Wick-quantized polynomials $b \in \mathcal{P}_{alg}(\mathcal{Z})$.*

**Proof.** We set:

$$\tilde{\varrho}_{\varepsilon_n}(t) := e^{i\frac{t}{\varepsilon_n}\mathrm{d}\Gamma(A)} e^{-i\frac{t}{\varepsilon_n}H_{\varepsilon_n}} \varrho_\varepsilon e^{i\frac{t}{\varepsilon_n}H_{\varepsilon_n}} e^{-i\frac{t}{\varepsilon_n}\mathrm{d}\Gamma(A)}.$$

It is worth noticing that for all $t \in \mathbb{R}$, the sequence $(\tilde{\varrho}_{\varepsilon_n}(t))_{n\in\mathbb{N}}$ is localized in the ball with radius $R$.

For a *fixed* $b \in \mathcal{P}_{p,q}(\mathcal{Z})$, differentiating with respect to $t$ the quantity $\mathrm{Tr}[\tilde{\varrho}_\varepsilon(t)\, b^{Wick}]$, we obtain:

$$\mathrm{Tr}\big[\tilde{\varrho}_{\varepsilon_n}(t) b^{Wick}\big] = \mathrm{Tr}\big[\tilde{\varrho}_{\varepsilon_n}(0) b^{Wick}\big] + \frac{i}{\varepsilon_n} \int_0^t \mathrm{Tr}\big[\tilde{\varrho}_{\varepsilon_n}(s)\big[Q_s^{Wick}, b^{Wick}\big]\big] \, ds, \tag{41}$$

and replacing $b$ by $b_t$ we end up with,

$$\operatorname{Tr}\left[\varrho_{\varepsilon_n}(t)b^{Wick}\right] = \operatorname{Tr}\left[\varrho_{\varepsilon_n}(0)b_t^{Wick}\right] + i\int_0^t \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(s)\{Q_s, b_t\}^{Wick}\right] ds$$
$$+ i\sum_{j=2}^{r}\frac{\varepsilon_n^{j-1}}{j!}\int_0^t \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(s)\left(\{Q_s, b_t\}^{(j)}\right)^{Wick}\right] ds. \tag{42}$$

Consider now the case when $b \in \mathcal{P}^{\infty}_{p,q}(\mathcal{Z})$ with a compact kernel, $\tilde{b} \in \mathcal{L}^{\infty}(\bigvee^p \mathcal{Z}; \bigvee^q \mathcal{Z})$. Then we know that the left-hand side converges to $\int_{\mathcal{Z}} b(z)\, d\mu_t(z)$. The number estimate of Proposition 2.3 with $\operatorname{Tr}[\mathbf{N}^{\alpha}\varrho_{\varepsilon_n}] \leqslant R^{2\alpha}$ implies that the last term of the right-hand side converges to 0 as $n \to \infty$. Finally the first term of the right-hand side converges to $\int_{\mathcal{Z}} b(z)\, d\mu_0(z)$, even when $\tilde{b}$ is not compact.

We conclude that the limit of the second term of the r.h.s. exists with

$$\int_{\mathcal{Z}} b(z)\, d\mu_t(z) = \int_{\mathcal{Z}} b_t(z)\, d\mu_0(z) + \lim_{n\to\infty} i\int_0^t \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(s)\{Q_s, b_t\}^{Wick}\right] ds,$$

and this initiates our induction process.

Given $K > 1$, take the approximation $b^K(t) = \sum_{k=0}^{K-1} b_k(t)$ to $b(\mathbf{F}_t(z))$ given in (37), and assume:

$$\int_{\mathcal{Z}} b(z)\, d\mu_t(z) = \int_{\mathcal{Z}} b^K(t,z)\, d\mu_0(z) \tag{43}$$
$$+ \lim_{n\to\infty} i^K \int_0^t dt_1 \cdots \int_0^{t_{K-1}} dt_K \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(t_K)\left(\left\{Q_{t_K}, \ldots\{Q_{t_1}, b_t\}\ldots\right\}\right)^{Wick}\right]. \tag{44}$$

A simple differentiation with respect to $t_K$ gives for $\Theta \in \mathcal{P}_{alg}(\mathcal{Z})$,

$$\operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(t_K)\Theta^{Wick}\right] = \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(0)\Theta^{Wick}\right] + i\int_0^{t_K} dt_{K+1} \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(t_{K+1})\left(\{Q_{t_{K+1}}, \Theta\}\right)^{Wick}\right]$$
$$+ i\sum_{j=2}^{r}\frac{\varepsilon_n^{j-1}}{j!}\int_0^{t_K} \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(t_{K+1})\left(\{Q_{t_{K+1}}, \Theta\}^{(j)}\right)^{Wick}\right] dt_{K+1}.$$

Hence, choosing $\Theta = \{Q_{t_K}, \ldots\{Q_{t_1}, b_t\}\ldots\}$ yields:

$$\begin{aligned}\int_{\mathcal{Z}} b(z)\, d\mu_t(z) &= \int_{\mathcal{Z}} b^K(t,z)\, d\mu_0(z) + \lim_{n\to\infty}\Bigg\{ i^K \int_0^t dt_1 \cdots \int_0^{t_{K-1}} dt_K \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(0)\left(\left\{Q_{t_K}, \ldots\{Q_{t_1}, b_t\}\ldots\right\}\right)^{Wick}\right] \\ &\quad + i^{K+1}\sum_{j=2}^{r}\frac{\varepsilon_n^{j-1}}{j!}\int_0^t dt_1 \cdots \int_0^{t_K} dt_{K+1} \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(t_{K+1})\left(\left\{Q_{t_{K+1}}, \ldots\{Q_{t_1}, b_t\}\ldots\right\}\right)^{Wick}\right] \\ &\quad + i^{K+1}\int_0^t dt_1 \cdots \int_0^{t_K} dt_{K+1} \operatorname{Tr}\left[\tilde{\varrho}_{\varepsilon_n}(t_{K+1})\left(\left\{Q_{t_{K+1}}, \ldots\{Q_{t_1}, b_t\}\ldots\right\}\right)^{Wick}\right]\Bigg\} \\ &=: \mathrm{I} + \lim_{n\to\infty}(\mathrm{II} + \mathrm{III} + \mathrm{IV}).\end{aligned}$$

For any $K$, when $n \to \infty$, the second term (II) converges to $\int_{\mathcal{Z}} \Theta(z)\, d\mu_0(z)$ because the initial states $\tilde{\varrho}_{\varepsilon_n}(0) = \varrho_{\varepsilon_n}$ satisfies $\lim_{\varepsilon\to 0}\operatorname{Tr}[\varrho_{\varepsilon_n} c^{Wick}] = \int_{\mathcal{Z}} c(z)\, d\mu_0(z)$ according to Proposition 2.12. Moreover, the third term (III) vanishes, when $n \to \infty$, thanks to the number estimate in Proposition 2.3 and the fact that $\operatorname{Tr}[\varrho_{\varepsilon_n}\mathbf{N}^{\alpha}] \leqslant R^{2\alpha}$. Therefore, we have:

$$\int_{\mathcal{Z}} b(z)\,d\mu_t(z) = \int_{\mathcal{Z}} b^{K+1}(t,z)\,d\mu_0(z) + \lim_{n\to\infty} i^{K+1} \int_0^t dt_1 \cdots \int_0^{t_K} dt_{K+1}\, \mathrm{Tr}\big[\tilde{\varrho}_{\varepsilon_n}(t_{K+1})\big(\big\{Q_{t_{K+1}}, \dots \{Q_{t_1}, b_t\}\dots\big\}\big)^{Wick}\big].$$

By Proposition 3.5 and the fact that $\mu_0$ is supported in $\{|z| \leqslant R\}$, we deduce:

$$\begin{aligned}\bigg|\int_{\mathcal{Z}} b(z)\,d\mu_t(z) - \int_{\mathcal{Z}} b(\mathbf{F}_t(z))\,d\mu_0\bigg| &\leqslant \langle R\rangle^{p+q} 2^{\frac{p+q}{2(r-1)}}(p+q)|b|_{\mathcal{P}_{p,q}}\big[4r^3\|Q\|\langle R\rangle^{2(r-1)}|t|\big]^K \\ &+ \bigg|\lim_{n\to\infty}\int_0^t dt_1 \cdots \int_0^{t_{K-1}} dt_K\, \mathrm{Tr}\big[\tilde{\varrho}_{\varepsilon_n}(t_K)\big(\big\{Q_{t_K}, \dots \{Q_{t_1}, b_t\}\dots\big\}\big)^{Wick}\big]\bigg|. \end{aligned} \tag{45}$$

The number estimate of Proposition 2.3 with the inequality (39) of Proposition 3.4 implies:

$$\big|\langle \mathbf{N}\rangle^{-\frac{q+K(r-1)}{2}}\big(\big\{Q_{t_K}, \dots \{Q_{t_1}, b_t\}\dots\big\}\big)^{Wick}\langle \mathbf{N}\rangle^{-\frac{p+K(r-1)}{2}}\big|_{\mathcal{L}(\mathcal{H})} \leqslant 2^{\frac{p+q}{2(r-1)}}(p+q)\big(4r^3\big)^K \|Q\|^K |b|_{\mathcal{P}_{p,q}}.$$

This provides for the last term in the r.h.s. of (45) the upper bound:

$$\langle R\rangle^{\frac{p+q}{2}+K(r-1)} 2^{\frac{p+q}{2(r-1)}}(p+q)\big(4r^3\big)^K\|Q\|^K|b|_{\mathcal{P}_{p,q}}|t|^K.$$

For small times, $|t| \leqslant T_\delta = \frac{\delta}{(4r^3)\|Q\|\langle R\rangle^{r-1}}$ with $\delta < 1$, taking the limit as $K \to \infty$ now gives:

$$\forall b \in \mathcal{P}^\infty_{p,q}(\mathcal{Z}), \quad \int_{\mathcal{Z}} b(z)\,d\mu_t(z) = \int_{\mathcal{Z}} b\big(\mathbf{F}_t(z)\big)\,d\mu_0(z).$$

But according to Proposition 3.6, the measure $\mu_t$ is a Borel probability measure supported in the ball $\{|z| \leqslant R\}$ which is weakly compact. Meanwhile cylindrical polynomials which are contained in $\mathcal{P}^\infty_{alg}(\mathcal{Z})$, because they are associated with finite rank kernels, make a dense set in the $\mathcal{C}^0(B(0,R)_{weak}, \mathbb{C})$ and therefore in $L^1(\mathcal{Z}, d\mu)$. Thus, we have proved that

$$\forall t \in [-T_\delta, T_\delta], \quad \mu_t = (\mathbf{F}_t)_*\mu_0.$$

Finally, since $|\mathbf{F}_t(z)| = |z|$ and $[H_\varepsilon, \mathbf{N}] = 0$, the pair $((\varrho_{\varepsilon_n}(t))_{n\in\mathbb{N}}, \mu_t)$ satisfies the same assumptions as $((\varrho_{\varepsilon_n})_{n\in\mathbb{N}}, \mu_0)$. Since the time $T_\delta$ depends only on $Q$ and $R$ the result extends to all $t \in \mathbb{R}$. $\square$

### *3.4. Proof of the main result*

Gathering all the information of Sections 2 and 3, we are now in position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states satisfying hypothesis of Theorem 1.1 and let $\chi \in \mathcal{C}^0([0,\infty), \mathbb{R})$ be a continuous cutoff function such that $0 \leqslant \chi \leqslant 1$, $\chi(x) = 1$ if $x \leqslant 1/2$ and $\chi(x) = 0$ if $x \geqslant 1$. For $R > 0$, consider the family of normal states,

$$\varrho^R_\varepsilon = \frac{\chi(\mathbf{N}/R^2)\varrho_\varepsilon\chi(\mathbf{N}/R^2)}{\mathrm{Tr}[\chi(\mathbf{N}/R^2)\varrho_\varepsilon\chi(\mathbf{N}/R^2)]},$$

localized in the ball of radius $R$. By Proposition 2.16, we know that

$$\text{(i)}\quad \mathcal{M}\big(\varrho^R_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big) = \left\{\frac{\chi^2(|z|^2/R^2)}{\int_{\mathcal{Z}}\chi^2(|z|^2/R^2)\,d\mu_0}\mu_0\right\} =: \big\{\mu^R_0\big\},$$

$$\text{(ii)}\quad \forall \alpha \in \mathbb{N}, \quad \lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho^R_\varepsilon \mathbf{N}^\alpha\big] = \int_{\mathcal{Z}} |z|^{2\alpha}\,d\mu^R_0(z).$$

Next, we use the notations:

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon} \quad \text{and} \quad \varrho_\varepsilon^R(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon^R e^{i\frac{t}{\varepsilon}H_\varepsilon}.$$

For any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ there exists by Proposition 3.3 a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a family of Borel probability measures $(\mu_t^R)_{t\in\mathbb{R}}$ such that

$$\text{(i)}' \quad \mathcal{M}\big(\varrho_{\varepsilon_{n_k}}^R(t), k\in\mathbb{N}\big) = \big\{\mu_t^R\big\},$$

$$\text{(ii)}' \quad \forall\alpha\in\mathbb{N}, \quad \lim_{k\to\infty}\operatorname{Tr}\big[\varrho_{\varepsilon_{n_k}}^R \mathbf{N}^\alpha\big] = \int_{\mathcal{Z}} |z|^{2\alpha}\, d\mu_0^R(z).$$

Applying now Proposition 3.6 with (i)′–(ii)′, we obtain that

$$\mathcal{M}\big(\varrho_{\varepsilon_{n_k}}^R(t),\ k\in\mathbb{N}\big) = \big\{(\mathbf{F}_t)_*\mu_0^R\big\}, \tag{46}$$

for any time $t\in\mathbb{R}$. Since for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ we can extract a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that (46) holds we conclude that

$$\mathcal{M}\big(\varrho_\varepsilon^R(t),\ \varepsilon\in(0,\bar\varepsilon)\big) = \big\{(\mathbf{F}_t)_*\mu_0^R\big\}, \tag{47}$$

for any $R>0$ and $t\in\mathbb{R}$. Again applying Proposition 3.3 for $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$, there exists for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a family of Borel probability measures $(\mu_t)_{t\in\mathbb{R}}$ such that

$$\mathcal{M}\big(\varrho_{\varepsilon_{n_k}}(t),\ k\in\mathbb{N}\big) = \{\mu_t\}.$$

The identification of the measures $(\mu_t)_{t\in\mathbb{R}}$ follows by a $\delta/3$ argument. For any $b\in\mathcal{S}_{cyl}(\mathcal{Z})$ based in $p\mathcal{Z}$, $p\in\mathbb{P}$, we write:

$$\left|\operatorname{Tr}\big[\varrho_{\varepsilon_{n_k}}(t)b^{Weyl}\big] - \int_{\mathcal{Z}} b(z)\, d(\mathbf{F}_t)_*\mu_0\right| \leqslant \big|\operatorname{Tr}\big[\varrho_{\varepsilon_{n_k}}(t)b^{Weyl}\big] - \operatorname{Tr}\big[\varrho_{\varepsilon_{n_k}}^R(t)b^{Weyl}\big]\big| \tag{48}$$

$$+ \left|\operatorname{Tr}\big[\varrho_{\varepsilon_{n_k}}^R(t)b^{Weyl}\big] - \int_{\mathcal{Z}} b(z)\, d\mu_t^R\right| \tag{49}$$

$$+ \left|\int_{\mathcal{Z}} b\big(\mathbf{F}_t(z)\big)\, d\mu_0^R - \int_{\mathcal{Z}} b\big(\mathbf{F}_t(z)\big)\, d\mu_0\right|. \tag{50}$$

Each term (48)–(50) can be made arbitrarily small by choosing $R$ and $k$ large enough and respectively using the bound (21), the relation (47) and the dominated convergence theorem. So, we conclude that $\mu_t = (\mathbf{F}_t)_*\mu_0$ and hence we have proved:

$$\mathcal{M}\big(\varrho_\varepsilon,\ \varepsilon\in(0,\bar\varepsilon)\big) = \big\{(\mathbf{F}_t)_*\mu_0\big\}.$$

Finally, the use of Proposition 2.12 with $\varrho_\varepsilon(t)$ yields,

$$\lim_{\varepsilon\to 0}\operatorname{Tr}\big[\varrho_\varepsilon(t)b^{Wick}\big] = \int_{\mathcal{Z}} b\circ\mathbf{F}_t(z)\, d\mu_0(z),$$

since $\lim_{\varepsilon\to 0}\operatorname{Tr}[\varrho_\varepsilon(t)\mathbf{N}^\alpha] = \lim_{\varepsilon\to 0}\operatorname{Tr}[\varrho_\varepsilon\mathbf{N}^\alpha] = \int_{\mathcal{Z}}|z|^{2\alpha}\, d\mu_0 = \int_{\mathcal{Z}}|z|^{2\alpha}\, d\mu_t$, for all $\alpha\in\mathbb{N}$. The reformulation of this result in terms of BBGKY hierarchy of reduced matrices is a consequence of Proposition 2.11. □

### *3.5. Additional results*

Although it was not written in Theorem 1.1, remember that the existence of Wigner measures contains a result for Weyl observables.

**Corollary 3.7.** *Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ *be a family of normal states on* $\mathcal{H}$ *satisfying the hypothesis of Theorem* 1.1.
*The limit,*

$$\lim_{\varepsilon\to 0}\mathrm{Tr}\big[e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}b^{Weyl}\big]=\int_{\mathcal{Z}} b\circ \mathbf{F}_t(z)\,d\mu_0,$$

*holds for any* $b$ *in the cylindrical Schwartz space* $\mathcal{S}_{cyl}(\mathcal{Z})$, *any* $t\in\mathbb{R}$ *and any* $b\in S^{\nu}_{p\mathcal{Z}}$, $\nu\in[0,1]$, $p\in\mathbb{P}$.

The next result shows that the class of observables can be extended to functions of Wick-quantized symbols.

**Corollary 3.8.** *Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ *be a family of normal states on* $\mathcal{H}$ *satisfying the hypothesis of Theorem* 1.1. *Then*

(i) *The limit,*

$$\lim_{\varepsilon\to 0}\mathrm{Tr}\big[e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}f\big(b^{\mathrm{Wick}}\big)\big]=\int_{\mathcal{Z}} f\big(b\circ \mathbf{F}_t(z)\big)\,d\mu_0, \tag{51}$$

*holds for any* $f\in\mathcal{F}^{-1}(\mathcal{M}_b(\mathbb{R}))$ *and any* $b\in\mathcal{P}_{p,p}(\mathcal{Z})$ *such that* $\tilde b^*=\tilde b$.

(ii) *If additionally* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ *is a family of localized states on a ball of radius* $R>0$, *then the limit* (51) *holds for any entire function* $f(x)=\sum_{k=0}^\infty a_k x^k$ *over* $\mathbb{C}$ *and any* $b\in\mathcal{P}_{p,p}(\mathcal{Z})$ *such that* $\tilde b^*=\tilde b$.

**Proof.** (i) Let $\chi\in\mathcal{C}^0([0,\infty),\mathbb{R})$ be a continuous cutoff function such that $0\leqslant\chi\leqslant 1$, $\chi(x)=1$ if $x\leqslant 1/2$ and $\chi(x)=0$ if $x\geqslant 1$. Consider the family $(\varrho_\varepsilon(t)=e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H^\varepsilon})_{\varepsilon\in(0,\bar\varepsilon)}$, with

$$\varrho_\varepsilon^R=\frac{\chi(\mathbf{N}/R^2)\varrho_\varepsilon\chi(\mathbf{N}/R^2)}{\mathrm{Tr}[\chi(\mathbf{N}/R^2)\varrho_\varepsilon\chi(\mathbf{N}/R^2)]},\qquad R>0.$$

Let $b\in\mathcal{P}_{p,p}(\mathcal{Z})$ such that $\tilde b^*=\tilde b$, then $b^{Wick}$ extends to a self-adjoint operator on $\mathcal{H}$ satisfying $[\mathbf{N},b^{Wick}]=0$. We claim that

$$\forall\theta\in\mathbb{R},\quad \mathrm{Tr}\big[\varrho_\varepsilon^R(t)e^{i\theta b^{Wick}}\big]=\sum_{k=0}^\infty\frac{i^k}{k!}\theta^k\,\mathrm{Tr}\big[\varrho_\varepsilon^R(t)\big(b^{Wick}\big)^k\big]. \tag{52}$$

Thanks to the estimate,

$$\begin{aligned}\big|\mathrm{Tr}\big[\varrho_\varepsilon^R(t)\big(b^{Wick}\big)^k\big]\big|&=\big|\mathrm{Tr}\big[\langle\mathbf{N}\rangle^{pk/2}\varrho_\varepsilon^R(t)\langle\mathbf{N}\rangle^{pk/2}\big(\langle\mathbf{N}\rangle^{-p/2}b^{Wick}\langle\mathbf{N}\rangle^{-p/2}\big)^k\big]\big|\\&\leqslant\langle R\rangle^{pk}|b|^k_{\mathcal{P}_{p,p}},\end{aligned} \tag{53}$$

the l.h.s. of (52) is an absolutely convergent series uniformly in $\varepsilon\in(0,\bar\varepsilon)$. Moreover, on can easily show the strong limit,

$$s-\lim_{N\to\infty}\sum_{k=0}^N\frac{i^k}{k!}\theta^k\big(b^{Wick}\big)^k 1_{[0,R^2]}(\mathbf{N})=e^{i\theta b^{Wick}}1_{[0,R^2]}(\mathbf{N}).$$

Therefore, we see that

$$\begin{aligned}\sum_{k=0}^\infty\frac{i^k}{k!}\theta^k\,\mathrm{Tr}\big[\varrho_\varepsilon^R(t)\big(b^{Wick}\big)^k\big]&=\sum_{k=0}^\infty\frac{i^k}{k!}\theta^k\,\mathrm{Tr}\big[\varrho_\varepsilon^R(t)\big(b^{Wick}\big)^k 1_{[0,R^2]}(\mathbf{N})\big]\\&=\mathrm{Tr}\big[\varrho_\varepsilon^R(t)e^{i\theta b^{Wick}}\big].\end{aligned}$$

This proves (52) and again by the uniform estimate (53) with respect to $\varepsilon\in(0,\bar\varepsilon)$, we obtain:

$$\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon^R(t)e^{i\theta b^{Wick}}\big]=\sum_{k=0}^\infty\frac{i^k}{k!}\theta^k\int_{\mathcal{Z}} b\big(\mathbf{F}_t(z)\big)^k\,d\mu_0=\int_{\mathcal{Z}} e^{-i\theta b(\mathbf{F}_t(z))}\,d\mu_0.$$

Now, a similar $\delta/3$ argument as in the proof of Theorem 1.1

$$\begin{aligned}&\left|\mathrm{Tr}\big[\varrho_{\varepsilon_{n_k}}(t)e^{i\theta b^{Wick}}\big]-\int_{\mathcal{Z}} e^{i\theta b(\mathbf{F}_t(z))}\,d\mu_0\right|\\&\quad\leqslant\big|\varrho_{\varepsilon_{n_k}}-\varrho^R_{\varepsilon_{n_k}}\big|_{\mathcal{L}^1(\mathcal{H})}+\left|\mathrm{Tr}\big[\varrho^R_{\varepsilon_{n_k}}(t)e^{i\theta b^{Wick}}\big]-\int_{\mathcal{Z}} e^{i\theta b(\mathbf{F}_t(z))}\,d\mu^R_0\right|\\&\qquad+\left|\int_{\mathcal{Z}} e^{i\theta b(\mathbf{F}_t(z))}\,d\mu^R_0-\int_{\mathcal{Z}} e^{i\theta b(\mathbf{F}_t(z))}\,d\mu_0\right|,\end{aligned}$$

using the bound (21), the relation (47) and the dominated convergence theorem, yields the limit,

$$\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_{\varepsilon_{n_k}}(t)e^{i\theta b^{Wick}}\big]=\int_{\mathcal{Z}} e^{i\theta b(\mathbf{F}_t(z))}\,d\mu_0.$$

By integrating with respect to $\mathcal{F}(f)\in\mathcal{M}_b(\mathbb{R})$, we end the proof.

(ii) The proof is similar to (i). Indeed, one shows

$$\mathrm{Tr}\big[\varrho_\varepsilon(t)f\big(b^{Wick}\big)\big]=\sum_{k=0}^{\infty}a_k\,\mathrm{Tr}\big[\varrho_\varepsilon(t)\big(b^{Wick}\big)^k\big],\tag{54}$$

with an l.h.s. absolutely convergent series uniformly in $\varepsilon\in(0,\bar\varepsilon)$. Letting $\varepsilon\to 0$ in (54) yields the result. □

## 4. Examples

We review a series of examples. Firstly, the propagation of coherent states and Hermite states is recalled. Secondly, bounded interactions occur naturally within the modeling of rapidly rotating Bose–Einstein condensates, owing to some hypercontractivity property. Thirdly, the tensor decomposition of the Fock space allows to specify some Wigner measures for which the propagation cannot be translated in terms of the reduced density matrices without writing all the BBGKY hierarchy. Finally, the result of Theorem 1.1 provides a new way to consider the Hartree–von Neumann limit in the mean field regime.

### *4.1. Coherent and Hermite states*

The coherent states on the Fock space, $\Gamma_s(\mathcal{Z})$ are given by $E(\xi)=W(\frac{\sqrt{2}}{i\varepsilon}\xi)\Omega=e^{\frac{a^*(\xi)-a(\xi)}{\varepsilon}}\Omega$, where $\Omega$ is the vacuum vector of $\Gamma_s(\mathcal{Z})$, $\xi\in\mathcal{Z}$ and $[a(f),a^*(g)]=\varepsilon\langle f,g\rangle I$. The Hepp method [36,30,31] consists in studying the propagation of squeezed coherent states a slightly larger class which includes covariance deformations. The normal state made with $E(\xi)$ is

$$\varrho_\varepsilon(\xi):=\left|W\left(\frac{\sqrt{2}}{i\varepsilon}\xi\right)\Omega\right\rangle\left\langle W\left(\frac{\sqrt{2}}{i\varepsilon}\xi\right)\Omega\right|.$$

We proved in [7] that $\mathcal{M}(\varrho_\varepsilon(\xi),\varepsilon\in(0,\bar\varepsilon))=\{\delta_\xi\}$ and a simple computation shows that the property (PI) is satisfied:

$$\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon\mathbf{N}^k\big]=|\xi|^{2k}=\delta_\xi\big(|z|^{2k}\big).$$

A second example is given by Hermite states, also well studied within the propagation of chaos technique or other works (e.g., [44,13,23]). They are given by:

$$\varrho_N(\varphi):=\big|\varphi^{\otimes N}\big\rangle\big\langle\varphi^{\otimes N}\big|,\tag{55}$$

with $\varphi \in \mathcal{Z}$, $|\varphi|_{\mathcal{Z}} = 1$ and discrete values for $\varepsilon = \frac{1}{N}$. We know from [7] that $\mathcal{M}(\varrho_N(\varphi), N \in \mathbb{N}) = \{\frac{1}{2\pi}\int_0^{2\pi} \delta_{e^{i\theta}\varphi}\, d\theta\}$ where the rotation invariance is the phase-space translation of the gauge invariance of the Hermite states $\varphi \to e^{i\theta}\varphi$. One easily checks the property (PI):

$$\lim_{N\to\infty} \mathrm{Tr}\big[\varrho_N(\varphi)\mathbf{N}^k\big] = 1 = \frac{1}{2\pi}\int_0^{2\pi} |z|^{2k}\delta_{e^{i\theta}\varphi}(z)\, d\theta.$$

It is convenient to introduce a notation for this Wigner measure.

**Definition 4.1.** For $\varphi \in \mathcal{Z}$, the symbol $\delta_\varphi^{S^1}$ denotes the Borel probability measure:

$$\delta_\varphi^{S^1} = \frac{1}{2\pi}\int_0^{2\pi} \delta_{e^{i\theta}\varphi}\, d\theta.$$

Theorem 1.1 applies and the Wigner measures associated with,

$$\big(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon(\xi)e^{i\frac{t}{\varepsilon}H_\varepsilon}\big)_{\varepsilon\in(0,\bar{\varepsilon})} \quad \text{and} \quad \big(e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_N(\varphi)e^{i\frac{t}{\varepsilon}H_\varepsilon}\big)_{\varepsilon=1/N,\ N\in\mathbb{N}^*}$$

are respectively $\delta_{\xi_t}$ and $\delta_{\varphi_t}^{S^1}$, where $\xi_t$ or $\varphi_t$ evolves according to the classical flow.

For example, when

$$H_\varepsilon = \mathrm{d}\Gamma(-\Delta) + \frac{1}{2}\int_{\mathbb{R}^{2d}} V(x-y)a^*(x)a^*(y)a(x)a(y)\, dx\, dy,$$

with $\mathcal{Z} = L^2(\mathbb{R}^d)$ the classical flow is the Hartree equation:

$$i\partial_t\psi = -\Delta\psi + \big(V * |\psi|^2\big)\psi.$$

We conclude by noticing that for such states ($\varrho_N(\varphi)$ and $\varrho_\varepsilon(\xi)$) the asymptotic one particle reduced density matrix $\gamma_0^{(1)}(t)$ solves the equation:

$$\begin{cases} i\partial_t\gamma_0^{(1)} = \big[-\Delta + (V * n_{\gamma_0^{(1)}}), \gamma_0^{(1)}\big], \\ \gamma_0^{(1)}(t=0) = |\xi\rangle\langle\xi| \quad \text{for } \varrho_\varepsilon(\xi), \\ \big(\text{resp. } \gamma_0^{(1)}(t=0) = |\varphi\rangle\langle\varphi| \quad \text{for } \varrho_N(\varphi)\big), \end{cases} \tag{56}$$

with $n_{\gamma_0^{(1)}}(x) = \gamma_0^{(1)}(x,x)$.

### 4.2. LLL-mean field dynamics for rapidly rotating Bose–Einstein condensates

The case of bounded interaction terms occurs exactly in the modeling of rapidly rotating Bose–Einstein condensates in the Lowest Landau Level (LLL) regime. The (LLL) one particle states can be described (see [2]) within the Bargmann space

$$\mathcal{Z} = \big\{f \in L^2\big(\mathbb{C}_{\zeta_1}, e^{-\frac{|\zeta_1|^2}{h}}L(d\zeta_1)\big),\ \partial_{\bar{\zeta}_1}f = 0\big\},$$

where $L(d\zeta_1)$ is the Lebesgue measure on $\mathbb{C}$, $h > 0$ is a parameter which is small in the rapid rotation regime and where the norm on $\mathcal{Z}$ is given by,

$$|f|_{\mathcal{Z}}^2 = \int_{\mathbb{C}} \big|f(\zeta_1)\big|^2 e^{-\frac{|\zeta_1|^2}{h}}\frac{L(d\zeta_1)}{(\pi h)} = \frac{1}{\pi h}|u|_{L^2}^2, \quad u(\zeta_1) = f(\zeta_1)e^{-\frac{|\zeta_1|^2}{2h}}.$$

The multiparticle bosonic problem has been considered in [41] and the LLL-model has been justified for the stationary states of such a system not only in the mean field asymptotics. The $k$-particle states are elements of,

$$\bigvee^k \mathcal{Z} = \left\{F \in L^2\left(\mathbb{C}^k_\zeta, e^{-\frac{|\zeta|^2}{h}} L(d\zeta)\right),\ \partial_{\bar\zeta} F = 0,\ F(\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(k)}) = F,\ \forall \sigma \in \mathfrak{S}_k\right\},$$

with the norm

$$|F|^2_{\bigvee^k \mathcal{Z}} = \int_{\mathbb{C}^k} \left|F(\zeta)\right|^2 \frac{L(d\zeta)}{(\pi h)^k}.$$

With or without the symmetry condition, $\bigotimes^k \mathcal{Z}$ and $\bigvee^k \mathcal{Z}$ are closed subspaces of $L^2(\mathbb{C}^k_\zeta, e^{-\frac{|\zeta|^2}{h}} L(d\zeta))$ and they are the image of the orthogonal projection (add the symmetry for $\bigvee^k \mathcal{Z}$)

$$\left(\Pi^k_h G\right)(\zeta) = \int_{\mathbb{C}^k} e^{\frac{\zeta.\tau - |\tau|^2}{h}} G(\tau) \frac{L(d\tau)}{(\pi h)^k}.$$

Within the modeling of rapidly rotating Bose–Einstein condensates, the one particle kinetic energy term is $A = h\zeta_1 \partial_{\zeta_1}$ and it is associated with,

$$0 \leqslant E_{kin}(f) = \langle f, h\zeta_1 \partial_{\zeta_1} f\rangle_{\mathcal{Z}}.$$

The standard one particle nonlinear energy is given by,

$$\alpha \int_{\mathbb{C}} |u|^4 L(d\zeta_1), \qquad u(\zeta_1) = f(\zeta_1) e^{-\frac{|\zeta_1|^2}{2h}},$$

where $\alpha > 0$ is another parameter provided by the physics (see [2]), but more general energies can be considered:

$$E_{NL}(f) = \sum_{p=2}^r \alpha_p \int_{\mathbb{C}} |u|^{2p} L(d\zeta_1), \qquad u(\zeta_1) = f(\zeta_1) e^{-\frac{|\zeta_1|^2}{2h}}, \quad \alpha_p > 0. \tag{57}$$

The mean field Hamiltonian is thus given by:

$$h(f) = E_{kin}(f) + E_{NL}(f) = \langle f, h\zeta_1 \partial_{\zeta_1} f\rangle + \sum_{p=2}^r \alpha_p \int_{\mathbb{C}} \left|f(\zeta_1)\right|^{2p} e^{-\frac{p|\zeta_1|^2}{h}} L(d\zeta_1).$$

An important property of these nonlinear energies comes from the hypercontractivity of the semigroup $(e^{-th\xi\partial_\xi})_{t\geqslant 0}$ proved in [16] which can be written as

$$|U|_{L^{2p}} \leqslant C_{p,h,d} |U|_{L^2} \quad \text{if } U(\zeta) = F(\zeta) e^{-\frac{|\zeta|^2}{2h}}, \quad F \in \bigotimes^k \mathcal{Z},\ p \in [2, +\infty]. \tag{58}$$

This implies that the nonlinear energy is a norm continuous polynomial with respect to $f \in \mathcal{Z}$ and therefore the nonlinear mean field, equation,

$$i\partial_t f = h\zeta_1 \partial_{\zeta_1} f + \sum_{p=2}^r p\alpha_p \Pi^1_h\left(|u|^{2(p-1)} u\right) \Pi^1_h f, \tag{59}$$

defines a nonlinear flow on the phase-space $\mathcal{Z}$ according to Section 3.2 (we refer the reader to [45] for a more detailed analysis of the nonlinear dynamics of the LLL-model).

Let us consider the second quantized version $H_\varepsilon$ of the energy $h$ in $\Gamma_s(\mathcal{Z})$. The kinetic energy is nothing but $\mathrm{d}\Gamma(A)$:

$$\mathrm{d}\Gamma(A)|_{\bigvee^k \mathcal{Z}} = \varepsilon \sum_{j=1}^k h\zeta_j \partial_{\zeta_j} = \varepsilon h \zeta.\partial_\zeta,$$

and the quantum Hamiltonian $H_\varepsilon$ is then,

$$H_\varepsilon = \mathrm{d}\Gamma(A) + \sum_{p=2}^r \alpha_p Q^{Wick}_p \tag{60}$$

with

$$Q_p(f)=\int_{\mathbb{C}}\left|u(\zeta_1)\right|^{2p}L(d\zeta_1)=\int_{\mathbb{C}}\left|f(z)\right|^{2p}e^{-\frac{p|\zeta_1|^2}{h}}L(d\zeta_1)=\left\langle f^{\otimes p},\tilde{Q}_p f^{\otimes p}\right\rangle. \tag{61}$$

The operator $\tilde{Q}_p$ is easily identified after removing the center of mass in multiple integrals (see [41] for details) as

$$\tilde{Q}_pF(\zeta)=\Pi_p^h\left(\left[\prod_{j=1}^{p-1}\delta(\zeta_j')\right]F\right)(\zeta)=\frac{1}{(\pi h)^p}F\left(\frac{\zeta_1+\cdots+\zeta_p}{p},\ldots,\frac{\zeta_1+\cdots+\zeta_p}{p}\right),$$

with $\zeta_j'=\zeta_j-\frac{\zeta_1+\cdots+\zeta_p}{p}$. One easily checks as well, by using additionally the hypercontractivity estimate (58) with $p=+\infty$, that $\tilde{Q}_p\in\mathcal{L}(\bigvee^p\mathcal{Z})$.

The propagation result of Theorem 1.1 applies for such a model for all initial states which fulfill its assumptions (boundedness of all moments and condition (PI)).

### *4.3. Fock tensorization*

We have already used, and it is the basis of the introduction of cylindrical observables, the fact that $\Gamma_s(\mathcal{Z})\sim\Gamma_s(\mathcal{Z}_1)\otimes\Gamma_s(\mathcal{Z}_2)$ when $\mathcal{Z}=\mathcal{Z}_1\overset{\perp}{\oplus}\mathcal{Z}_2$. The definition of Wigner measures introduced via cylindrical observables, yields the next result.

**Lemma 4.2.** *Assume $\mathcal{Z}=\mathcal{Z}_1\overset{\perp}{\oplus}\mathcal{Z}_2$ and let $(\varrho_\varepsilon^1)_{\varepsilon\in(0,\bar\varepsilon)}$, $(\varrho_\varepsilon^2)_{\varepsilon\in(0,\bar\varepsilon)}$ be two families of normal states on $\Gamma_s(\mathcal{Z}_1)$ and $\Gamma_s(\mathcal{Z}_2)$ such that $\mathrm{Tr}[\varrho_\varepsilon^\ell\mathbf{N}_\ell^\delta]\leqslant C_\delta$ holds uniformly for some $\delta>0$ and $\mathcal{M}(\varrho_\varepsilon^\ell,\ \varepsilon\in(0,\bar\varepsilon))=\{\mu^\ell\}$ for $\ell=1,2$. Let $\varrho_\varepsilon$ be the state on $\Gamma_s(\mathcal{Z})$ identified with $\varrho_\varepsilon^1\otimes\varrho_\varepsilon^2$ in the decomposition $\Gamma_s(\mathcal{Z})\sim\Gamma_s(\mathcal{Z}_1)\otimes\Gamma_s(\mathcal{Z}_2)$. Then the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ admits the unique Wigner measure $\mu=\mu^1\times\mu^2$ on the phase-space $\mathcal{Z}=\mathcal{Z}_1\times\mathcal{Z}_2$.*

Before giving applications and variations on this result it is worth to notice that the identification of the "tensor" state $\varrho_\varepsilon$ requires some care. It is not equal in general to $\varrho_\varepsilon^1\otimes\varrho_\varepsilon^2$ since such a states does not preserve the symmetric Fock space $\Gamma_s(\mathcal{Z})$.

Here is a simple example, take $\varphi_1\in\mathcal{Z}_1$ and $\varphi_2\in\mathcal{Z}_2$ with $|\varphi_\ell|_{\mathcal{Z}_\ell}=1$, $N_1,N_2\in\mathbb{N}$, and set $\varrho^\ell=|\varphi_\ell^{\otimes N_\ell}\rangle\langle\varphi_\ell^{\otimes N_\ell}|$ for $\ell=1,2$. The tensor states $\varrho^1\otimes\varrho^2$ is the pure state $|\varphi_1^{\otimes N_1}\otimes\varphi_2^{\otimes N_2}\rangle\langle\varphi_1^{\otimes N_1}\otimes\varphi_2^{\otimes N_2}|$ in $\Gamma_s(\mathcal{Z}_1)\otimes\Gamma_s(\mathcal{Z}_2)$. It suffices to identify the vector $\varphi^{\vee(N_1,N_2)}\in\Gamma_s(\mathcal{Z})$ associated with $\varphi_1^{\otimes N_1}\otimes\varphi_2^{\otimes N_2}$. It is the symmetric vector in $\bigvee^{N_1+N_2}\mathcal{Z}$ made with $N_1$-times $\varphi_1$ and $N_2$-times $\varphi_2$ and we can summarize the situation with,

$$\varphi_\ell^{\otimes N_\ell}=\frac{1}{\sqrt{\varepsilon^{N_\ell}N_\ell!}}\overbrace{a^*(\varphi_\ell)\cdots a^*(\varphi_\ell)}^{N_\ell\text{ times}}|\Omega_\ell\rangle\quad\text{in }\Gamma_s(\mathcal{Z}_\ell),\ \ell=1,2,$$

$$\begin{aligned}\varphi^{\bigvee(N_1,N_2)}&=\sqrt{\frac{(N_1+N_2)!}{\varepsilon^{(N_1+N_2)}N_1!N_2!}}\mathcal{S}_{N_1+N_2}\left(\varphi_1^{\otimes N_1}\otimes\varphi_2^{\otimes N_2}\right)\\&=\frac{1}{\sqrt{\varepsilon^{N_1+N_2}N_1!N_2!}}\overbrace{a^*(\varphi_1)\cdots a^*(\varphi_1)}^{N_1\text{ times}}\overbrace{a^*(\varphi_2)\cdots a^*(\varphi_2)}^{N_2\text{ times}}|\Omega\rangle\quad\text{in }\Gamma_s(\mathcal{Z}).\end{aligned}$$

The tensor decomposition is especially useful when $\mathcal{Z}$ is endowed with a Hilbert basis $(e_j)_{j\in\mathbb{N}^*}$. An Hilbert basis of $\Gamma_s(\mathcal{Z})$ is $(e^{\bigvee\alpha})_{\alpha\in\bigcup_{j=0}^\infty(\mathbb{N}^*)^j}$ given by:

$$e^{\bigvee\alpha}=\sqrt{\frac{|\alpha|!}{\alpha!}}\mathcal{S}_{|\alpha|}\left(e^{\otimes\alpha}\right)=\frac{1}{\sqrt{\varepsilon^{|\alpha|}|\alpha|!}}\left[a^*(e)\right]^\alpha|\Omega\rangle,$$

with a natural multi-index notation $\alpha=(\alpha_1,\ldots,\alpha_k)$, $|\alpha|=\alpha_1+\cdots+\alpha_k$, $e^{\otimes\alpha}=e_1^{\otimes\alpha_1}\otimes\cdots\otimes e_k^{\alpha_k}$, and

$$\left[a^*(e)\right]^\alpha=a^*(e_1)^{\alpha_1}\cdots a^*(e_k)^{\alpha_k}.$$

For example, the identification between $\Gamma_s(\mathbb{C}e_1) \otimes \Gamma_s((\mathbb{C}e_1)^\perp)$ and $\Gamma_s(\mathcal{Z})$ is done via the mapping defined by $e_1^{\vee\alpha_1} \otimes e^{\vee\alpha'} \to e^{\vee(\alpha_1,\alpha')}$, for all $\alpha_1 \in \mathbb{N}$ and all $\alpha' \in \bigcup_{k=0}^{\infty}(\mathbb{N}\setminus\{0,1\})^k$. This can be iterated but remember that the definition of infinite tensor products requires the additional specification of one vector per component which is hopefully rather canonical for Fock spaces endowed with a vacuum vector (see [34]).

Below is a notation convenient to the definition of tensor states and which allows some extensions. Consider the linear isometry $C_j$ on $\mathcal{H} = \Gamma_s(\mathcal{Z})$ defined by its action on the Hilbert basis $(e^{\vee\alpha})_{\alpha\in\bigcup_{k=0}^{\infty}(\mathbb{N}^*)^k}$,

$$C_j e^{\vee\alpha} = \frac{1}{|a^*(e_j)e^{\vee\alpha}|} a^*(e_j)e^{\vee\alpha} = \frac{1}{\sqrt{\varepsilon(\alpha_j+1)}} a^*(e_j)e^{\vee\alpha} = e^{\vee(\alpha+1_j)}, \tag{62}$$

with $|1_j| = 1$ and $(1_j)_j = 1$. In the tensor decomposition $\Gamma_s(\mathcal{Z}) \sim \Gamma_s(\mathbb{C}e_j) \otimes \Gamma_s((\mathbb{C}e_j)^\perp)$, this isometry $C_j$ is nothing but the tensor product $[\frac{1}{\sqrt{\mathbf{N}_j}} a^*(e_j)] \otimes I$.

**Definition 4.3.** Let $\mathcal{Z}$ be endowed with a Hilbert basis $(e_j)_{j\in\mathbb{N}^*}$, for $j \in \mathbb{N}^*$, and take the isometries $(C_j)_{j\in\mathbb{N}^*}$ defined in $\mathcal{H}$ by (62). For $j \in \mathbb{N}^*$, the operator $E_j$ is defined on $\mathcal{L}^1(\mathcal{H})$ by:

$$E_j \varrho = C_j \varrho C_j^*, \quad \forall \varrho \in \mathcal{L}^1(\mathcal{H}).$$

For $\lambda = (\lambda_j)_{j\in\mathbb{N}^*} \in \ell^1([0,+\infty))$ such that $\sum_{j=1}^{\infty}\lambda_j = 1$, the notation $\lambda.E$ means:

$$\lambda.E = \sum_{j=1}^{\infty} \lambda_j E_j.$$

The operators $E_j$ and $\lambda.E$ transform normal states on $\bigvee^{k-1}\mathcal{Z}$ into normal states on $\bigvee^k \mathcal{Z}$ and they all commute. After taking $\varphi_1 = e_1$ and $\varphi_2 = e_2$ the tensor state on $\Gamma_s(\mathcal{Z})$ identified with $\varrho^1 \otimes \varrho^2$ and studied above with $\mathcal{Z}_1 = \mathbb{C}e_1$ and $\mathcal{Z}_2 = (\mathbb{C}e_1)^\perp$ is nothing but

$$E^{(N_1,N_2)}|\Omega\rangle\langle\Omega| = E_1^{N_1}E_2^{N_2}|\Omega\rangle\langle\Omega| = E_2^{N_2}E_1^{N_1}|\Omega\rangle\langle\Omega|.$$

Moreover the multinomial formula holds:

$$(\lambda.E)^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!}\lambda^\alpha E^\alpha. \tag{63}$$

We use these notion to formulate the propagation of nontrivial Wigner measures. The Hamiltonian is

$$H_\varepsilon = \mathrm{d}\Gamma(A) + \left(\sum_{j=2}^{r}\langle z^{\otimes j}, \tilde{Q}_j z^{\otimes j}\rangle\right)^{Wick}, \quad \varepsilon = \frac{1}{N},$$

with $(A, \mathcal{D}(A))$ self-adjoint and $\tilde{Q}_j = \tilde{Q}_j^* \in \mathcal{L}(\bigvee^j \mathcal{Z})$. It is associated with the mean field Hamiltonian,

$$h(z,\bar{z}) = \langle z, Az\rangle + \sum_{j=2}^{r} Q_j(z),$$

and the flow $(\mathbf{F}_t)_{t\in\mathbb{R}}$ in the phase space $\mathcal{Z}$.

**Proposition 4.4.** *Let $\mathcal{Z}$ be endowed with an orthonormal basis $(e_j)_{j\in\mathbb{N}^*}$ and let the family $(E_j)_{j\in\mathbb{N}^*}$ be as in Definition* 4.3. *Once $\varrho_\varepsilon(0)$ is fixed $\varrho_\varepsilon(t)$ is defined by $\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}\varrho_\varepsilon(0)e^{i\frac{t}{\varepsilon}H_\varepsilon}$.*

(1) *For $k \in \mathbb{N}^*$ and $(\nu_1,\ldots,\nu_k) \in [0,1]^k$ fixed such that $\sum_{\ell=1}^{k}\nu_\ell = 1$, assume that $N_\ell$ equals the integer part $[\nu_\ell N]$ for $\ell \in \{1,\ldots,k\}$. Then the family of states $(\varrho_\varepsilon(t))_{\varepsilon=1/N}$ given by $\varrho_\varepsilon(0) = E^{(N_1,\ldots,N_k)}|\Omega\rangle\langle\Omega|$ admits a unique Wigner measure,*

$$\mu_t = (\mathbf{F}_t)_*\mu_0 = (\mathbf{F}_t)_*\big(\delta^{S^1}_{\sqrt{\nu_1}e_1} \times \cdots \times \delta^{S^1}_{\sqrt{\nu_k}e_k}\big).$$

*The reduced density matrices* ${\gamma_\varepsilon}^{(p)}(t)$ *converge in* $\mathcal{L}^1(\bigvee^p \mathcal{Z})$ *to*

$$\gamma_0^{(p)}(t) = \int_{\mathcal{Z}} \big|z_t^{\otimes p}\big\rangle\big\langle z_t^{\otimes p}\big|\, d\mu_0(z), \tag{64}$$

*by setting* $z_t = \mathbf{F}_t z$.

(2) *Let* $\lambda = (\lambda_j)_{j\in\mathbb{N}^*} \in \ell^1([0,+\infty))$ *be such that* $\sum_{j=1}^\infty \lambda_j = 1$. *Then the family of states* $(\varrho_\varepsilon(t))_{\varepsilon=1/N}$ *given by* $\varrho_\varepsilon = (\lambda.E)^N|\Omega\rangle\langle\Omega|$ *satisfies the same properties as above, with*

$$\mu_0 = \bigtimes_{j=1}^\infty \delta^{S^1}_{\sqrt{\lambda_j}e_j}.$$

**Proof.** Actually it suffices to identify the measure $\mu_0$ and to check the assumptions of Theorem 1.1 at time $t=0$.

(1) It is a simple application of Lemma 4.2 with the decomposition,

$$\Gamma_s(\mathcal{Z}) \sim \Gamma_s(\mathbb{C}e_1)\otimes\cdots\otimes\Gamma_s\big((\mathbb{C}e_{k-1})\big)\otimes\Gamma_s\big((\mathbb{C}e_1\oplus\cdots\oplus\mathbb{C}e_{k-1})^\perp\big).$$

In this decomposition $E^{(N_1,\ldots,N_k)}|\Omega\rangle\langle\Omega|$ is nothing but a tensor product of Hermite states. $|e_\ell^{\otimes N_\ell}\rangle\langle e_\ell^{\otimes N_\ell}|$ and the result is a simple tensorization of the result for Hermite states with $\varepsilon = \frac{\nu_\ell}{N_\ell}$.

(2) The state $\varrho_\varepsilon(0) = (\lambda.E)^N|\Omega\rangle\langle\Omega|$ belongs to $\mathcal{L}^1(\bigvee^N \mathcal{Z})$. It is therefore localized in the ball with radius 1. According to Proposition 2.15, its Wigner measures are completely determined if we know the limits of,

$$\mathrm{Tr}\big[\varrho_\varepsilon(0)b^{Wick}\big],$$

for all the $b\in\mathcal{P}^\infty_{alg}(\mathcal{Z})$. Due to Pythagorean summation, the measure $\mu_0 = \bigtimes_{j=1}^\infty \delta^{S^1}_{\sqrt{\lambda_j}e_j}$ is supported in the ball of radius 1. The estimates,

$$\begin{aligned}\big|\mathrm{Tr}\big[\varrho_\varepsilon(0)\big(b-b'\big)^{Wick}\big]\big| &= \big|\mathrm{Tr}\big[\varrho_\varepsilon(0)\chi(\mathbf{N})\big(b-b'\big)^{Wick}\chi(\mathbf{N})\big]\big|\\ &\leqslant C_{p,q}\big|b-b'\big|_{\mathcal{P}_{p,q}},\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\mathcal{Z}}\big(b(z)-b'(z)\big)\,d\mu_0(z)\right| &= \left|\int_{\mathcal{Z}}\big(b(z)-b'(z)\big)\chi^2\big(|z|^2\big)\,d\mu_0(z)\right|\\ &\leqslant C_{p,q}\big|b-b'\big|_{\mathcal{P}_{p,q}},\end{aligned}$$

with the first one deduced from the number estimate (10) in Proposition 2.3, hold for all $b,b'\in\mathcal{P}^\infty_{p,q}(\mathcal{Z})$, $p,q\in\mathbb{N}$, as soon as $\chi\in\mathcal{C}_0^\infty([0,+\infty))$ is chosen such that $\chi\equiv 1$ on $[0,1]$. Hence it suffices to prove $\lim_{\varepsilon\to 0}\mathrm{Tr}[\varrho_\varepsilon(0)b^{Wick}] = \int_{\mathcal{Z}} b(z)\,d\mu_0(z)$ for a total set of $\mathcal{P}^\infty_{alg}(\mathcal{Z})$. With the compact kernel condition, any $\tilde b\in\mathcal{L}^\infty(\bigvee^p\mathcal{Z},\bigvee^q\mathcal{Z})$ can be approximated by a linear combination of rank one operators of the form $|e^{\vee\gamma}\rangle\langle e^{\vee\beta}| = \sqrt{\frac{\beta!\gamma!}{|\beta|!|\gamma|!}}\mathcal{S}_{|\gamma|}|e^{\otimes\gamma}\rangle\langle e^{\otimes\beta}|\mathcal{S}_{|\beta|}$, $|\beta|=p$, $|\gamma|=q$. With

$$\big(\big\langle z^{\otimes q}, e^{\otimes\gamma}\big\rangle\big\langle e^{\otimes\beta}, z^{\otimes p}\big\rangle\big)^{Wick} = \big[a^*(e)\big]^\gamma\big[a(e)\big]^\beta \quad\text{and}\quad \varrho_\varepsilon(0) = \sum_{|\alpha|=N}\lambda^\alpha\frac{N!}{\alpha!}\big|e^{\vee\alpha}\big\rangle\big\langle e^{\vee\alpha}\big|,$$

we can compute directly:

$$\mathrm{Tr}\big[\varrho_\varepsilon(0)\big(\big\langle z^{\otimes q}, e^{\otimes\gamma}\big\rangle\big\langle e^{\otimes\beta}, z^{\otimes p}\big\rangle\big)^{Wick}\big] = \sum_{|\alpha|=N}\frac{N!}{\alpha!}\lambda^\alpha\big\langle a(e)^\gamma e^{\vee\alpha}, a(e)^\beta e^{\vee\alpha}\big\rangle.$$

Actually

$$a(e)^\beta e^{\vee\alpha} = \begin{cases}\sqrt{\varepsilon^p\frac{\alpha!}{\alpha'!}}e^{\vee\alpha'} & \text{if } \alpha=\alpha'+\beta,\\ 0 & \text{else,}\end{cases}$$

with a similar identity for $\gamma$ yields:

$$\begin{aligned}\mathrm{Tr}\big[\varrho_\varepsilon(0)\big(\langle z^{\otimes q}, e^{\otimes\gamma}\rangle\langle e^{\otimes\beta}, z^{\otimes p}\rangle\big)^{Wick}\big] &= \delta_{\beta,\gamma}\varepsilon^p\frac{N!}{(N-p)!}\bigg(\sum_{|\alpha'|=N-p}\frac{(N-p)!}{\alpha'!}\lambda^{\alpha'}\bigg)\lambda^\beta \\ &= \delta_{\beta,\gamma}\varepsilon^p N(N-1)\dots(N-p+1)\lambda^\beta.\end{aligned}$$

With $\varepsilon = 1/N$ and $(p,q)$ fixed, we obtain:

$$\begin{aligned}\mathrm{Tr}\big[\varrho_\varepsilon(0)\big(\langle z^{\otimes q}, e^{\otimes\gamma}\rangle\langle e^{\otimes\beta}, z^{\otimes p}\rangle\big)^{Wick}\big] &= \delta_{\beta,\gamma}\lambda^\beta \\ &= \int_{\mathcal{Z}}\langle z^{\otimes q}, e^{\otimes\gamma}\rangle\langle e^{\otimes\beta}, z^{\otimes p}\rangle\,d\mu_0(z). \qquad\square\end{aligned}$$

We conclude with two remarks:

- The tensorized Hermite state $E^{(N_1,\dots,N_\ell,\dots)}|\Omega\rangle\langle\Omega|$ with $N_\ell = [\lambda_\ell N]$ and $\sum_{j=1}^\infty \lambda_j = 1$ can be studied and behaves asymptotically like $(\lambda.E)^N\ |\Omega\rangle\langle\Omega|$.
- When those tensor states are not Hermite states, the reduced density matrices satisfy no closed equation and all the hierarchy has to be considered. In the example leading to (56) for Hermite states the general equation for $\gamma_0^{(1)}(t)$ writes,

$$\begin{aligned}i\partial_t\gamma_0^{(1)}(x,y) &= \big[-\Delta, \gamma_0^{(1)}\big](x,y) \\ &\quad + \int_{\mathbb{R}^d} V\big(x-x'\big)\gamma_0^{(2)}\big(x',x,x',y\big) - \gamma_0^{(2)}\big(x',x,x',y\big)V\big(y-x'\big)\,dx',\end{aligned}$$

  and the equation for $\gamma_0^{(2)}$ involves $\gamma_0^{(3)}$ and so on. The propagation of Wigner measures gathers all the asymptotic information in this case. Geometrically it is interesting to notice that if the initial Wigner measure is $\delta_{\sqrt{\lambda_1}e_1}^{S^1} \times \delta_{\sqrt{\lambda_2}e_1}^{S^1}$, with $\lambda_1 + \lambda_2 = 1$, it is supported by a 2-dimensional torus. After the action of the continuous flow, the support of $\mu_t$ remains topologically a 2-dimensional torus but in general deformed in the infinite dimensional phase space with no exact finite dimensional reduction.

### *4.4. Condition (PI) for Gibbs states*

For $\sigma(\varepsilon) \in \mathcal{L}^1(\mathcal{Z})$, which is a non-negative strict contraction:

$$\sigma(\varepsilon) = \sum_{i=1}^\infty \sigma_i(\varepsilon)\big|e_i(\varepsilon)\big\rangle\big\langle e_i(\varepsilon)\big|, \quad 0 \leqslant \sigma_i(\varepsilon) < 1, \quad \sum_{i=1}^\infty \sigma_i(\varepsilon) < +\infty,$$

where $(e_i(\varepsilon))_{i\in\mathbb{N}^*}$ is a Hilbert basis of $\mathcal{Z}$, the operator $\Gamma(\sigma(\varepsilon))$ belongs to $\mathcal{L}^1(\mathcal{H})$. It equals $\Gamma(\sigma(\varepsilon)) = \sum_{n=0}^\infty \mathcal{S}_n(\sigma(\varepsilon))^{\otimes n}\mathcal{S}_n$ and the tensor decomposition gives:

$$\mathrm{Tr}\big[\Gamma\big(\sigma(\varepsilon)\big)\big] = \prod_{i=1}^\infty \frac{1}{1-\sigma_i(\varepsilon)} \in \mathbb{R}_+.$$

Hence we can consider the quasi-free state:

$$\varrho_\varepsilon = \frac{1}{\mathrm{Tr}[\Gamma(\sigma(\varepsilon))]}\Gamma\big(\sigma(\varepsilon)\big).$$

It is more convenient to write

$$\sigma_i(\varepsilon) = \frac{\nu_i(\varepsilon)}{\nu_i(\varepsilon)+\varepsilon} \quad \text{with } \nu_i(\varepsilon) \in [0,+\infty),$$

and the condition $\sum_{i=1}^\infty \sigma_i(\varepsilon) < +\infty$ is equivalent to $\sum_{j=1} \nu_i(\varepsilon) < +\infty$.

**Lemma 4.5.** *For $\sigma(\varepsilon)=\sum_{i=1}^{\infty}\frac{\nu_i(\varepsilon)}{\nu_i(\varepsilon)+\varepsilon}|e_i(\varepsilon)\rangle\langle e_i(\varepsilon)|\in\mathcal{L}^1(\mathcal{Z})$, the quasi-free state $\varrho_\varepsilon=\frac{1}{\mathrm{Tr}[\Gamma(\sigma(\varepsilon))]}\Gamma(\sigma(\varepsilon))$ satisfies,*

$$\forall k\in\mathbb{N},\qquad \sup_{\varepsilon\in(0,\bar\varepsilon)}\mathrm{Tr}\big[\varrho_\varepsilon\mathbf{N}^k\big]<+\infty,$$

*if and only if there exists $C>0$ such that $\sum_{i=1}^{\infty}\nu_i(\varepsilon)\leqslant C$. In such a case, the quantity $\mathrm{Tr}[\varrho_\varepsilon\mathbf{N}^k]$, $k\in\mathbb{N}$, is equivalent to*

$$k!\sum_{|\alpha|=k}\nu(\varepsilon)^\alpha$$

*as $\varepsilon\to0$, with the usual multi-index convention, $\nu(\varepsilon)^\alpha=\prod_{k=1}^{\infty}\nu_k(\varepsilon)^{\alpha_k}$.*

**Proof.** Consider for $x\in[-c,c]$, $c>0$, the quantity:

$$\mathrm{Tr}\big[\varrho_\varepsilon(1+\varepsilon x)^{\frac{\mathbf{N}}{\varepsilon}}\big]=\frac{\prod_{i=1}^{\infty}\frac{1}{1-\frac{\nu_i(\varepsilon)}{\nu_i+\varepsilon}(1+\varepsilon x)}}{\prod_{i=1}^{\infty}\frac{\nu_i(\varepsilon)+\varepsilon}{\varepsilon}}=\prod_{i=1}^{\infty}\frac{1}{1-\nu_i(\varepsilon)x}.$$

When $\mathrm{Tr}[\varrho_\varepsilon\mathbf{N}^k]$ is uniformly bounded w.r.t. $\varepsilon\in(0,\bar\varepsilon)$, for all $k\in\mathbb{N}$ it is a $\mathcal{C}^\infty$ function around $x=0$, with

$$\begin{aligned}\partial_x^k\,\mathrm{Tr}\big[\varrho_\varepsilon(1+\varepsilon x)^{\frac{\mathbf{N}}{\varepsilon}}\big]\big|_{x=0}&=\mathrm{Tr}\big[\varrho_\varepsilon\mathbf{N}(\mathbf{N}-\varepsilon)\cdots\big(\mathbf{N}-(k-1)\varepsilon\big)\big]\\&\sim\mathrm{Tr}\big[\varrho_\varepsilon\mathbf{N}^k\big]\quad\text{as }\varepsilon\to0.\end{aligned}$$

But the first derivative is nothing but,

$$\partial_x\,\mathrm{Tr}\big[\varrho_\varepsilon(1+\varepsilon x)^{\frac{\mathbf{N}}{\varepsilon}}\big]\big|_{x=0}=\sum_{i=1}^{\infty}\nu_i(\varepsilon),$$

which says that the uniform bound $\sum_{i=1}^{\infty}\nu_i(\varepsilon)\leqslant C$ is a necessary condition.

Reciprocally when $\sum_{i=1}^{\infty}\nu_i(\varepsilon)\leqslant C$, then the function $\prod_{i=1}^{\infty}(1-\nu_j(\varepsilon)x)^{-1}$ is analytic with respect to $x$ in a disc of radius $R_C$ and equals,

$$\prod_{i=1}^{\infty}\big(1-\nu_j(\varepsilon)x\big)^{-1}=\prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty}\nu_i(\varepsilon)^jx^j\right)=\sum_{k=0}^{\infty}x^k\left[\sum_{|\alpha|=k}\nu(\varepsilon)^\alpha\right],$$

which yields the result. □

A Gibbs state is a quasi-free state with $\sigma(\varepsilon)=e^{-\varepsilon L(\varepsilon)}$ where $L(\varepsilon)$ is a strictly positive operator assumed here with a discrete spectrum:

$$L(\varepsilon)=\sum_{i=1}^{\infty}\ell_i(\varepsilon)|e_i\rangle\langle e_i|,\qquad \ell_i(\varepsilon)\leqslant\ell_{i+1}(\varepsilon),\tag{65}$$

where the basis $(e_j)_{j\in\mathbb{N}^*}$ is assumed independent of $\varepsilon\in(0,\bar\varepsilon)$ for the sake of simplicity. There is a simple translation of the assumptions of Theorem 1.1, the non-obvious one being the condition (PI) hidden in the assumption (2).

**Proposition 4.6.** *The Gibbs state $\varrho_\varepsilon=\frac{\Gamma(e^{-\varepsilon L(\varepsilon)})}{\mathrm{Tr}[\Gamma(e^{-\varepsilon L(\varepsilon)})]}$ with $L(\varepsilon)$ given in* (65) *satisfies the assumptions of Theorem* 1.1*, if and only if*:

- *For all $i\in\mathbb{N}^*$ the limit $\lim_{\varepsilon\to0}\ell_i(\varepsilon)=\ell_i(0)$ exists in $(0,+\infty]$.*
- *If $J\in\mathbb{N}^*\cup\{\infty\}$ denotes the largest element in $\mathbb{N}^*\cup\{\infty\}$ such that $\ell_i(0)<+\infty$ for all $i\leqslant J$, the two conditions are verified*:

$$\sum_{i=1}^{J}\frac{1}{\ell_i(0)}<+\infty,\tag{66}$$

*and*

$$\lim_{\varepsilon\to 0}\sum_{i>J}\frac{\varepsilon e^{-\varepsilon\ell_i(\varepsilon)}}{(1-e^{-\varepsilon\ell_i(\varepsilon)})}=0. \tag{67}$$

**Proof.** First of all, writing $\sigma(\varepsilon)=e^{-\varepsilon L(\varepsilon)}$ allows to apply Lemma 4.5 with $\nu_i(\varepsilon)=\frac{\varepsilon e^{-\varepsilon\ell_i(\varepsilon)}}{1-e^{-\varepsilon\ell_i(\varepsilon)}}$. From $e^{-\varepsilon\ell_i(\varepsilon)}\geqslant 1-\varepsilon\ell_i(\varepsilon)$ we deduce:

$$\nu_i(\varepsilon)\geqslant\frac{e^{-\varepsilon\ell_i(\varepsilon)}}{\ell_i(\varepsilon)}.$$

Hence the uniform boundedness of $\mathrm{Tr}[\varrho_\varepsilon\mathbf{N}^k]$ for $k\in\mathbb{N}$, which is equivalent to $\sum_{i=1}^\infty\nu_i(\varepsilon)\leqslant C$ implies:

$$\inf_{j\in\mathbb{N}^*,\varepsilon\in(0,\bar\varepsilon)}\ell_j(\varepsilon)=\kappa>0. \tag{68}$$

We now use the assumption that the family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ admits a unique Wigner measure $\mu_0$. As a quasi-free state, $\varrho_\varepsilon$ is given by its characteristic function (see for example [15] and [7] for the $\varepsilon$-dependent version),

$$\mathrm{Tr}\big[\varrho_\varepsilon W(f)\big]=e^{-\frac{\varepsilon}{4}\langle f,\frac{1+e^{-\varepsilon L(\varepsilon)}}{1-e^{-\varepsilon L(\varepsilon)}}f\rangle}.$$

But the Wigner measure is characterized by its characteristic function,

$$G(\xi)=\int_{\mathcal{Z}}e^{-2i\pi S(z,\xi)}\,d\mu_0(z)=\lim_{\varepsilon\to 0}\mathrm{Tr}\big[\varrho_\varepsilon W(\sqrt{2}\pi\xi)\big].$$

By taking $\xi=e_i,\ i\in\mathbb{N}^*$, this implies that the limit,

$$\lim_{\varepsilon\to 0}e^{-\frac{\varepsilon\pi^2}{2}\frac{1+e^{-\varepsilon\ell_i(\varepsilon)}}{1-e^{-\varepsilon\ell_i(\varepsilon)}}},$$

exists in $\mathbb{R}$. With the constraint (68) there are two possibilities: either $\lim_{\varepsilon\to 0}\ell_i(\varepsilon)=\ell_i(0)\in[\kappa,+\infty)$ and $G(e_i)=e^{-\frac{\pi^2}{\ell_i(0)}}$ or $\lim_{\varepsilon\to 0}\ell_i(\varepsilon)=+\infty$ and $G(e_i)=1$. After recalling that the $\ell_i(\varepsilon)$ are ordered and by introducing the index $J$ like in our statement, we get for $\xi=\sum_{i=1}^\infty\xi_i e_i\in\mathcal{Z}$:

$$G(\xi)=e^{-\pi^2\sum_{i=1}^J\frac{|\xi|^2}{\ell_i(0)}}.$$

The measure $\mu_0$ has to be the Gaussian measure:

$$\mu_0=\bigtimes_{i=1}^J\left[\frac{\ell_i(0)}{\pi}e^{-\ell_i(0)|z_i|^2}L(dz_i)\right],\qquad z=\sum_{i=1}^\infty z_ie_i.$$

Our assumptions imply that the integral $\int_{\mathcal{Z}}|z|^2\,d\mu_0(z)$ equals:

$$\sum_{i=1}^J\frac{1}{\ell_i(0)}=\int_{\mathcal{Z}}|z|^2\,d\mu_0(z)=\lim_{\varepsilon\to 0}\mathrm{Tr}[\varrho_\varepsilon\mathbf{N}].$$

After Lemma 4.5 we know that

$$\sum_{i=1}^J\frac{1}{\ell_i(0)}=\lim_{\varepsilon\to 0}\sum_{i=1}^\infty\nu_i(\varepsilon)=\lim_{\varepsilon\to 0}\sum_{i=1}^\infty\frac{\varepsilon e^{-\varepsilon\ell_i(\varepsilon)}}{1-\varepsilon\ell_i(\varepsilon)},$$

which enforces the two conditions (66) and (67).

Conversely assume that all the conditions are satisfied. Reconsidering the final argument in the proof of Lemma 4.5 says that the function,

$$\prod_{i=J+1}^\infty\big(1-\nu_i(\varepsilon)x\big)^{-1},$$

converges to 1 in a given neighborhood of $x=0$. Hence

$$\lim_{\varepsilon\to 0}\operatorname{Tr}\big[\varrho_\varepsilon \mathbf{N}^k\big]=\lim_{\varepsilon\to 0} k!\sum_{\substack{|\alpha|=k,\\ \alpha_i=0 \text{ for } i>J}}\left(\frac{\varepsilon e^{-\varepsilon\ell_i(\varepsilon)}}{1-e^{-\varepsilon\ell_i(\varepsilon)}}\right)^{\alpha}=k!\sum_{\substack{|\alpha|=k,\\ \alpha_i=0 \text{ for } i>J}}\ell(0)^{-\alpha},$$

which is easily checked to be equal to $\int_{\mathcal{Z}}|z|^{2k}\,d\mu_0(z)$. □

In the Bose–Einstein condensation of the free Bose gas in dimension 3, considered in [7], the first eigenvalue is tuned so that $\ell_1(0)\in(0,+\infty)$ and all the other eigenvalues are such that $\ell_i(0)=+\infty$. The condition which fails and gives rise to a physical example of dimensional defect of compactness is (67).

### *4.5. The Hartree–von Neumann limit*

Let $\varrho_0$ be a non-negative trace class operator on $L^2(\mathbb{R}^d)$ satisfying $\operatorname{Tr}[\varrho_0]=1$ and let:

$$\varrho^{\otimes N}=\varrho\otimes\cdots\otimes\varrho.$$

Consider the time-dependent von Neumann equation for a system of $N$ particles

$$\begin{cases} i\partial_t\varrho_N(t)=\big[\mathbb{H}_N,\varrho_N(t)\big],\\ \varrho_N(0)=\varrho_0^{\otimes N},\end{cases}\tag{69}$$

with $\varrho_N(t)$ is a trace class operator on $L^2(\mathbb{R}^d)^{\otimes N}\sim L^2(\mathbb{R}^{dN})$. Here $\mathbb{H}_N$ is the Hamiltonian of the $N$ particles system,

$$\mathbb{H}_N=\sum_{i=1}^N 1\otimes\cdots\otimes A\otimes\cdots\otimes 1+\frac{1}{N}\sum_{i<j}V(x_i-x_j),$$

with $A$ is a self-adjoint operator and $V\in L^\infty(\mathbb{R}^d)$ real-valued satisfying $V(x)=V(-x)$. As will appear in the proof, more general interactions could be considered in the spirit of Theorem 1.1, but we prefer to stick to the usual presentation for an example.

The next result concerns the limit of the von Neumann dynamics (69) in the mean field regime $N\to\infty$ already studied in [10,9]. We shall see that although the particles are not assumed to be bosons, our bosonic mean field result apply to this case due to the symmetry of the tensorized initial state $\varrho_0^{\otimes N}$.

**Proposition 4.7.** *Let $(\varrho_N(t))$ denote the solution to* (69)*, and consider the trace class operator $\sigma_N^{(k)}(t)\in\mathcal{L}^1(L^2(\mathbb{R}^{kd}))$ defined by relation*:

$$\forall B\in\mathcal{L}\big(L^2\big(\mathbb{R}^{kd}\big)\big),\quad \operatorname{Tr}\big[\sigma_N^{(k)}(t)B\big]=\operatorname{Tr}\big[\varrho_N(t)(B\otimes I_{L^2(\mathbb{R}^{d(N-k)})})\big].$$

*Then the convergence,*

$$\lim_{N\to\infty}\sigma_N^{(k)}=\varrho(t)^{\otimes k},\tag{70}$$

*holds in $\mathcal{L}^1(L^2(\mathbb{R}^{dk}))$ for all $t\in\mathbb{R}$ and when $\varrho(t)$ solves the Hartree–von Neumann equation*:

$$\begin{cases} i\partial_t\varrho(t)=\big[A+(V*n_{\varrho(t)}),\varrho(t)\big],\\ \varrho(0)=\varrho_0,\end{cases}\tag{71}$$

*with $n_\varrho(x,t):=\varrho(x;x,t)$.*

**Proof.** The proof will be done in three steps: Bosonization, Liouvillian and mean field limit.

**Bosonization.** The phase space that we will consider is not the one particle space $L^2(\mathbb{R}^d)$ but,

$$\mathcal{Z}=\mathcal{L}^2\big(L^2\big(\mathbb{R}^d\big)\big),$$

the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^d)$. It is endowed with the inner product,

$$\langle \omega_1, \omega_2 \rangle_{\mathcal{Z}} = \mathrm{Tr}_{L^2}\big[\omega_1^* \omega_2\big],$$

where $\mathrm{Tr}_{L^2}[.]$ here denotes the trace on $L^2(\mathbb{R}^d)$ and $\omega_1^*$ is the adjoint of $\omega_1$.

The cyclicity of the trace leads to,

$$\mathrm{Tr}_{(L^2)^{\otimes N}}\big[\varrho_N(t)(B \otimes I_{L^2(\mathbb{R}^{d(N-k)})})\big] = \big\langle \Psi_N(t), (B \otimes I_{L^2(\mathbb{R}^{d(N-k)})})\Psi_N(t)\big\rangle_{\mathcal{Z}^{\otimes N}}, \tag{72}$$

with $\Psi_N(t) = e^{-it\mathbb{H}_N}\sqrt{\varrho_0}^{\otimes N} e^{it\mathbb{H}_N}$.

The important point is that at time $t = 0$, $\Psi_N(0) = \sqrt{\varrho_0}^{\otimes N}$, is a Hermite state in $\bigvee^N \mathcal{Z}$ and that the evolution preserves this symmetry so that

$$\forall t \in \mathbb{R},\ \Psi_N(t) \in \bigvee^N \mathcal{Z}, \quad \Psi_N(0) = \sqrt{\varrho_0}^{\otimes N}.$$

With any bounded operator $B : L^2(\mathbb{R}^{dk}) \to L^2(\mathbb{R}^{dk})$, the action by left (resp. right) multiplication is defined by:

$$\begin{aligned} L_B \text{ (resp. } R_B) : \bigvee^k \mathcal{Z} &\to \bigvee^k \mathcal{Z}, \\ \omega^{\otimes k} &\mapsto \mathcal{S}_k\big(B\omega^{\otimes k}\big) \quad \big(\text{resp. } \mathcal{S}_k\big(\omega^{\otimes k}B\big)\big), \end{aligned}$$

where $\mathcal{S}_k$ is the orthogonal projection from $\bigotimes^k \mathcal{Z}$ onto $\bigvee^k \mathcal{Z}$. Since $(\omega^{\otimes k})_{\omega\in\mathcal{Z}}$ is a total family in $\bigvee^k \mathcal{Z}$ this defines a bounded operator $L_B \in \mathcal{L}(\bigvee^k \mathcal{Z})$, (resp. $R_B \in \mathcal{L}(\bigvee^k \mathcal{Z})$) such that $L_B^* = L_{B^*}$ (resp. $R_B^* = R_{B^*}$). When $B(x_1, \ldots, x_k, y_1, \ldots, y_k)$ is the Schwartz kernel of $B \in \mathcal{L}(L^2(\mathbb{R}^{dk}))$, $L_B$ (resp. $R_B$) is the left (resp. right) multiplication by the operator with kernel:

$$\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} B(x_{\sigma(1)}, \ldots, x_{\sigma(k)}, y_{\sigma(1)}, \ldots, y_{\sigma(k)}).$$

Hence the trace (72) equals:

$$\mathrm{Tr}_{(L^2)^{\otimes N}}\big[\varrho_N(t)(B \otimes I_{L^2(\mathbb{R}^{d(N-k)})})\big] = \big\langle \Psi_N(t), L_{[B\otimes I^{\otimes(N-k)}]}\Psi_N(t)\big\rangle_{\bigvee^N \mathcal{Z}}.$$

With an operator $B \in \mathcal{L}(L^2(\mathbb{R}^{dk}))$, we can now associate a symbol:

$$b_B(\omega) = \big\langle \omega^{\otimes k}, L_B\omega^{\otimes k}\big\rangle_{\bigvee^k \mathcal{Z}} = \mathrm{Tr}_{(L^2)^{\otimes k}}\big[\big(\omega^*\big)^{\otimes k} B\omega^{\otimes k}\big] \in \mathcal{P}_{k,k}(\mathcal{Z}).$$

Since $L_{[B\otimes I^{\otimes(N-k)}]}$ is nothing but $L_B \vee I_{\bigvee^{N-k}\mathcal{Z}}$ we get (with $\varepsilon = \frac{1}{N}$),

$$\mathrm{Tr}_{(L^2)^{\otimes N}}\big[\varrho_N(t)(B \otimes I_{L^2(\mathbb{R}^{d(N-k)})})\big] = \frac{(N-k)!}{N!\varepsilon^k}\big\langle \Psi_N(t), b_B^{Wick}\Psi_N(t)\big\rangle_{\bigvee^N \mathcal{Z}}.$$

**Liouvillian.** Let us now determine the appropriate Hamiltonian $H_\varepsilon$ of this problem which is actually a Liouvillian. The map,

$$\mathbb{R} \ni t \mapsto e^{-itA}\omega e^{itA},$$

defines a continuous unitary group on $\mathcal{Z}$ with a self-adjoint generator,

$$\begin{aligned} \mathfrak{L}_A : \mathcal{Z} &\to \mathcal{Z}, \\ \omega &\mapsto [A, \omega]. \end{aligned}$$

The interaction is a bounded self-adjoint operator $\tilde{Q} : \bigvee^2 \mathcal{Z} \to \bigvee^2 \mathcal{Z}$ given by $\tilde{Q} = \frac{1}{2}(L_V - R_V) \in \mathcal{L}(\bigvee^2 \mathcal{Z})$ and we associate the symbol $Q(\omega) = \langle \omega^{\otimes 2}, \tilde{Q}\omega^{\otimes 2}\rangle$. For any $\omega \in \mathcal{Z}$ the kernel of $\tilde{Q}\omega^{\otimes 2} \in \bigvee^2 \mathcal{Z}$ is given by:

$$\big(\tilde{Q}\omega^{\otimes 2}\big)(x_1, y_1; x_2, y_2) = \frac{1}{2}V(x_1 - x_2)\omega(x_1, y_1)\omega(x_2, y_2) - \frac{1}{2}V(y_1 - y_2)\omega(x_1, y_1)\omega(x_2, y_2).$$

After introducing the Hamiltonian:

$$H_\varepsilon = \mathrm{d}\Gamma(\mathfrak{L}_A) + Q^{Wick},$$

acting as a self-adjoint operator on $\Gamma_s(\mathcal{Z})$, we get for $\Theta \in \bigvee^N \mathcal{Z} \cap \mathcal{D}(\mathrm{d}\Gamma(\mathfrak{L}_A))$,

$$\varepsilon^{-1} H_\varepsilon \Theta = [\mathbb{H}_N, \Theta] \quad \text{with } \varepsilon = 1/N.$$

This implies:

$$\Psi_N(t) = e^{-it\mathbb{H}_N} (\sqrt{\varrho_0})^{\otimes N} e^{it\mathbb{H}_N} = e^{-i\frac{t}{\varepsilon}H_\varepsilon} (\sqrt{\varrho_0})^{\otimes N} \in \bigvee^N \mathcal{Z}.$$

**Mean field limit.** The initial data $\varrho_\varepsilon(0) = |\sqrt{\varrho_0}^{\otimes N}\rangle\langle\sqrt{\varrho_0}^{\otimes N}|$ is a Hermite state which fulfills the assumptions of Theorem 1.1, with

$$\mu_0 = \delta^{S^1}_{\sqrt{\varrho_0}}.$$

The classical energy associated with the Hamiltonian $H_\varepsilon$ is,

$$h(\omega) = \langle \omega, \mathfrak{L}_A \omega\rangle_{\mathcal{Z}} + \frac{1}{2}\big\langle \omega^{\otimes 2}, (L_V - R_V)\omega^{\otimes 2}\big\rangle_{\mathcal{Z}},$$

and the mean field flow $\mathbf{F}_t$ is nothing but the one given by:

$$i\partial_t \omega = \partial_{\bar{\omega}} h(\omega) = [A, \omega] + \big(V * n^1_\omega\big)\omega - \omega\big(V * n^2_\omega\big),$$

where $V * n^i_\omega$ are multiplication operators and $n^1_\omega(x) = \int_{\mathbb{R}^d} |\omega(x,y)|^2\,dy$, $n^1_\omega(y) = \int_{\mathbb{R}^d} |\omega(x,y)|^2\,dx$ when $\omega(x,y)$ denotes the kernel of $\omega$. Beside the invariance $|\mathbf{F}_t(\omega)|_{\mathcal{Z}} = |\omega|_{\mathcal{Z}}$ and $\mathbf{F}_t(e^{-i\theta}\omega) = e^{-i\theta}\mathbf{F}_t(\omega)$, the flow $\mathbf{F}_t$ also satisfies:

$$\mathbf{F}_t\big(\omega^*\big) = \mathbf{F}_t(\omega)^*. \tag{73}$$

Thus previous equation becomes equivalent to the Hartree–von Neumann equation (71) with $\varrho(t) = \omega(t)^2$ when $\omega(0) = \sqrt{\varrho_0}$. Theorem 1.1 says:

$$\forall b \in \mathcal{P}_{k,k}(\mathcal{Z}), \quad \lim_{N\to\infty} \mathrm{Tr}_{\bigvee^N \mathcal{Z}}\big[\big|\Psi_N(t)\big\rangle\big\langle\Psi_N(t)\big| b^{Wick}\big] = \int_{\mathcal{Z}} b(\omega_t)\delta^{S^1}_{\sqrt{\varrho_0}} = b\big(\sqrt{\varrho(t)}\big).$$

In particular when $B \in \mathcal{L}(L^2(\mathbb{R}^{dk}))$, this implies:

$$\lim_{N\to\infty} \mathrm{Tr}\big[\varrho_N(t)(B \otimes I_{L^2(\mathbb{R}^{d(N-k)})})\big] = \mathrm{Tr}_{L^2(\mathbb{R}^{dk})}\big[\varrho(t)^{\otimes k} B\big].$$

This proves the weak convergence in (70), but since it is concerned with non-negative trace class operator and $\mathrm{Tr}[\sigma^{(k)}_N(t)] = 1 = \mathrm{Tr}[\varrho(t)^{\otimes k}]$ the convergence holds in the $\mathcal{L}^1$-norm. □

We end with three remarks:

- When $\varrho$ is a pure state, the result of Proposition 4.7 is the same as (56).
- When $\varrho$ is not a pure state Section 4.3 has already shown that one has to be very careful with tensor products. Actually $\varrho^{\otimes N} \in \mathcal{L}^1(\bigotimes^N \mathcal{Z})$ commutes with the symmetrization projection $\mathcal{S}_N$ (or the antisymmetrization $\mathcal{A}_N$ for fermions) but the corresponding states in $\mathcal{L}^1(\bigvee^N \mathcal{Z})$ (resp. $\mathcal{L}^1(\bigwedge^N \mathcal{Z})$) are

  $$\mathcal{S}_N \varrho^{\otimes N} \mathcal{S}_N \quad \big(\text{resp. } \mathcal{A}_N \varrho^{\otimes N} \mathcal{A}_N\big).$$

  But as shows the formula $\mathrm{Tr}[\Gamma_s(\varrho)] = \prod_{\lambda\in\sigma(\varrho)} \frac{1}{1-\lambda}$ (resp. $\mathrm{Tr}[\Gamma_a(\varrho)] = \prod_{\lambda\in\sigma(\varrho)}(1+\lambda)$), the trace of $\mathcal{S}_N\varrho^{\otimes N}\mathcal{S}_N$ (resp. $\mathcal{A}_N\varrho^{\otimes N}\mathcal{A}_N$) converges to 0 as $N\to\infty$. We leave for subsequent works, the question whether normalizing these states would lead to the same asymptotics as in Proposition 4.7.
- We recall that a tensorization based on the tensor decomposition of Fock spaces in Section 4.3 led to the evolution of Wigner measures which cannot be translated in terms of Hartree–von Neumann equations.

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