Continuous collections of hereditarily indecomposable continua

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Abstract

We present a survey of work on the title topic. Several questions are also posed.

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A continuum is a compact, connected, metric space.

A continuum is indecomposable if it is not the union of two proper subcontinua, and hereditarily indecomposable if each subcontinuum of it is indecomposable.

A nondegenerate continuum $X$ is chainable if it is homeomorphic to an inverse limit of arcs or, equivalently, if for every $\varepsilon > 0$ there exists an open cover of $X$ by an $\varepsilon$-chain, i.e., a cover $\mathcal{C} = \{C_0, C_1, \ldots, C_n\}$ where $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leq 1$ and where $diam(C_i) < \varepsilon$ for each $i$.

A continuous decomposition $G$ of a locally compact space $X$ is a collection $G$ of pairwise disjoint subsets of $X$, with $\bigcup \{g \mid g \in G\} = X$, such that for each $g \in G$ and each $\varepsilon > 0$, $\bigcup \{g' \in G \mid g \subset N_\varepsilon(g') \text{ and } g' \subset N_\varepsilon(g)\}$ is an open subset of $X$. Thus $G$ is both upper semicontinuous and lower semicontinuous. Equivalently, the natural projection map $\pi : X \to X/G$ obtained by identifying each $g \in G$ to a point is both open and closed.

The first example of a nondegenerate hereditarily indecomposable continuum was given by Knaster [16] in 1922. This was later shown by Bing [5] in 1951 to be the same continuum which Moise [28] in 1948 termed a pseudo-arc and showed to be hereditarily equivalent (i.e., homeomorphic to each of its nondegenerate subcontinua) and which Bing
[4] in 1948 showed to be homogeneous. Bing [5] has characterized the pseudo-arc as (up to homeomorphism) the only nondegenerate, hereditarily indecomposable, chainable continuum.

The first example of a nontrivial continuous collection of hereditarily indecomposable continua was announced by Knaster [27,29] in the early 1940's. Apparently there were at least two announcements of related results by Knaster. He [29] announced a continuous decomposition of the plane into hereditarily indecomposable continua, with decomposition space a ray, thus showing that a continuous real-valued function of two variables need not be constant on any arc. At a topology conference in Kiev in the early 1940's Knaster [27] announced a decomposition result much closer to a result subsequently announced by Anderson [1] and proven by Lewis and Walsh [26]. (A more comprehensive treatment of the early history of the pseudo-arc and hereditarily indecomposable continua will be included as a chapter in a manuscript under preparation by this author.)

In 1950 Anderson [1] announced (without proof) that there exists a continuous decomposition of the plane into pseudo-arcs. In 1952 he [2] provided an example of a continuous decomposition of the plane into nondegenerate cellular sets (i.e., sets each of which is the intersection of a nested sequence of cells) and showed that there does not exist a continuous decomposition of the plane into cellular sets each of which is a nondegenerate Peano continuum. In 1955 Dyer [10] extended this result by showing that there is no continuous decomposition of the plane into cellular sets each of which is a decomposable continuum.

The first published construction of a continuous decomposition of the plane into pseudo-arcs was produced by Lewis and Walsh [26] in 1978. The construction can be readily modified to give a continuous decomposition of any compact 2-manifold without boundary into pseudo-arcs.

Several other two dimensional continuous collections of hereditarily indecomposable continua have been constructed.

Brown [8] has constructed a continuous decomposition of \( \mathbb{R}^n - \{0\} \) into “concentric” \((n - 1)\)-dimensional hereditarily indecomposable continua, each of which has exactly 2 complementary domains and irreducibly separates the origin from the point at infinity. The decomposition space is thus the halfline, giving a result for \( n = 2 \) similar to that announced by Knaster [29].

In an unpublished manuscript, Connor [9] uses the construction of Brown [8] and the assumption that for \( n = 2 \) these continua can be constructed so as to all be pseudo-circles to obtain a continuous decomposition of the plane into pseudo-arcs.

Prims [30] has announced a continuous collection of pseudo-arcs filling up the annulus with decomposition space a simple closed curve, a result earlier announced (without proof) by Anderson [1] (a “selectible” circle of pseudo-arcs). He has also suggested general conditions under which a continuum admits a continuous decomposition into pseudo-arcs, as well as conditions equivalent to the existence of such a decomposition.

Villareal [38] has constructed new examples of homogeneous two dimensional continua as fibered products obtained from continuous curves of pseudo-arcs.
Seaquist [35] has constructed a new continuous decomposition of the disk into non-degenerate cellular elements, as well as such a decomposition [36] for the Sierpiński universal plane curve. For the Sierpiński curve $S$ this decomposition yields the first known example of a monotone open map of $S$ onto itself which is not a homeomorphism. While the elements of these decompositions are not specifically constructed to be hereditarily indecomposable, techniques of construction similar to those for the other results mentioned in this paper are used. For the Sierpiński curve $S$ these techniques hold the promise of significantly increasing our knowledge of its mapping properties. In a forthcoming paper, Seaquist will use these techniques with a noncellular decomposition element to show that the Sierpiński curve is homogeneous with respect to the class of open monotone maps.

The following theorem, or an appropriate variation, has proven useful in constructing continuous collections.

**Theorem 1** [26]. Let $X$ be a compactum and $\{P_n\}_{n=1}^\infty$ be a sequence satisfying:

1. For each $n$, $P_n$ is a finite collection of nonempty closed subsets of $X$ with $\bigcup\{p_n \mid p_n \in P_n\} = X$, with the elements of $P_n$ having pairwise disjoint interiors, and with $\text{cl}(\text{int}(p_n)) = p_n$ for each $p_n \in P_n$;
2. For each $p_{n-1} \in P_{n-1}$, $\text{st}^*(p_{n-1}, P_n) \subseteq \text{st}(p_{n-1}, P_{n-1})^*$;
3. There is a positive number $L$ such that for each pair $P_n, P_{n'} \in P_n$ with $P_n \cap P_{n'} \neq \emptyset$, $p_n \subseteq N_{L/2^n}(p_{n'})$;
4. There is a positive number $K$ such that for each $p_n \in P_n$, there is a $p_{n-1} \in P_{n-1}$ with $p_n \cap p_{n-1} \neq \emptyset$ and $p_{n-1} \subseteq N_{K/2^n}(p_n)$.

Let $G$ be defined by $g \in G$ if $g = \bigcap_{n=1}^\infty \text{st}(p_n, P_n)^*$ where $\bigcap_{n=1}^\infty p_n \neq \emptyset$. Then $G$ is a continuous decomposition of $X$.

Conditions (1) and (2) in the above theorem guarantee that the decomposition $G$ is upper semicontinuous; the addition of conditions (3) and (4) insures that $G$ is continuous.

Several questions about decompositions of spaces of dimension at least two present themselves.

**Question 1.** Does there exist a continuous decomposition of the plane into pseudo-arcs with all decomposition elements equivalently embedded in the plane? such that every (orientation preserving) homeomorphism of the plane lifts to a homeomorphism respecting the decomposition? (This would be a homogeneous decomposition.)

While the decomposition constructed by Lewis and Walsh [26] has much translational symmetry, it does not appear to have either of the above properties.

**Question 2.** Does there exist a continuous decomposition of the plane into pseudo-arcs such that no two decomposition elements are equivalently embedded? (This would be a rigid decomposition, though it is possible that a rigid decomposition—in the sense of no nonidentity homeomorphism of the plane lifting to a homeomorphism respecting the decomposition—exists without satisfying the above condition.)
Note that the existence of such a decomposition is independent of a rigid embedding of the pseudo-arc in the plane, the existence of which has been announced by this author [25].

**Question 3.** For a continuous decomposition of the plane into pseudo-arcs, must the preimage of a homogeneous continuum be homogeneous?

**Question 4.** Does there exist a universal continuous decomposition of the plane into pseudo-arcs, in the sense of a decomposition with elements of every embedding type of the pseudo-arc among the decomposition elements?

**Question 5.** Does there exist a continuous decomposition of the plane into nonchainable, tree-like, hereditarily indecomposable continua? With all decomposition elements mutually homeomorphic?

Such would be atriodic and hence such a result would not violate the classic result of Moore about the nonexistence of uncountable many triods in the plane. Perhaps the continua constructed by Ingram [13] could serve as the basis for such a decomposition.

**Question 6.** For every nonseparating homogeneous plane continuum X does there exist a continuous decomposition of the plane into copies of X.

This is true for the two known examples of homogeneous nonseparating plane continua, the point and the pseudo-arc.

**Question 7.** Does there exist a continuous decomposition of $\mathbb{R}^3$ into pseudo-arcs with decomposition space homeomorphic to $\mathbb{R}^3$ which is nowhere a local product of a continuous decomposition of the plane into pseudo-arcs?

In posing this question, we are looking for a tame decomposition of $\mathbb{R}^3$ which is fundamentally different from a product of decompositions of the plane, not simply a modification of such a product by inserting a countable dense set of wild pseudo-arcs.

We will now consider some results on one dimensional continuous collections.

Bing and Jones [7] produced the *circle of pseudo-arcs*, a circle-like continuum with a continuous decomposition into pseudo-arcs such that the decomposition space is a simple closed curve. (This should not be confused with the pseudo-circle, a circle-like continuum constructed by Bing [5] which is hereditarily indecomposable and hence does not admit a monotone map onto a simple closed curve.)

They also proved several properties of the circle of pseudo-arcs and of the corresponding arc of pseudo-arcs, including the uniqueness of each. Similar results can be obtained for the various solenoids of pseudo-arcs [12,21,34].

The circle of pseudo-arcs is homogeneous, and is the only known homogeneous continuum other than the simple closed curve which separates the plane.
It is known from results of Jones [14] and Rogers [33] that every homogeneous continuum which separates the plane has a continuous decomposition into mutually homeomorphic, homogeneous, hereditarily indecomposable, nonseparating, plane continua with decomposition space a simple closed curve. It is not known whether for every homogeneous nonseparating plane continuum such a continuous circle of continua exists. For the only two known homogeneous nonseparating plane continua, the point and the pseudo-arc, such planar continuous circles do exist which are homogeneous and unique.

The arc of pseudo-arcs and the circle of pseudo-arcs have the property that every homeomorphism of either of them projects to a homeomorphism of an arc or circle, and every homeomorphism (or map) of the arc or circle lifts to a homeomorphism (or map) of the curve of pseudo-arcs. In the latter case, motion within individual decomposition elements is free. Reed [31] has established more general conditions on when homeomorphisms between closed subsets can be extended to homeomorphisms of the circle of pseudo-arcs.

Using these, one can construct other continuous curve of pseudo-arcs.

**Theorem 2 [24].** For every one dimensional continuum \( M \) there exist a one dimensional continuum \( M' \) such that \( M \) has a continuous decomposition into pseudo-arcs with the decomposition space homeomorphic to \( M \). Every homeomorphism (self-map) of \( M \) can be lifted to a homeomorphism (self-map) of \( M' \), with motion within decomposition elements being free.

In the above construction, all decomposition elements are terminal continua. (A sub-continuum \( H \) of the continuum \( X \) is terminal if for any subcontinuum \( K \) of \( X \) with \( H \cap K \neq \emptyset \) either \( H \subseteq K \) or \( K \subseteq H \).)

Prajs [30] has announced a continuous decomposition of the Sierpiński universal plane curve into pseudo-arcs. Such decomposition elements would not be terminal continua. He has also announced the existence of one-dimensional continua which are neither chainable nor circularly chainable (and in fact contain triods) which admit continuous decompositions into pseudo-arcs with decomposition space an arc or a simple closed curve.

This author [22] has observed that any homogeneous continuum containing a terminal subcontinuum which is a pseudo-arc admits a continuous decomposition into maximal terminal pseudo-arcs such that the decomposition space is homogeneous and contains no terminal pseudo-arc.

Combining this observation with the above theorem, one has that homogeneous one-dimensional continua can be divided into two classes, with a one-to-one correspondence between the classes: those with terminal pseudo-arcs and those without terminal pseudo-arcs. For each continuum in one class there is a unique corresponding continuum in the other class, obtained either by shrinking out maximal terminal pseudo-arcs or by enlarging each point to a terminal pseudo-arc.

Any continuum all of whose sufficiently small subcontinua are hereditarily indecomposable admits a continuous decomposition into such hereditarily indecomposable con-
tinua. One method of obtaining such a decomposition is by taking the elements of a Whitney level in the hyperspace of subcontinua.

Any continuum all of whose sufficiently small nondegenerate subcontinua are pseudo-arcs admits a continuous decomposition into terminal pseudo-arcs. Any such continuum also has the property that it is locally homeomorphic to the pseudo-arc. Many properties of homeomorphisms and of the structure of the homeomorphism group for the pseudo-arc are also valid for any such continuum, depending on local properties, e.g., the existence of stable homeomorphisms [19] and the nonpseudo-contractibility of the continuum [37].

Hereditarily indecomposable continua of dimension greater than one have been constructed by Bing [6] and shown to be abundant. Kelley [15] has shown that every hereditarily indecomposable continuum of dimension greater than one admits a continuous decomposition into nondegenerate subcontinua such that the decomposition space is infinite dimensional. Thus the hyperspace of any such continuum is infinite dimensional. Eberhart and Nadler [11] have shown that the hyperspace of subcontinua of any nondegenerate hereditarily indecomposable continuum has dimension either two or infinity.

This author [20] has shown that every hereditarily indecomposable continuum can be obtained as a quotient from a monotone continuous decomposition of a one dimensional hereditarily indecomposable continuum. Thus there exist one dimensional hereditarily indecomposable continua with infinite dimensional hyperspaces. Levin [17] has recently shown that such one dimensional hereditarily indecomposable continua which admit monotone dimension raising maps are very abundant. Specifically, he has shown that every hereditarily indecomposable continuum of dimension at least two contains a one dimensional (hereditarily indecomposable) subcontinuum with infinite dimensional hyperspace.

Levin and Sternfeld [18] have recently used hereditarily indecomposable continua to provide a very nice proof that every continuum (whether decomposable or indecomposable) of dimension at least two has an infinite dimensional hyperspace.

We conclude with several questions about one dimensional continuous collections.

**Question 8.** Is there any (homogeneous) one dimensional continuum $H$ other than the pseudo-arc such that for every (homogeneous) one dimensional continuum $M$ there is a (homogeneous) one dimensional continuum $\tilde{M}$ with a continuous decomposition into copies of $H$, such that the decomposition space is homeomorphic to $\tilde{M}$?

**Question 9.** If the answer to Question 8 is yes, must $H$ be hereditarily indecomposable? tree-like? Does every homeomorphism (self-map) of $\tilde{M}$ lift to a homeomorphism (self-map) of $\tilde{M}$?

**Question 10.** If $T$ is a homogeneous nonseparating plane continuum, does there exist a homogeneous separating plane continuum $C_T$ with a continuous decomposition into copies of $T$?

**Question 11.** Which one dimensional continua admit a continuous decomposition into nonterminal pseudo-arcs?
**Question 12.** Does the pseudo-arc admit an hereditarily equivalent embedding in the plane, i.e., an embedding where all nondegenerate subcontinua are equivalently embedded?

**Question 13.** If the nondegenerate indecomposable continuum $X$ admits an hereditarily equivalent embedding in the plane, must $X$ be a pseudo-arc?

Hagopian has posed the above two questions as a way of attacking the classification of hereditarily equivalent continua, a problem which has been resistant to progress in recent years. The "standard" Moise embedding of the pseudo-arc in the plane is not hereditarily equivalent.

**Question 14.** Does there exist an embedding of the arc of pseudo-arcs or circle of pseudo-arcs in the plane such that every homeomorphism (or self-map) of the arc or circle lifts to a homeomorphism (or self-map) of the arc or circle of pseudo-arcs which extends to the plane? Which homeomorphisms (or self-maps) of the arc or circle of pseudo-arcs extend to homeomorphisms (or self-maps) of the plane for which embeddings?

**Question 15.** What is a characterization, either topological or algebraic, of the one-dimensional hereditarily indecomposable continua which have infinite dimensional hyperspaces? Does every nondegenerate planar hereditarily indecomposable continuum have a two dimensional hyperspace?

**References**

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