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ORIGINAL ARTICLE

The fractional complex transformation for nonlinear fractional partial differential equations in the mathematical physics

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Abstract In this article, the modified extended tanh-function method is employed to solve fractional partial differential equations in the sense of the modified Riemann–Liouville derivative. Based on a nonlinear fractional complex transformation, certain fractional partial differential equations can be turned into nonlinear ordinary differential equations of integer orders. For illustrating the validity of this method, we apply it to four nonlinear equations namely, the space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations, the space–time fractional nonlinear Whitham–Broer–Kaup equations, the space–time fractional nonlinear coupled Burgers equations and the space–time fractional nonlinear coupled mKdV equations.

1. Introduction

Fractional differential equations are the generalizations of classical differential equations with integer orders. In recent years, nonlinear fractional differential equations in mathematical physics are playing a major role in various fields, such as physics, biology, engineering, signal processing, and control theory, finance and fractal dynamics (Miller and Ross, 1993; Kilbas et al., 2006; Podlubny, 1999). Finding approximate and exact solutions to the fractional differential equations is an important task. A large amount of literatures were developed concerning the solutions of the fractional differential equations in nonlinear dynamics (El-sayed et al., 2009). Many powerful and efficient methods have been proposed to obtain the numerical and exact solutions of fractional differential equations. For example, these methods include the variational iteration method (Safari et al., 2009); Wu and Lee, 2010; Yang and Baleanu, 2012; Guo and Mei, 2011), the Lagrange characteristic method (Jumarie, 2006a), the homotopy analysis method (Song and Zhang, 2009), the Adomian decomposition method (El-Sayed and Gaber, 2006; El-sayed et al., 2010), the homotopy perturbation method (He, 1999; He, 2000; Yildirim and Gulkanat, 2010), the differential transformation method (Odibat and Momani, 2008), the finite difference method (Cui, 2009), the finite element method (Huang et al., 2009), the fractional sub-equation method (Zhang and Zhang, 2011; Guo et al., 2012; Lu, 2012), the (G'/G)-expansion method (Zheng, 2012; Gepreel and Omran, 2011; Younis and Zafar, 2013), the modified extended tanh-function method (El-Wakil et al., 2005; El-Wakil et al., 2002; Soliman, 2006; Dai and Wang, 2014), the fractional complex transformation

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method (Li and He, 2010; Li, 2010; He and Li, Li et al., 2012; Hristov, 2010), the exp-function method (He, 2013), the similarity transformation method (Dai et al., 2013; Zhu, 2013), the Hirota method (Liu et al., 2013) and so on.

The objective of this paper is to apply the modified extended tanh-function method for solving fractional partial differential equations in the sense of the modified Riemann–Liouville derivative which has been derived by (Jumarie, 2006b). These equations can be reduced into nonlinear ordinary differential equations (ODE) with integer orders using some fractional complex transformations. Jumarie's modified Riemann–Liouville derivative of order α is defined by the following expression:

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\eta)^{-\alpha} [f(\eta) - f(0)] d\eta, 0 < \alpha \leqslant 1, \\ \left[f^{(n)}(t) \right]^{(\alpha - n)}, n \leqslant \alpha < n+1, n \geqslant 1 \end{cases}$$

We list some important properties for the modified Riemann–Liouville derivative as follows:

$$D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, r > 0 \tag{1}$$

$$D_t^{\alpha}[f(t)g(t)] = f(t) D_t^{\alpha} g(t) + g(t) D_t^{\alpha} f(t) \tag{2}$$

$$D_t^{\alpha}[f(g(t))] = f\prime_g(g(t)) D_t^{\alpha} g(t) \tag{3}$$

$$D_t^{\alpha}[f(g(t))] = D_g^{\alpha} f(g(t)) [g\prime(t)]^{\alpha} \tag{4}$$

where Γ denotes the Gamma function.

The rest of this paper is organized as follows: In Section 2, the description of the modified extended tanh-function method for solving nonlinear fractional partial differential equations is given. In Section 3, we apply this method to establish the exact solutions for the space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations, the space–time fractional nonlinear Whitham–Broer– Kaup equations, the space–time fractional nonlinear coupled Burgers equations and the space–time fractional nonlinear coupled mKdV equations. In Section 4 physical explanations of some obtained solutions are given. In Section 5, some conclusions are obtained.

2. Description the modified extended tanh-function method for solving nonlinear fractional partial differential equations

Suppose we have the following nonlinear fractional partial differential equation:

$$F(u, D_t^{\alpha} u, D_x^{\alpha} u, \ldots) = 0, 0 < \alpha \leqslant 1, \tag{5}$$

where $D_t^{\alpha} u$ and $D_x^{\alpha} u$ are the modified Riemann–Liouville derivatives and F is a polynomial in $u = u(x,t)$ and its fractional derivatives. In the following, we give the main steps of this method:

Step 1: Using the nonlinear fractional complex transformation (Li and He, 2010; Li, 2010; He and Li, 2012; Li et al., 2012; Hristov, 2010).

$$u(x,t) = u(\xi), \xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} + \frac{ct^{\alpha}}{\Gamma(1+\alpha)} + \xi_0, \tag{6}$$

where k,c,ξ_0 are constants with k,$c \neq 0$, to reduce Eq. (5) to the following ODE of integer order with respect to the variable ξ :

$$P(u, u', u'', \ldots) = 0, \tag{7}$$

where P is a polynomial in $u(\xi)$ and its total derivatives u',u'',u''',... such that $u' = \frac{du}{d\xi}, u'' = \frac{d^2u}{d\xi^2}, \ldots$...

Step 2: We suppose that the formal solution of the ODE (7) can be expressed as follows:

$$u(\xi) = a_0 + \sum_{i=1}^{N} [a_i \phi^i(\xi) + b_i \phi^{-i}(\xi)], \tag{8}$$

where $\phi(\xi)$ satisfies the Riccati equation

$$\phi' = b + \phi^2, \tag{9}$$

where b is a constant. Fortunately, Eq. (9) admits several types of the following solutions:

(i) If $b < 0$, we have the hyperbolic solutions;

$$\phi(\xi) = -\sqrt{-b} \tanh(\sqrt{-b}\xi), \phi(\xi) = -\sqrt{-b} \coth(\sqrt{-b}\xi). \tag{10}$$

(ii) If $b > 0$, we have the trigonometric solutions;

$$\phi(\xi) = \sqrt{b} \tan(\sqrt{b}\xi), \phi(\xi) = -\sqrt{b} \cot(\sqrt{b}\xi). \tag{11}$$

(iii) If $b = 0$, we have the rational solutions;

$$\phi(\xi) = \frac{-1}{\xi + d}, \tag{12}$$

where d is a constant.

Step 3: We determine the positive integer N in (8) by balancing the highest nonlinear terms and the highest order derivatives of $u(\xi)$ in Eq. (7).

Step 4: We substitute (8) along with Eq. (9) into Eq. (7) and equate all the coefficients of $\phi^i (i = 0, \pm 1, \pm 2, \ldots)$ to zero to yield a system of algebraic equations for a_i,b_i,c,k, b.

Step 5: We solve the algebraic equations obtained in Step 4 using Mathematica or Maple, and use the well- known solutions (10)–(12) of Eq. (9) to obtain the exact solutions of Eq. (5).

3. Applications

In this section, we construct the exact solutions of the following four nonlinear fractional partial differential equations using the proposed method of Section 2:

Example 1. The Space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations.

These equations are well-known (Guo et al., 2012; Zheng, 2012) and have the forms:

$$D_t^{\alpha} u - \frac{1}{2} D_x^{3\alpha} u + 3u D_x^{\alpha} u - 3 D_x^{\alpha}(vw) = 0, \tag{13}$$

$$D_t^{\alpha} v + D_x^{3\alpha} v - 3u D_x^{\alpha} v = 0, \tag{14}$$

$$D_t^{\alpha} w + D_x^{3\alpha} w - 3u D_x^{\alpha} w = 0, \tag{15}$$

where $0 < \alpha \leqslant 1$. Eqs. (13)–(15) can be used to describe the iteration of two long waves with different dispersion relations (Abazari and Abazari, 2012). When $\alpha = 1$, Eqs. (13)–(15) were first proposed in (Wu and Geng, 1999). When $0 < \alpha \leqslant 1$, Eqs. (13)–(15) have been discussed in (Zheng, 2012) using the (G'/G)-expansion method and in (Guo et al, 2012) using the fractional sub-equation method. Let us now solve Eqs. (13)–(15) using the proposed method of Section 2. To this end, we suppose that $u(x,t) = U(\xi), v(x,t) = V(\xi), w(x,t) = W(\xi)$ where ξ is given by the fractional complex transformation (6). Then by use of Eqs. (1) and (3), the system (13)–(15) can be turned into the following system of ODEs with integer orders:

$$cU - \frac{1}{2}k^3U'' + \frac{3}{2}kU^2 - 3kVW = 0, \tag{16}$$

$$cV' + k^3V''' - 3kUV' = 0, \tag{17}$$

$$cW' + k^3W''' - 3kUW' = 0. \tag{18}$$

Balancing the order of U'' with U^2, V''' with UV' and W''' with UW' in Eqs. (16)–(18), we deduce that the formal solutions of Eqs. (16)–(18) have the forms:

$$U(\xi) = a_0 + a_1\phi + a_2\phi^{-1} + a_3\phi^2 + a_4\phi^{-2}, \tag{19}$$

$$V(\xi) = b_0 + b_1\phi + b_2\phi^{-1} + b_3\phi^2 + b_4\phi^{-2}, \tag{20}$$

$$W(\xi) = c_0 + c_1\phi + c_2\phi^{-1} + c_3\phi^2 + c_4\phi^{-2}, \tag{21}$$

where a_i, b_i, c_i $(i = 0,1,2,3,4)$ are constants to be determined later. We substitute (19)–(21) along with Eq. (9) into Eqs. (16)–(18) and collect all the terms with the same power of $\phi^j, (j = 0, \pm 1, \pm 2, \pm 3, \ldots)$. Equating each coefficient to zero yields a set of the following algebraic equations:

$$\begin{aligned}
\phi^5:\quad & 24c_3k^3 - 6kc_3a_3 = 0\\
& 24b_3k^3 - 6kb_3a_3 = 0
\end{aligned}$$

$$\begin{aligned}
\phi^4:\quad & -3a_3k^3 + \tfrac{3}{2}ka_3^2 - 3kb_3c_3 = 0\\
& 6b_1k^3 - 3k(2a_1b_3 + b_1a_3) = 0\\
& 6c_1k^3 - 3k(2a_1c_3 + b_1c_3) = 0
\end{aligned}$$

$$\begin{aligned}
\phi^3:\quad & -a_1k^3 + 3a_1a_3k - 3k(b_1c_3 + c_1b_3) = 0\\
& 2cb_3 + 40b_3bk^3 - 3k(2a_0b_3 + a_1b_1 + 2a_3b_3b) = 0\\
& 2cc_3 + 40c_3bk^3 - 3k(2a_0c_3 + a_1c_1 + 2a_3c_3b) = 0
\end{aligned}$$

$$\begin{aligned}
\phi^2:\quad & ca_3 - 4a_3bk^3 + \tfrac{3}{2}k(a_1^2 + 2a_0a_3) - 3k(b_0c_3 + b_1c_1 + b_3c_0) = 0\\
& b_1c + 8b_1bk^3 - 3k(a_0b_1 + 2a_1b_3b + 2a_2b_3 + a_3b_1b - a_3b_2) = 0\\
& c_1c + 8c_1bk^3 - 3k(a_0c_1 + 2a_1c_3b + 2a_2c_3 + a_3c_1b - a_3c_2) = 0
\end{aligned}$$

$$\begin{aligned}
\phi:\quad & ca_1 - a_1bk^3 + 3k(a_0a_1 + a_2a_3) - 3k(b_0c_1 + b_1c_0 + b_2c_3 + b_3c_2) = 0\\
& 2bcb_3 + 16b_3b^2k^3 - 3k(2a_0b_3b + a_1b_1b - b_2a_1 + a_2b_1 - 2a_3b_4 + 2a_4b_3) = 0\\
& 2bcc_3 + 16c_3b^2k^3 - 3k(2a_0c_3b + a_1c_1b - c_2a_1 + a_2c_1 - 2a_3c_4 + 2a_4c_3) = 0
\end{aligned}$$

$$\begin{aligned}
\phi^0:\quad & ca_0 - k^3(a_3b^2 + a_4) + \tfrac{3}{2}k(a_0^2 + 2a_1a_2 + a_2a_4) - 3k(b_0c_0 + b_1c_2 + b_2c_1 + b_3c_4 + b_4c_3) = 0\\
& c(b_1b - b_2) + k^3(2b_1b^2 - 2b_2b) - 3k(a_0b_1b - a_0b_2 - 2a_1b_4 + 2a_2b_3b - a_3b_2b + a_4b_1) = 0\\
& c(c_1b - c_2) + k^3(2c_1b^2 - 2c_2b) - 3k(a_0c_1b - a_0c_2 - 2a_1c_4 + 2a_2c_3b - a_3c_2b + a_4c_1) = 0
\end{aligned}$$

$$\begin{aligned}
\phi^{-1}:\quad & ca_2 - a_2bk^3 + 3k(a_0a_2 + a_1a_4) - 3k(b_0c_2 + b_1c_4 + b_2c_0 + b_4c_2) = 0\\
& 2cb_4 + 16b_4bk^3 + 3k(-2a_0b_4 - a_1b_2b + a_2b_1b - b_2a_2 - 2a_3b_4b + 2a_4b_3b) = 0\\
& 2cc_4 + 16c_4bk^3 + 3k(-2a_0c_4 - a_1c_2b + a_2c_1b - c_2a_2 - 2a_3c_4b + 2a_4c_3b) = 0
\end{aligned}$$

$$\begin{aligned}
\phi^{-2}:\quad & ca_4 - 4a_4bk^3 + \tfrac{3}{2}k(a_2^2 + 2a_0a_4) - 3k(b_0c_4 + b_2c_2 + b_4c_0) = 0\\
& -b_2bc - 8b_2b^2k^3 + 3k(a_0b_2b + 2a_1b_4b + 2a_2b_4 - a_4b_1b + a_4b_2) = 0\\
& -c_2bc - 8c_2b^2k^3 + 3k(a_0c_2b + 2a_1c_4b + 2a_2c_4 - a_4c_1b + a_4c_2) = 0
\end{aligned}$$

$$\begin{aligned}
\phi^{-3}:\quad & a_2b^2k^3 - 3a_2a_4k + 3k(b_2c_4 + c_2b_4) = 0\\
& 2cbb_4 + 40b_4b^2k^3 - 3k(a_2b_2b + 2a_4b_4 + 2a_0b_4b) = 0\\
& 2cbc_4 + 40c_4b^2k^3 - 3k(a_2c_2b + 2a_4c_4 + 2a_0c_4b) = 0
\end{aligned}$$

$$\begin{aligned}
\phi^{-4}:\quad & 3a_4b^2k^3 - \tfrac{3}{2}ka_4^2 + 3kb_4c_4 = 0\\
& 6b_2b^3k^3 - 3k(2a_2b_4b + a_4b_2b) = 0\\
& 6c_2b^3k^3 - 3k(2a_2c_4b + a_4c_2b) = 0
\end{aligned}$$

$$\begin{aligned}
\phi^{-5}:\quad & 24b_4b^3k^3 - 6kb_4a_4 = 0\\
& 24c_4b^3k^3 - 6kc_4ba_4 = 0
\end{aligned}$$

Solving the above set of algebraic equations by using Maple or Mathematica , we get the following results:

$$b = b, c = \tfrac{k^3}{4c_3}\left[\tfrac{-64}{3}bc_3 + 32c_0 + \tfrac{8}{3}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}\right],$$

$$k = k, a_3 = 4k^2,$$

$$a_0 = \tfrac{k^2}{8c_3}\left[\tfrac{64}{9}bc_3 + \tfrac{64}{3}c_0 + \tfrac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}\right],$$

$$b_3 = \tfrac{4k^4}{c_3}, c_0 = c_0,$$

$$a_1 = a_2 = a_4 = b_1 = b_2 = b_4 = c_1 = c_2 = c_4 = 0, c_3 = c_3$$

$$b_0 = \tfrac{k4}{c_3^2}\left[\tfrac{-80}{9}bc_3 + \tfrac{52}{3}c_0 + \tfrac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}\right]$$

Now, we have the following exact solutions:

(i) If $b < 0$, we have the hyperbolic solutions

$$\begin{cases} u(x,t) = \frac{k^2}{8c_3}[\frac{64}{9}bc_3 + \frac{64}{3}c_0 + \frac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}] - 4k^2b\tanh^2(\sqrt{-b}\xi) \\ v(x,t) = \frac{k^4}{c_3^2}[\frac{-80}{9}bc_3 + \frac{52}{3}c_0 + \frac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}] - \frac{4bk^4}{c_3}\tanh^2(\sqrt{-b}\xi) \\ w(x,t) = c_0 - c_3b\tanh^2(\sqrt{-b}\xi) \end{cases} \tag{22}$$

$$\begin{cases} u(x,t) = \frac{k^2}{8c_3}[\frac{64}{9}bc_3 + \frac{64}{3}c_0 + \frac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}] - 4k^2b\coth^2(\sqrt{-b}\xi) \\ v(x,t) = \frac{k^4}{c_3^2}[\frac{-80}{9}bc_3 + \frac{52}{3}c_0 + \frac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}] - \frac{4bk^4}{c_3}\coth^2(\sqrt{-b}\xi) \\ w(x,t) = c_0 - c_3b\coth^2(\sqrt{-b}\xi) \end{cases} \tag{23}$$

(ii) If $b > 0$, we have the trigonometric solutions

$$\begin{cases} u(x,t) = \frac{k^2}{8c_3}[\frac{64}{9}bc_3 + \frac{64}{3}c_0 + \frac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}] + 4k^2b\tan^2(\sqrt{b}\xi) \\ v(x,t) = \frac{k^4}{c_3^2}[\frac{-80}{9}bc_3 + \frac{52}{3}c_0 + \frac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}] + \frac{4bk^4}{c_3}\tan^2(\sqrt{b}\xi) \\ w(x,t) = c_0 + c_3b\tan^2(\sqrt{b}\xi) \end{cases} \tag{24}$$

$$\begin{cases} u(x,t) = \frac{k^2}{8c_3}[\frac{64}{9}bc_3 + \frac{64}{3}c_0 + \frac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}] + 4k^2b\cot^2(\sqrt{b}\xi) \\ v(x,t) = \frac{k^4}{c_3^2}[\frac{-80}{9}bc_3 + \frac{52}{3}c_0 + \frac{16}{9}\sqrt{34b^2c_3^2 - 120bc_0c_3 + 90c_0^2}] + \frac{4bk^4}{c_3}\cot^2(\sqrt{b}\xi) \\ w(x,t) = c_0 + c_3b\cot^2(\sqrt{b}\xi) \end{cases} \tag{25}$$

(iii) If $b = 0$, we have the rational solutions

$$\begin{cases} u(x,t) = \frac{k^2}{3c_3}(8 + 2\sqrt{10})c_0 + \frac{4k^2}{(\xi+d)^2} \\ v(x,t) = \frac{k^4}{3c_3^2}(52 + 16\sqrt{10})c_0 + \frac{4bk^4}{c_3(\xi+d)^2} \\ w(x,t) = c_0 + \frac{c_3}{(\xi+d)^2} \end{cases} \tag{26}$$

On comparing our results (22)–(26) with the results in (Younis and Zafar, 2013) and (El-Wakil et al., 2005) we conclude that our results are new.

Example 2. The Space–time fractional nonlinear Whitham–Broer–Kaup equations

These equations are well-known (Xu et al., 2007; Xu et al., 2007; Ping, 2010) and have the forms:

$$D_t^\alpha u + uD_x^\alpha u + D_x^\alpha v + \beta D_x^{2\alpha} u = 0, \tag{27}$$

$$D_t^\alpha v + D_x^\alpha(uv) - \beta D_x^{2\alpha} v + \gamma D_x^{3\alpha} u = 0, \tag{28}$$

where $0 < \alpha \leqslant 1$, and γ,β are constants. In these equations $u(x,t)$ is the field of horizontal velocity , $v(x,t)$ is the height deviating from the equilibrium position of liquid, while γ and β represent different diffusion powers. When $\alpha = 1$, Eqs. (27) and (28) are the generalization of nonlinear Whitham–Broer–Kaup equations, which can be used to describe the dispersive long wave in shallow water (Xu et al, 2007; Ping, 2010). Eqs. (27) and (28) have been discussed in Guo and Mei (2012) using the improved fractional sub-equation method. Let us now solve Eqs. (27), (28) using the proposed method of Section 2. To this end, we suppose that $u(x,t) = U(\xi), v(x,t) = V(\xi)$. Then by the use of Eqs. (1) and (3), the system (27), (28) after integrating once can be turned into the following system of ODEs with integer orders:

$$cU + \frac{1}{2}kU^2 + kV + \beta k^2 U' = 0, \tag{29}$$

$$cV + kUV - \beta k^2 V' + \gamma k^3 U'' = 0, \tag{30}$$

with zero constants of integration. Balancing the highest order derivatives and highest nonlinear terms in Eqs. (29) and (30), we have the following formal solutions:

$$U(\xi) = a_0 + a_1\phi + a_2\phi^{-1}, \tag{31}$$

$$V(\xi) = b_0 + b_1\phi + b_2\phi^{-1} + b_3\phi^2 + b_4\phi^{-2}, \tag{32}$$

where $a_i(i = 0,1,2), b_i(i = 0,1,2,3,4)$ are constants to be determined later. We substitute (31) and (32) along with Eq. (9) into Eqs. (29) and (30) and collect all the terms with the same power of $\phi^j, (j = 0, \pm 1, \pm 2, \pm 3)$. Equating each coefficient to zero yields a set of the following algebraic equations:

$$\phi^3 : a_1b_3k - 2b_3\beta k^2 + 2\gamma a_1 k^3 = 0$$

$$\phi^2 : \quad \tfrac{1}{2}a_1^2 k + kb_3 + a_1\beta k^2 = 0$$
$$b_3c + k(a_0b_3 + a_1b_1) - b_1\beta k^2 = 0$$

$$\phi : \quad a_1c + a_0a_1k + kb_1 = 0$$
$$b_1c + k(a_0b_1 + a_1b_0 + a_2b_3) - 2\beta k^2 bb_3 + 2a_1\gamma k^3 b = 0$$

$$\phi^0 : \quad a_0c + \tfrac{1}{2}k(a_0^2 + 2a_1a_2) + kb_0 + \beta k^2(a_1b - a_2) = 0$$
$$b_0c + k(a_0b_0 + a_1b_2 + a_2b_1) - \beta k^2(bb_1 - b_3) = 0$$

$$\phi^{-1} : \quad a_2c + a_0a_2k + kb_2 = 0$$
$$b_2c + k(a_0b_2 + a_1b_4 + a_2b_0) + 2\beta k^2 b_4 + 2\gamma a_2 bk^3 = 0$$

$$\phi^{-2} : \quad b_4c + k(a_0b_4 + a_2b_2) + b_2\beta k^2 = 0$$
$$\tfrac{1}{2}a_2^2k + kb_4 - a_2b\beta k^2 = 0$$

$$\phi^{-3} : a_2b_4k + 2bb_4\beta k^2 + 2a_2b^2\gamma k^3 = 0$$

On solving the above algebraic equations by using Maple or Mathematica, we have the following results:

Case 1

$$b = b, c = \frac{4k^2\beta(b-1)\sqrt{-b}}{3b}, a_0 = \frac{-4k\beta(b-1)\sqrt{-b}}{3b}, b < 0, a_1 = \frac{-2k\beta(b-1)}{3b},$$
$$a_2 = \frac{2k\beta(b-1)}{3}, b_1 = b_2 = 0, b_0 = \frac{4k^2\beta^2(2b^2-b-1)}{9b}, b_3 = \frac{2k^2\beta^2(2b^2-b-1)}{9b^2},$$
$$b_4 = \frac{2k^2\beta^2(2b^2-b-1)}{9}, \gamma = -\frac{\beta^2(8b^2+2b-1)}{9b^2}$$

In this case we have the hyperbolic solutions

$$\begin{cases} u(x,t) = \frac{-2k\beta(b-1)\sqrt{-b}}{3b}\{1 - [\tanh(\sqrt{-b}\xi) + \coth(\sqrt{-b}\xi)]\} \\ v(x,t) = \frac{2k^2\beta^2(2b^2-b-1)}{3b}\{2 - [\tanh^2(\sqrt{-b}\xi) + \coth^2(\sqrt{-b}\xi)]\} \end{cases} \tag{33}$$

Case 2

$$b = b, \beta = \beta, c = \frac{2\sqrt{2}k^2\beta(b-1)\sqrt{b}}{3b}, a_0 = \frac{-2\sqrt{2}k\beta(b-1)\sqrt{b}}{3b}, b > 0,$$
$$a_1 = \frac{2}{3b}k\beta(b-1), a_2 = \frac{2k\beta(b-1)}{3},$$
$$b_0 = b_1 = b_2 = 0, b_3 = \frac{-2k^2\beta^2(4b^2-5b-1)}{9b^2}, \gamma = -\frac{\beta^2(8b^2+2b-1)}{9b^2},$$
$$b_4 = \frac{2k^2\beta^2(2b^2-b-1)}{9}$$

In this case we have the trigonometric solutions

$$\begin{cases} u(x,t)=\frac{2k\beta(b-1)}{3\sqrt{b}}\{-\sqrt{2}+[\tan(\sqrt{b}\xi)+\cot(\sqrt{b}\xi)]\} \\ v(x,t)=\frac{-2k^2\beta^2(b-1)}{9b}\{(4b-1)\tan^2(\sqrt{b}\xi)-(2b+1)\cot^2(\sqrt{b}\xi)]\} \end{cases} \tag{34}$$

and

$$\begin{cases} u(x,t)=\frac{-2k\beta(b-1)}{3\sqrt{b}}\{\sqrt{2}-[\tan(\sqrt{b}\xi)+\cot(\sqrt{b}\xi)]\} \\ v(x,t)=\frac{-2k^2\beta^2(b-1)}{9b}\{(4b-1)\cot^2(\sqrt{b}\xi)-(2b+1)\tan^2(\sqrt{b}\xi)]\} \end{cases} \tag{35}$$

On comparing our results (33)–(35) with the results obtained in (Younis and Zafar, 2013; Dai et al., 2013; Zhu, 2013) we conclude that our results are new.

Example 3. The space–time fractional nonlinear coupled Burgers equations

These equations are well-known (Zhao et al, 2012) and have the forms:

$$D_t^\alpha u - D_x^{2\alpha} u + 2uD_x^\alpha u + pD_x^\alpha(uv) = 0, \tag{36}$$

$$D_t^\alpha v - D_x^{2\alpha} v + 2vD_x^\alpha v + qD_x^\alpha(uv) = 0, \tag{37}$$

where $0 < \alpha \leqslant 1$ and p, q are constants. Eqs. (36) and (37) have been discussed in (Zhao et al, 2012) using the extended fractional sub-equation method. Let us now solve Eqs. (36) and (37) using the proposed method of Section 2. To this end we suppose that $u(x,t) = U(\xi)$, $v(x,t) = V(\xi)$. Then by the use of Eqs. (1) and (3), the system (36) and (37) can be turned into the following system of ODEs with integer orders:

$$cU' - k^2U'' + 2kUU' + pk(UV)' = 0, \tag{38}$$

$$cV' - k^2V'' + 2kVV' + qk(UV)' = 0, \tag{39}$$

Integrating Eqs. (38) and (39) with vanishing the constants of integration, we get

$$cU - k^2U' + kU^2 + pkUV = 0, \tag{40}$$

$$cV - k^2V' + kV^2 + qkUV = 0, \tag{41}$$

Balancing the highest order of derivatives and highest nonlinear terms in Eqs. (40) and (41), we have the following formal solutions:

$$U(\xi) = a_0 + a_1\phi + a_2\phi^{-1}, \tag{42}$$

$$V(\xi) = b_0 + b_1\phi + b_2\phi^{-1}, \tag{43}$$

where $a_i, b_i (i = 0,1,2)$ are constants to be determined later. We substitute (42) and (43) along with Eq. (9) into Eqs. (40) and

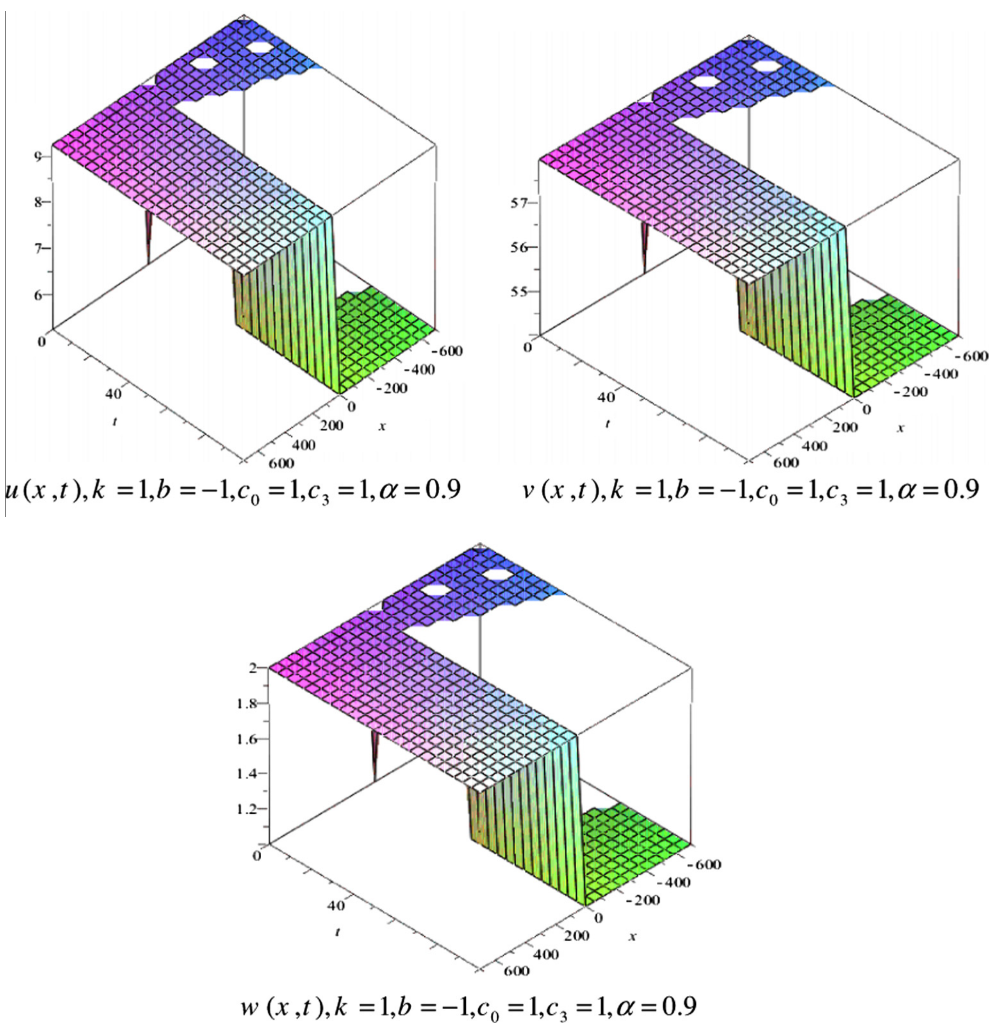


Fig. 1 The plots of solutions (22) of the space time fractional generalized nonlinear Hirota–Statsuma coupled KdV equations.

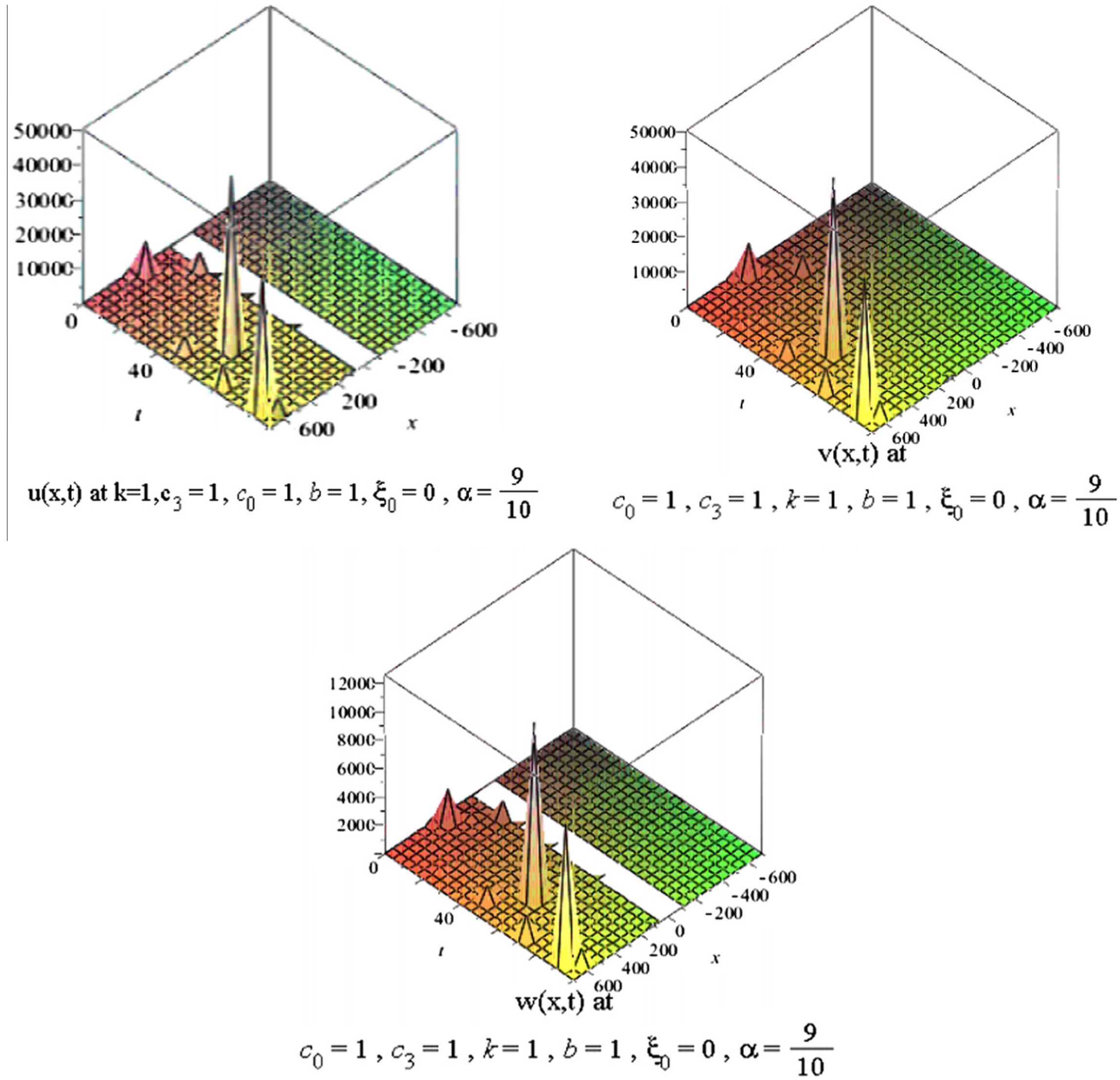


Fig. 2 The plots of solutions (24) of the space time fractional generalized nonlinear Hirota–Statsuma coupled KdV equations.

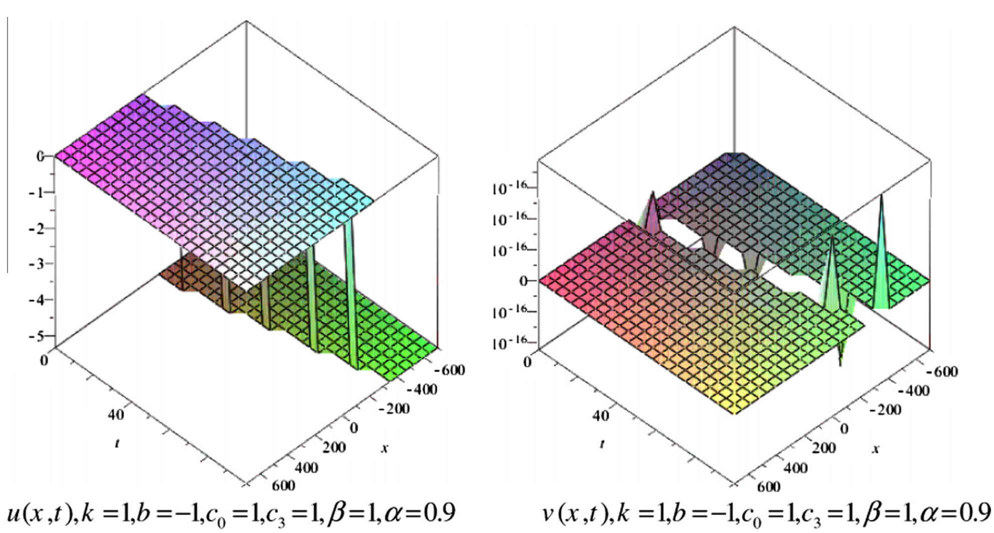


Fig. 3 The plots of solutions (33) of the space time fractional nonlinear Whitham–Broer–Kaup equations.

(41) and collect all the terms with the same power of $\phi^j, (j = 0, \pm 1, \pm 2)$. Equating each coefficient to zero yields a set of the following algebraic equations:

$$\phi^2: \quad -k^2a_1 + ka_1^2 + a_1b_1pk = 0$$
$$-k^2b_1 + kb_1^2 + a_1b_1qk = 0$$

$$\phi: \quad b_1c + 2b_0b_1k + qk(a_0b_1 + a_1b_0) = 0$$
$$a_1c + 2a_0a_1k + pk(a_0b_1 + a_1b_0) = 0$$
$$\phi^0: \quad a_0c - k^2(a_1b - a_2) + k(a_0^2 + 2a_1a_2) + pk(a_0b_0 + a_1b_2 + a_2b_1) = 0$$
$$b_0c - k^2(b_1b - b_2) + k(b_0^2 + 2b_1b_2) + qk(a_0b_0 + a_1b_2 + a_2b_1) = 0$$

$$\phi^{-1}: \quad a_2c + 2a_0a_2k + pk(a_0b_2 + a_2b_0) = 0$$
$$b_2c + 2b_0b_2k + qk(a_0b_2 + a_2b_0) = 0$$

$$\phi^{-2}: \quad k^2b_2b + kb_2^2 + a_2b_2qk = 0$$
$$k^2a_2b + ka_2^2 + a_2b_2pk = 0$$

By solving the above set of algebraic equations by using Maple or Mathematica, we get the results

Case 1

$$b < 0, c = -2k^2\sqrt{-b}, a_0 = k\sqrt{-b}, p = q = a_1 = b_2 = 0,$$
$$b_0 = k\sqrt{-b}, a_2 = -bk, b_1 = k$$

In this case, we have the hyperbolic solutions:

$$\begin{cases} u(x,t) = k\sqrt{-b} + \frac{bk}{\sqrt{-b}}\coth(\sqrt{-b}\xi) \\ v(x,t) = k\sqrt{-b}\{1 - \tanh(\sqrt{-b}\xi)\} \end{cases} \tag{44}$$

and

$$\begin{cases} u(x,t) = k\sqrt{-b} + \frac{bk}{\sqrt{-b}}\tanh(\sqrt{-b}\xi) \\ v(x,t) = k\sqrt{-b}\{1 - \coth(\sqrt{-b}\xi)\} \end{cases} \tag{45}$$

Case 2

$$b < 0, p = 0, q = 2, c = -2k^2\sqrt{-b}, a_0 = k\sqrt{-b},$$
$$p = a_1 = b_0 = 0, a_2 = -kb, b_2 = bk, b_1 = k$$

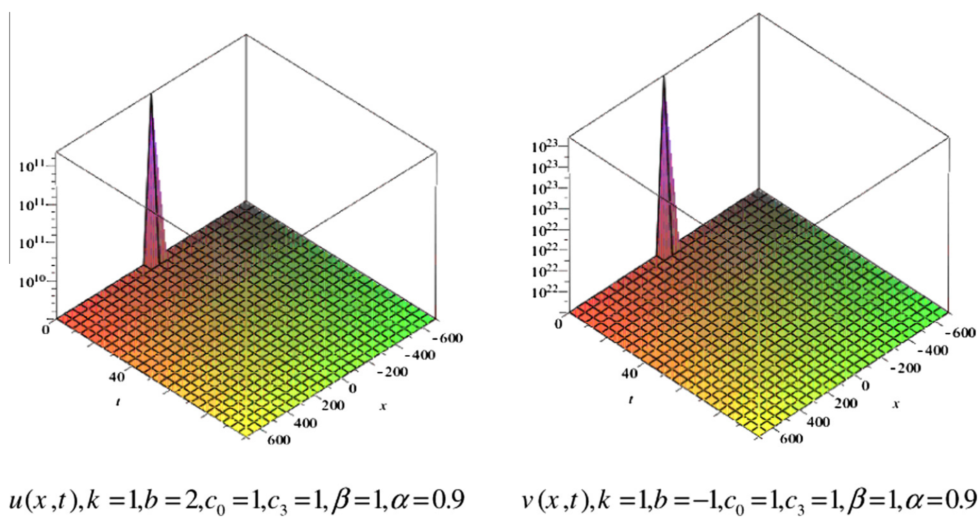


Fig. 4 The plots of solutions (34) of the space time fractional nonlinear Whitham–Broer–Kaup equations.

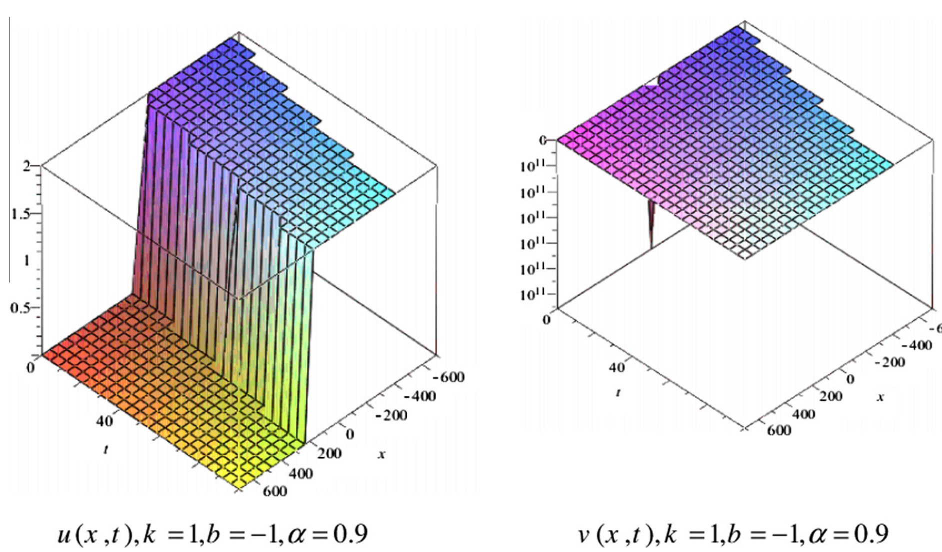


Fig. 5 The plots of solutions (44) of the space time fractional nonlinear coupled Burgers equations.

In this case the following solutions

$$\begin{cases} u(x,t) = k\sqrt{-b} - \frac{bk}{\sqrt{-b}}\coth(\sqrt{-b}\xi) \\ v(x,t) = -k\sqrt{-b}\coth(\sqrt{-b}\xi) - \frac{bk}{\sqrt{-b}}\tanh(\sqrt{-b}\xi) \end{cases} \tag{46}$$

and

$$\begin{cases} u(x,t) = -k\sqrt{-b} - \frac{bk}{\sqrt{-b}}\tanh(\sqrt{-b}\xi) \\ v(x,t) = -k\sqrt{-b}\tanh(\sqrt{-b}\xi) - \frac{bk}{\sqrt{-b}}\coth(\sqrt{-b}\xi) \end{cases} \tag{47}$$

On comparing our results (44)–(47) with the results obtained in (Liu et al., 2013) we conclude that our results are new.

Example 4. The space–time fractional nonlinear coupled mKdV equations

These equations are well-known (Zhao et al., 2012) and have the forms:

$$D_t^\alpha u = \frac{1}{2}D_x^{3\alpha}u - 3u^2 D_x^\alpha u + \frac{3}{2}D_x^{2\alpha}u + 3D_x^\alpha(uv) - 3\lambda D_x^\alpha u, \tag{48}$$

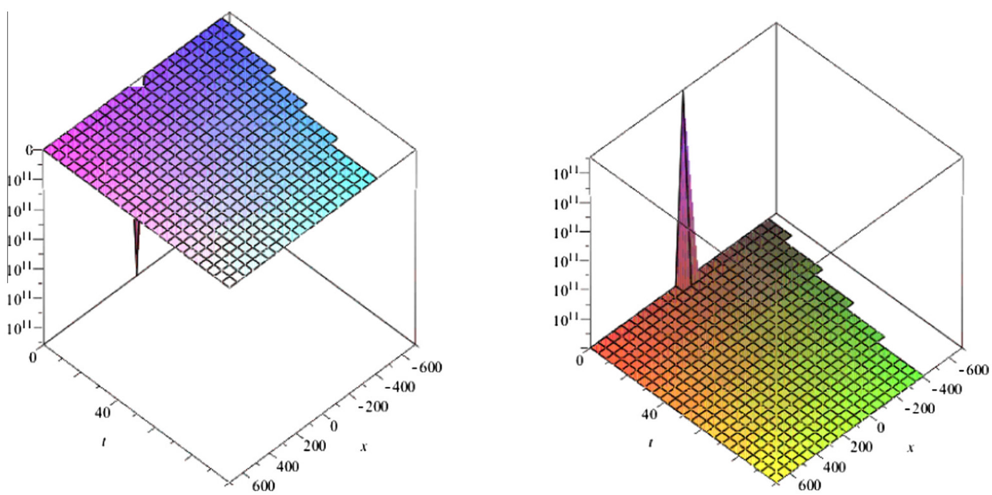
$$D_t^\alpha v = -D_x^{3\alpha}v - 3vD_x^\alpha v - 3D_x^\alpha u D_x^\alpha v + 3u^2 D_x^\alpha v + 3\lambda D_x^\alpha v, \tag{49}$$

where $0 < \alpha \leqslant 1$ and λ is constant. Eqs. (48) and (49) have been discussed in (Zhao et al, 2012) using the extended fractional sub-equation method. Let us now solve Eqs. (48) and (49) using the proposed method of Section 2. To this end we suppose that $u(x,t) = U(\xi), v(x,t) = V(\xi)$. Then by the use of Eqs. (1) and (3), the system (48) and (49) can be turned into the following system of ODEs with integer orders:

$$(c + 3\lambda k)U - \frac{1}{2}k^3 U'' + kU^3 - \frac{3}{2}k^2 U' - 3kUV = 0, \tag{50}$$

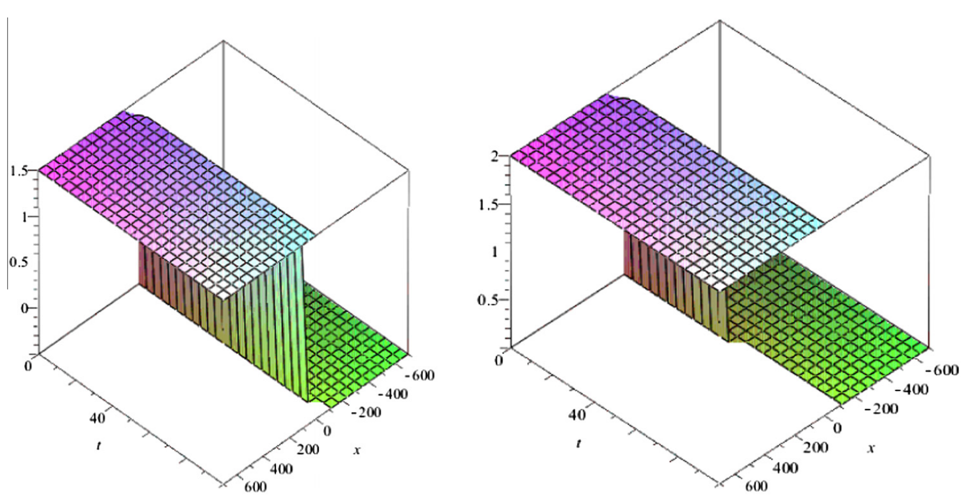
$$cV' = -k^3 V''' - 3kVV' - 3k^2 U'V' + 3kU^2 V' + 3\lambda k V', \tag{51}$$

Balancing the highest order of derivatives and highest nonlinear terms in Eqs. (50) and (51), we have the following formal solutions:



$u(x,t), k = 1, b = -1, \alpha = 0.9$ $\quad$ $v(x,t), k = 1, b = -1, \alpha = 0.9$

Fig. 6 The plots of solutions (46) of the space time fractional nonlinear coupled Burgers equations.



$u(x,t), k = 1, b = -1, c_0 = 1, c_3 = 1, \alpha = \frac{9}{10}$ $\quad$ $v(x,t), k = 1, b = -1, c_0 = 1, c_3 = 1, \alpha = \frac{9}{10}$

Fig. 7 The plots of solutions (54) of the space time fractional nonlinear coupled mKdV equations.

$$U(\xi) = a_0 + a_1\phi + a_2\phi^{-1}, \tag{52}$$

$$V(\xi) = b_0 + b_1\phi + b_2\phi^{-1}, \tag{53}$$

where $a_i, b_i (i = 0,1,2)$ are constants to be determined later. We substitute (52) and (53) along with Eq. (9) into Eqs. (50) and (51) and collect all the terms with the same power of $\phi^j, (j = 0, \pm 1, \pm 2, \pm 3)$. Equating each coefficient to zero yields the following set of algebraic equations:

$$\phi^4 : 6a_1k^3 + 3a_1b_1k^3 - 3a_1^2b_1k = 0$$

$$\phi^3 : \quad -k^3a_1 + ka_1^3 = 0$$
$$3b_1^2k - 6a_0a_1b_1k = 0$$

$$\phi^2 : \quad (c - 3\lambda k)b_1 + 8a_1bk^3 + 3b_0b_1k + 3k^2(2a_1b_1b - a_1b_2 - a_2b_1) = 0$$
$$3a_0a_1^2k - \tfrac{3}{2}k^2a_1 - 3a_1b_1k = 0$$

$$\phi : \; 3kb_1^2b - 3k(2a_0a_1b_1b - 2a_0a_1b_2 + 2a_0a_2b_1) = 0$$
$$(c + 3\lambda k)a_1 - a_1bk^3 + k(3a_0^2a_1 + 3a_2a_1^2) - 3k(a_0b_1 + a_1b_0) = 0$$

$$\phi^0 : \quad (c - 3\lambda k)(b_1b - b_2) + k^3(2a_1b^2 - 2a_2b) + 3k^2(-2a_1b_2b + a_1b_1b^2 - 2a_2b_1b + a_2b_2) + 3k(b_0b_1b - b_0b_2) - 3k(-a_1^2b_2b + a_0^2b_1b - a_0^2b_2 + 2a_1a_2b_1b - 2a_1a_2b_2 + a_2^2b_1) = 0$$
$$(c + 3\lambda k)a_0 + k(3a_0^3 + 6a_0a_2a_1) - \tfrac{3}{2}k^2(a_1b - a_2) - 3k(a_0b_0 + a_1b_2 + a_2b_1) = 0$$

$$\phi^{-1} : -3b_2^2k + 3k(2a_0a_1bb_2 - 2a_0a_2bb_1 + 2a_0a_2b_2) = 0$$
$$(c + 3\lambda k)a_2 - a_2bk^3 + k(3a_0^2a_2 + 3a_1a_2^2) - 3k(a_0b_2 + a_2b_0) = 0$$

$$\phi^{-2} : \quad (c - 3\lambda k)b_2b + 8a_2b^2k^3 + 3b_0b_2bk + 3k(a_1b_2b^2 - 2a_2b_2b + a_2b_1b^2) + 3k(-a_0^2b_2b - 2a_1a_2b_2b + a_2^2b_1b - a_2^2b_2) = 0$$
$$3a_0a_2^2k + \tfrac{3}{2}k^2a_2b - 3a_2b_2k = 0$$

$$\phi^{-3} : \quad -3kbb_2^2 + 6a_0a_2bb_2k = 0$$
$$-k^3a_2b^2 + ka_2^3 = 0$$

$$\phi^{-4} : -6a_2b^3k^3 + 3a_2b_2b^2k^2 + 3a_2^2b_2bk = 0$$

By solving the above set of algebraic equations by using Maple or Mathematica, we get the results

Case 1

$$b = -\frac{1}{4k^2}, c = \frac{k}{2}, k = k, \lambda = \lambda, a_0 = \frac{1}{2},$$
$$a_1 = -k, a_2 = b_2 = 0, b_0 = \lambda, b_1 = -k.$$

Hence the solutions of the space–time fractional coupled (mKdV) Eqs. (48), (49) are given by:

$$\begin{cases} u(x,t) = \frac{1}{2} + k\sqrt{-b}\tanh(\sqrt{-b}\xi), \\ v(x,t) = \frac{\lambda}{2} + k\sqrt{-b}\tanh(\sqrt{-b}\xi) \end{cases} \tag{54}$$

$$\begin{cases} u(x,t) = \frac{1}{2} + k\sqrt{-b}\coth(\sqrt{-b}\xi), \\ v(x,t) = \frac{\lambda}{2} + k\sqrt{-b}\coth(\sqrt{-b}\xi) \end{cases} \tag{55}$$

Case 2.

$$b = -\frac{1}{4}, c = \frac{1}{2}, k = 1, \lambda = \lambda, a_0 = \frac{1}{2},$$
$$a_1 = b_1 = 0, b_0 = \lambda, b_2 = -\frac{1}{4}.$$

Hence the solutions of the space–time fractional coupled mKdV Eqs. (48), (49) are given by:

$$\begin{cases} u(x,t) = \frac{1}{4}[2 + \frac{1}{\sqrt{-b}}\coth(\sqrt{-b}\xi)], \\ v(x,t) = \frac{\lambda}{4}[4 + \frac{1}{\sqrt{-b}}\coth(\sqrt{-b}\xi)] \end{cases} \tag{56}$$

and

$$\begin{cases} u(x,t) = \frac{1}{4}[2 + \frac{1}{\sqrt{-b}}\tanh(\sqrt{-b}\xi)], \\ v(x,t) = \frac{\lambda}{4}[4 + \frac{1}{\sqrt{-b}}\tanh(\sqrt{-b}\xi)]. \end{cases} \tag{57}$$

On comparing our results (54)–(57) with the results obtained in (Liu et al., 2013) we conclude that our results are new.

4. Physical explanations of some obtained solutions

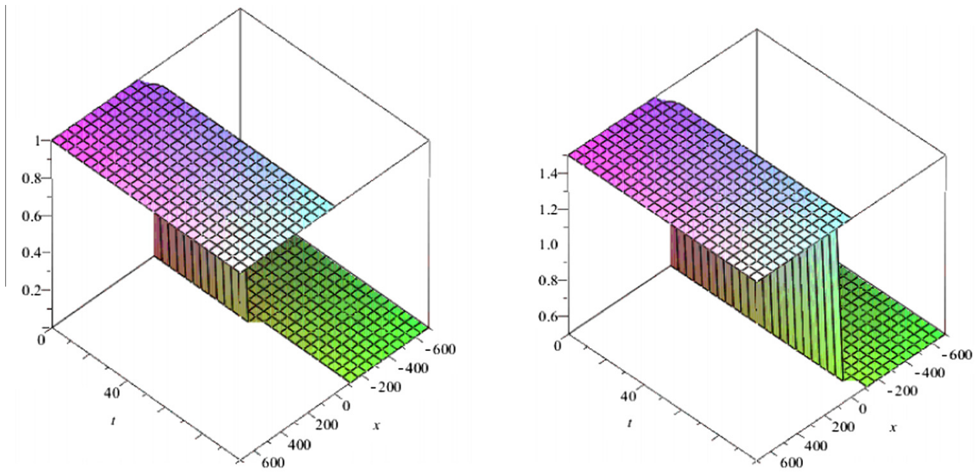
In this paper, we have obtained three types of solutions namely, hyperbolic, trigonometric and rational function solutions. In this section, we have presented some graphs of these types of solutions to visualize the underlying mechanism of the original equations. Using the mathematical software Maple 15, we will give some plots for these solutions.

4.1. The Space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations

The obtained solutions of these equations incorporate three types of explicit solutions namely hyperbolic, trigonometric and rational function solutions (22)–(26) respectively. From these explicit results, the solutions (22) are kink solutions, (23) are singular kink solutions, while (24) and (25) are periodic solutions and (26) are rational solutions. For more convenience the graphical representations of (22) and (24) are shown in Figs. 1 and 2 respectively.

4.2. The Space–time fractional nonlinear Whitham–Broer–Kaup equations

The obtained solutions of these equations incorporate two types of explicit solutions namely hyperbolic and trigonometric



$u(x,t), k=1, b=\frac{-1}{4}, c_0=1, c_3=1, \alpha=\frac{9}{10}$ $\quad$ $v(x,t), k=1, b=\frac{-1}{4}, c_0=1, c_3=1, \alpha=\frac{9}{10}$

Fig. 8 The plots of solutions (57) of the space time fractional nonlinear coupled mKdV equations.

function solutions (33)–(35) respectively. From these explicit results, the solutions (33) represent kink solutions while (34) and (35) are periodic solutions. For more convenience the graphical representations of (33) and (34) are shown in Figs. 3 and 4 respectively.

4.3. *The space–time fractional nonlinear coupled Burgers equations*

The obtained solutions of these equations incorporate hyperbolic function solutions (44)–(46) which represent the kink and singular kink solutions. For more convenience the graphical representations of (44) and (46) are shown in Figs. 5 and 6 respectively.

4.4. *The space–time fractional nonlinear coupled mKdV equations*

The obtained solutions of these equations incorporate hyperbolic function solutions (54)–(57) which represent the kink and singular kink solutions. For more convenience the graphical representations of (54) and (57) are shown in Figs. 7 and 8 respectively.

5. Some conclusions

In this paper, we have extended successfully the modified extended tanh- function method to solve four nonlinear fractional partial differential equations. As applications, abundant new exact solutions for the space–time fractional generalized nonlinear Hirota–Satsuma coupled KdV equations, the space–time fractional nonlinear Whitham–Broer–Kaup equations, the space–time fractional nonlinear coupled Burgers equations and the space–time fractional nonlinear coupled mKdV equations have been successfully found. As one can see, the nonlinear fractional complex transformation (6) for ξ is very important, which ensures that a certain fractional partial differential equation can be turned into another ordinary differential equation of integer order, whose solutions can be expressed in the form (8) where $\phi(\xi)$ satisfies the Riccati Eq. (9). Besides, as this method is based on the homogeneous balancing principle, it can also be applied to other nonlinear fractional partial differential equations, where the homogeneous balancing principle is satisfied.

Acknowledgments

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