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RHYMING SCHEMES: CROSSINGS AND COVERINGS

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Some recent results on the enumeration of relations on finite totally ordered sets are unified by establishing correspondences between these relations and some types of rhyming scheme which are characterized by the planarity of certain graphical representations. These representations themselves and a systematic method of enumeration involving them are sketched. Interpretations and proofs of a number of combinatorial identities are obtained.

1. Introduction

The purpose of this paper is to draw together, by means of rhyming schemes [5, 7], some recent results on the enumeration of relations on finite, totally ordered sets [4, 5, 7, 12, 13, 14]. The rhyming scheme of an n line stanza may be regarded formally as a binary relation R on the set of lines $X_n = \{x_1, \ldots x_n\}$ of the stanza where x_iRx_j if and only if the lines x_i and x_j rhyme. R is reflexive, symmetric, and transitive and so an equivalence relation on X_n and conversely an equivalence relation may be interpreted as a rhyming scheme. It is convenient to define a second relation R^* on X_n in terms of R by taking $x_iR^*x_j$ whenever x_iRx_j but $x_i\not{R}x_k$ for k with $i<k<j$ and extending so that R^* is reflexive and symmetric. R^* is not in general transitive.

Two other types of relation, connective, and similarity relations, on finite, totally ordered sets, have attracted some attention recently. The relations R in question are again reflexive, symmetric relations on the totally ordered set $X_n = \{x_i : 1 \leq i \leq n\}$ where now x_i are not necessarily the lines of a stanza but may sometimes have this interpretation. R is then a connective relation on X_n if, in addition, $x_s\not{R}x_t$ for $i \leq s \leq j \leq t$ whenever x_jRx_j, $i<j$, [12, 13, 16] while R is a similarity relation on X_n if x_iRx_s, x_sRx_j for $i \leq s \leq j$ whenever x_iRx_j [4, 12, 14]. It is known [16, 14] that the number of relations of either kind on X_n is C_n, the nth Catalan number [15, Sequence 577], given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 1. \tag{1}$$

Here we establish an indirect biunique correspondence between the two sets of relations on X_n by first putting each set in correspondence with certain subsets of the set $\mathcal{B}_n$ of rhyming schemes on X_n and then, in turn, putting these subsets of $\mathcal{B}_n$ in correspondence (Secti[illegible]).

The two subsets of $\mathscr{B}_n$ are characterized in terms of the planarity of one other of two graphical representations (Section 2) and further enumerative resul are obtained by studying graph theoretical properties arising from these represe tations. A relation R on X_n is said to be of valence at least one if for each x_i in there is some $j \neq i$ for which $x_i R x_j$ (in terms of rhyming schemes, every li rhymes with at least one other). A relation of a given kind may often be made of relations of the same kind of valence at least one. Moreover, in many of t correspondences we establish, the property of being valence at least one preserved as is the number of edges in the various graphical representatio Another property of interest which, however, is not preserved, is that of spacir a relation R on X_n is spaced if $x_i R x_{i+1}$ for no i (no two adjacent lines rhyme

These sets may be enumerated systematically by colouring the edges of t associated graphs independently with any one of a range of colours and th obtaining a convolutive or multiplicative identity, (see, for example, (4)), for t number of such coloured graphs by considering the first occurrence of so distinguishing feature. Closed expressions for these numbers are obtained using Lagrange's inversion formula [17, pp. 132–133] in the following modifi form that if

$$A(x) = \sum_{n \geq 1} a_n x^n = xH(A(x)),$$

where, for example, $H(t)$ is a polynomial in t (as it is in the cases conside below), then

$$na_n = \text{coefficient of } t^{n-1} \text{ in } ((H(t))^n.$$

We obtain, in the process, the number of graphs of certain kinds with gi numbers of edges (Section 4).

The same considerations may be applied to the set $\mathscr{B}_n$ itself to obtain (Sect 5) combinatorial proofs of identities involving the Bell numbers [15, Seque 585], and the Stirling numbers $S(n, r)$ of the second kind [9, pp. 32–34], as wel other related numbers (see (14), (15), (17), (19)–(21)). The number B_n of rhym schemes (equivalence relations) on X_n is the nth Bell number, where (comp (16)),

$$\sum_{n \geq 0} B_n \frac{x^n}{n!} = e^{(e^x - 1)}.$$

Similarly the number V_n of rhyming schemes in $\mathscr{B}_n$ which are of valence at l one satisfies (see [15, Sequence 1387] and compare (18)),

$$\sum_{n \geq 0} V_n \frac{x^n}{n!} = e^{(e^x - x - 1)}.$$

The number of rhyming schemes on X_n was, according to [1], considere Sylvester, but I have failed to discover this work in a brief search in Sylvesl publications in the Bodleian Library, Oxford. The literary history and analysis of rhyming schemes is, however, very much older: Growney [7] draws attention to a 16th century work, Puttenham's *The Arte of English Poesie*, while an early Japanese novel *The Tale of Genji*, of Murasaki Shikibu, is cited in the extensive bibliography [6].

2. Graphical representations

Consider a reflexive, symmetric relation R on a finite, totally ordered set X_n as before. The (undirected) graph $\Gamma(R)$ of the relation R is the graph whose vertex set is X_n, identified with a set of n points on a line, labelled consecutively x_1 to x_n, in which x_i is jointed to x_j if and only if $x_i R x_j$, $i \neq j$, the graph being contained in a half plane bounded by the line of points [12]. Alternatively, we may take the vertices of $\Gamma(R)$ to be points on a circle and the edges chords joining (some of) these points so that, in the case where $\Gamma(R)$ is a planar graph, it is a ladder graph [2]. The bipartite graph $B(R)$ of the relation R is the bipartite graph with vertex set $X_n \cup Y_n$, where $Y_n = \{y_i : 1 \leq i \leq n\}$, with edges unordered pairs $\{x_i, y_j\}$ where $\{x_i, y_j\}$ is an edge of $B(R)$ if and only if $x_i R x_j$, $i<j$. X_n and Y_n may conveniently be identified with ordered sets of n points on two parallel lines. R is uniquely specified in terms of the adjacency relation of either graph with the addition that R is reflexive. The number of edges in the two representations of R is the same.

A relation R on X_n is of valence at least one if and only if every vertex of $\Gamma(R)$ has valence at least one or, equivalently, for every i, at least one of the vertices x_i, y_i of $B(R)$ has valence at least one.

The unipartite representation of a rhyming scheme R on X_n is the graph $\Gamma(R^*)$ [5, 7]. A rhyming scheme R is planar if its unipartite representation is a planar graph, that is $x_i R^* x_j$ and $x_s R^* x_t$ for no $i<s<j<t$; otherwise R has crossings. Growney [7] showed that the number of planar rhyming schemes on X_n is C_n, as given by (1). The graphs (R^*) of rhyming schemes R on X_3 are shown in Fig. 1.

The bipartite representation of a rhyming scheme R on X_n is the graph $B(R^*)$ [3]. A rhyming scheme R is uncovered if its bipartite representation is a planar graph, that is $x_i R^* x_j$ and $x_s R^* x_t$ for no $i<s<t<j$; otherwise R has coverings. The bipartite representation of an uncovered rhyming scheme on X_n is a decreasing bipartite graph on $X_n \cup Y_n$ in which $\{x_i, y_i\}$ is not an edge for any i and so the number of such schemes is also C_n [3]. The graphs $B(R^*)$ of rhyming schemes R on X_3 are shown in Fig. 2.

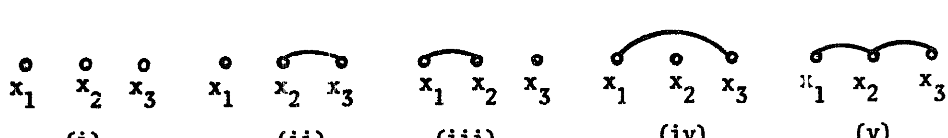

Fig. 1. $\Gamma(R^*)$ for rhyming schemes R on X_3.

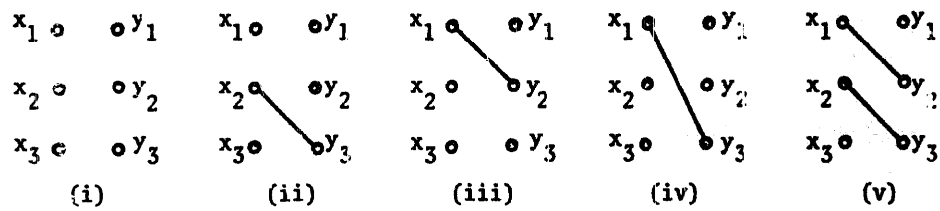

Fig. 2. $B(R^*)$ for rhyming schemes R on X_3.

We apply the terms "planar" and "uncovered", "has crossings" and "has coverings" interchangably to R and R^*. The planarity of a scheme depends on the choice of representation: there are schemes which are planar but not uncovered and vice versa, as is shown in Figs. 3 and 4. (Figs. 3(b) and 4(b) show the graphs $\Gamma(R^*)$ and $B(R^*)$ for the quatrain R in which the lines rhyme alternatively.)

The first of the correspondences which we now proceed to describe, that between planar and uncovered schemes, is obtained by a process of unravelling crossings and coverings suggested by Fig. 3 and 4. (The graphs of corresponding (indeed, in this case identical) schemes on X_3 have the same labels in Figs. 1 and 2.)

Fig. 3. $\Gamma(R^*)$: (a) R planar; (b) R uncovered.

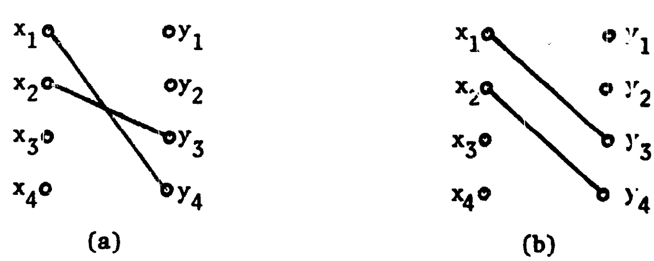

Fig. 4. $B(R^*)$ for the same R as in Fig. 3.

3. Correspondences

In establishing a correspondence between planar and uncovered schemes it is only necessary to show how planar schemes with coverings correspond to uncovered schemes with crossings, leaving those schemes which are both planar and uncovered unchanged. Consider then a non-planar uncovered scheme R on X_n and define a sequence of reflexive, symmetric relations R_k, $k \geq 0$ on X_n induc-

(a) (b)

Fig. 5. Correspondence between planar rhyming schemes and connective relations: (a) $\Gamma(R^*)$; (b) $\Gamma(R_c)$.

tively as follows:

(a) $R_0 = R^*$;

(b) for $k \geq 0$, if R_k has a crossing and $x_i R_k x_j$, $x_s R_k x_t$, with $i<s<j<t$ is the left most crossing (that is i and s are minimal subject to these conditions) then R_{k+1} is the same as R_k except that $x_i R_{k+1} x_t$, $x_s R_{k+1} x_j$, thereby replacing the left most crossing in R_k by a cover in R_{k+1}.

After a finite number of steps, say m, we obtain a relation $S^* = R_m$ which corresponds to a planar scheme S with covers; and we may reverse the procedure to obtain R by working at each step with the outermost and left most cover. (The relation illustrated in Fig. 3(b) is taken into that shown in Fig. 3(a).) This establishes the required correspondence and we may further note that the number of crossings in R^* is the same as the number of coverings in S^*.

Consider again a planar rhyming scheme R on X_n. We may turn this into a connective relation R_c on X_n by gathering each set of rhyming lines (equivalence class) at the final appearance of their rhyme: more precisely if, for each i, $x_{i_1}, \ldots, x_{i_{k(i)}}$, where $i_1 < i_2 < \cdots < i_{k(i)}$, are all the lines which rhyme under R with x_i so that $x_i = x_{i_j}$ for some j, $1 \leq j \leq k(i)$, and $x_{i_s} R^* x_{i_{s+1}}$, $1 \leq s < k(i)$, then we take $x_{i_s} R_c x_{i_{k(i)}}$, $1 \leq s \leq k(i)$, and extend R_c to be reflexive and symmetric. Then the condition of planarity on R ensures that R_c is connective; and the correspondence is reversible so that we may recover R uniquely from R_c, via R^*. An illustration of this correspondence is given in Fig. 5.

Finally, given an uncovered rhyming scheme R on X_n we may regard it as the unique "skeleton" of a similarity relation R_s by taking $x_p R_s x_q$ for $i \leq p \leq q \leq j$ whenever $x_i R^* x_j$ and extending R_s to be reflexive and symmetric. Conversely every similarity relation R_s has a unique skeleton R^* where R^* is reflexive and symmetric and $x_i R^* x_j$ whenever $x_i R_s x_j$ and $x_i \not{R}_s x_q$, $j<q$, $x_p \not{R}_s x_j$, $p<i$; and, in turn, R^* corresponds to an uncovered scheme R. This correspondence is illustrated in Fig. 6.

(a) (b)

Fig. 6. Correspondence between uncovered rhyming schemes and similarity relations: (a) $\Gamma(R^*)$; (b) $\Gamma(R_s)$.

The property of being spaced is not, in general, preserved under these correspondences. However, if a relation R on X_n is spaced then $B(R)$ may be regarded as a graph $B(R')$ on $X_{n-1}=X_n\backslash\{x_n\}$ and $Y'_{n-1}=Y_n\backslash\{y_1\}$ with the shift $y'_i=y_{i+1}$, $1<i\leq n$, and hence associated with a relation R' on X_{n-1} in a unique way. This provides a further one-to-one correspondence between rhyming schemes (respectively uncovered schemes) R on X_n for which R^* is spaced and rhyming schemes (respectively uncovered schemes) on X_{n-1}, a correspondence which, however, does not, in general, preserve the property of being of valence at least one.

In the first two correspondences established in this section, the number of edges in the graphical representations of corresponding relations is the same and the property of being of valence at least one is preserved. This is not the case with the third correspondence, that between uncovered schemes and similarity relations, in which additional edges are added and this explains in part the difference between the (additive) enumeration of similarity relations and the (convolutive or multiplicative) enumeration of connective relations remarked upon in [12, Section 9].

4. Enumerative methods and results

The enumeration of the set $\mathscr{C}_n$ of connective relations on X_n has been described fully elsewhere [13] and the enumeration of the sets $\mathscr{U}_n$ and $\mathscr{P}_n$ of respectively uncovered schemes (in the form of increasing bipartite graphs) and planar schemes on X_n may be treated similarly. Here we give a brief review of the methods and results for these sets concentrating on $\mathscr{U}_n$ and, more especially, on $\mathscr{P}_n$.

Let $f_k(n)$ be the number of graphs, associated with the relations in any one of these sets, whose edges are coloured independently with any of a range of k colours using $\Gamma(R)$ for R is $\mathscr{C}_n$, $B(R^*)$ for R in $\mathscr{U}_n$ or $\Gamma(R^*)$ for R in $\mathscr{P}_n$. In all three cases

$$f_k(1)=1, \qquad f_k(2)=1+k,$$

and, in [13], it is shown, by partitioning the set $\mathscr{C}_n$ according to the largest integer i such that x_1Rx_i for R in $\mathscr{C}_n$, that

$$f_k(n)=f_k(n-1)+k\sum_{i=2}^{n-1} f_k(i-1)f_i(n-i)+kf_k(n-1), \quad n\geq 3. \tag{6a}$$

Similarly, partitioning the set $\mathscr{U}_n$ according to the smallest integer i, if any, such that $x_iR^*x_{i+1}$ for R in $\mathscr{U}_n$, we have

$$f_k(n)=kf_k(n-1)+k\sum_{i=2}^{n-1} f_k(i-1)f_k(n-i)+f_k(n-1), \quad n\geq 3, \tag{6b}$$

noting that if there is no such i then R is a spaced relation in $\mathscr{U}_n$ and, by the remarks in Section 3, there are $f_k(n-1)$ of these.

Finally, partitioning the set $\mathscr{P}_n$ according to the largest integer i such that

$x_1 R^* x_i$, we find

$$f_k(x)=f_k(n-1)+kf_k(n-1)+k\sum_{i=3}^{n} f_k(i-2)f_k(n-i+1), \quad n\geq 3. \tag{6c}$$

We may rewrite (6a), (6b), (6c) in the form

$$f_k(n)=(1+k)f_k(n-1)+k\sum_{r=1}^{n-2} f_k(r)f_k(n-r-1), \quad n\geq 3,$$

and it then follows, in terms of generating functions, that (compare [13, (12)])

$$\begin{aligned} F_k(x)=\sum_{n\geq 1} f_k(n)x^n &= x(1+(1+k)f_k(x)+k(F_k(x))^2) \\ &= x(1+kF_k(x))(1+F_k(x)). \end{aligned} \tag{7}$$

Applying Lagrange's inversion formula, as in (2), (3), to (7), we have

$$nf_k(n)=\text{coefficient of } t^{n-1} \text{ in } (1+kt)^n(1+t)^n.$$

Now

$$(1+kt)^n(1+t)^n=\left(\sum_{r=0}^{n}\binom{n}{r}k^r t^r\right)\left(\sum_{s=0}^{n}\binom{n}{r}t^{n-s}\right),$$

so

$$\begin{aligned} f_k(n) &= \frac{i}{n}\sum_{r=0}^{n-1}\binom{n}{r}\binom{n}{r+1}k^r, & n\geq 1, \\ &= \frac{1}{n+1}\sum_{r=0}^{n-1}\binom{n+1}{r+1}\binom{n-1}{r}k^r, & n\geq 1. \end{aligned}$$

Hence the number $f(n, r)$ of graphs with r edges arising from any one of the sets $\mathscr{C}_n$, $\mathscr{U}_n$, or $\mathscr{P}_n$, is given by (compare [13, (1)])

$$f(n,r)=\frac{1}{n+1}\binom{n+1}{r+1}\binom{n-1}{r}, \quad 0\leq r\leq n-1.$$

Moreover, for $k=1$, writing (7) in the form

$$F_1(x)=x(1+F_1(x))^2,$$

another application of Lagrange's inversion formula yields (see (1)),

$$f(n)=f_1(n)=\frac{1}{n}\binom{2n}{n-1}=\frac{1}{n+1}\binom{2n}{n}=C_n, \quad n\geq 1.$$

A similar analysis holds for relations of valence at least one. For example, if $g_k(n)$ is the number of graphs associated with schemes in $\mathscr{P}_n$ of valence at least one, coloured as before, and $\hat{g}_k(x)$ is the number of these in which x_1 and x_2 are joined then, with the same partition as above, $g_k(2)=\hat{g}_k(2)=k$, $g_k(3)=\hat{g}_k(3)=k^2$, and

$$g_k(n)=\hat{g}_k(n)+\sum_{r=2}^{n-2}\hat{g}_k(n)g_k(n-r),\quad n\geqslant 4; \tag{8}$$

$$\hat{g}_k(n)=k(g_k(n-1)+g_k(n-2)),\quad n\geqslant 4. \tag{9}$$

Hence, combining (8) and (9) using generating functions, we have

$$\hat{G}_k(x)=\sum_{n\geqslant 2}\hat{g}_k(n)x^{n-1}=x[k(1+\hat{G}_k(x))+(G_k(x))^2],$$

from which it follows, as another application of Lagrange's inversion formula, that (compare [13, (14)])

$$\hat{g}_k(n+2)=\frac{1}{n+1}\sum_{r=0}^{[n/2]}\binom{n}{n}\binom{n-r}{n-1-2r}k^{n+1-r},\quad n\geqslant 0,$$
$$=\sum_{r=0}^{[n/2]}\binom{n}{2r}C_r k^{n+1-r},\quad n\geqslant 0.$$

In particular

$$g_1(n+2)=m_n,\quad n\geqslant 0,\qquad g_1(n)=\gamma_n,\quad n\geqslant 2,$$

where m_n is the nth Motzkin number [15, Sequence 456; 3] given by

$$m_n=\sum_{r=0}^{[n/2]}\binom{n}{2r}C_r,\quad n\geqslant 0, \tag{10}$$

and $\gamma_n+\gamma_{n+1}=m_n$, $n\geqslant 0$, $\gamma_0=1$ (see [11, p. 219; 3]). Finally for R in $\mathcal{P}_n$, considering those x_i which are not related under R^* to any other x_j, $j\neq i$, in X_n, we obtain the identities

$$C_{n+1}=\sum_{r=0}^{n}\binom{n}{r}m_r,\quad C_n=\sum_{r=0}^{n}\binom{n}{r}\gamma_r,\qquad n\geqslant 0. \tag{11}$$

Analogous interpretations of (8), (9), (10) hold for $\mathcal{C}_n$ and $\mathcal{U}_n$, and the sets of bipartite graphs considered in [3] may be enumerated in the same way.

The three partitions given above, when considered using the correspondences in Section 3 as partitions of the same set, are all distinct and so we may obtain further combinatorial results. For example transferring the partition of $\mathcal{C}_n$ to $\mathcal{P}_n$ (or $\mathcal{U}_n$) the number of planar (or uncovered) schemes R on X_n for which for each i, $1\leqslant i\leqslant n$ there are s, t with $s\leqslant i<t$ and $x_sR^*x_t$ is C_{n-1}, $n\geqslant 1$.

Arguing as for relations of valence at least one, we may show that the number of spaced planar schemes on X_{n+1} is m_n, $n\geqslant 0$, (from which we may deduce, by the correspondence of Section 3, that this is also the number of connective relations R_c on X_{n+1} for which $x_iR_cx_k$, $x_{i+1}R_cx_k$ for no i, k with $i<k$ (see also [13])). A direct proof of this may be based on Motzkin's original definition [8] of the numbers m_n as the number of ladder graphs $\Gamma(R^+)$ on X_n in which the

vertices are either isolated or joined in disjoint pairs. There are $\binom{n}{2r}$ ways of choosing $2r$ points from X_n and then, by a result of Errera [10], C_r ways of joining these in disjoint pairs by non-intersecting chords. We thus obtain a combinatorial interpretation of (10), the number of these graphs $\Gamma(R^+)$ with r edges being

$$\binom{n}{2r}C_r \quad 0 \leq r \leq \left[\frac{n}{2}\right]. \tag{12}$$

From a graph $\Gamma(R^+)$ of this kind, we obtain a reflexive, symmetric relation R^+ on X_n and we may, in turn, define a relation R^* on X_{n+1} by taking $x_i R^* x_{j+1}$ whenever $x_i R^+ x_j$, $i<j$, and extending R^* to be reflexive and symmetric. This correspondence between R^+ and R^* is one-to-one and then the rhyming scheme R which corresponds to R^* as in Section 1 is a planar spaced one on X_{n+1}. Hence the number of such schemes on X_{n+1} is m_n, $n \geq 0$, as stated and, moreover, since $\Gamma(R^+)$ and $\Gamma(R^*)$ have the same number of edges, the number of these schemes R on X_{n+1} whose graphs $\Gamma(R^*)$ have r edges is also given by (12).

The enumeration of similarity relations may be obtained via the correspondence of Section 3 but, as remarked there, a direct approach proceeds somewhat differently from the foregoing and is described in [12].

5. Further enumerative results

The Stirling numbers $S(n, r)$ of the second kind are the number of equivalence relations on X_n with r equivalence classes and are related to the Bell numbers B_n by

$$B_n = \sum_{r=0}^{n} S(n, r), \quad n \geq 0. \tag{13}$$

It follows that the number of $B(n, r)$ of schemes in $\mathcal{B}_n$ whose unipartite representation has r edges is given by

$$B(n, r) = S(n, n-r), \quad 0 \leq r \leq n, \tag{14}$$

but we may also show this using the colouring technique of the previous section.

Let $b_k(n)$ be the number of distinct k coloured graphs, $\Gamma(R^*)$, arising (as in that section) from schemes R in $\mathcal{B}_n$ (so $b_1(n) = B_n$). There are $\binom{n}{r}$ ways of choosing r lines to rhyme with the first line in a scheme on X_{n+1}. So $b_k(0) = 1$ and

$$b_k(n+1) = \sum_{r=0}^{n} \binom{n}{r} b_k(n-r)k^r, \quad n > 0. \tag{15}$$

Writing (compare (4))

$$B_k(x) = \sum_{n \geq 0} b_k(n) \frac{x^n}{n!}$$

we have

$$B_k'(x) = e^{kx} B_k(x),$$

and so

$$B_k(x) = e^{k^{-1}(e^{kx}-1)} = \sum_{n\geq 0} \frac{(kx)^n}{n!} \sum_{r=0}^{n} S(n,r)k^{-r} = \sum_{n\geq 0} \frac{x^n}{n!} \sum_{r=0}^{n} S(n,r)k^{n-r} \tag{16}$$

from which (13), (14) follow.

Similarly, if $V(n, r)$ is the number of schemes R of valence at least one in $\mathscr{B}_n$ whose graphs $\Gamma(R^*)$ have r edges and $v_k(n)$ is the total number of distinct k coloured graphs, $\Gamma(R^*)$, arising from schemes R of valence at least one in $\mathscr{B}_n$ then $v_k(0) = 1$, $v_k(1) = 0$, and

$$v_k(n+1) = \sum_{r=1}^{n} \binom{n}{r} v_k(n-r)k^r, \quad n \geq 1, \tag{17}$$

leading to (compare (5))

$$V_k(x) = \sum_{n\geq 0} v_k(n) \frac{x^n}{n!} = e^{k^{-1}(e^{kx}-kx-1)} = \sum_{n\geq 0} \frac{x^n}{n!} \sum_{r=0}^{n} b(n,r)k^{n-r} \tag{18}$$

and

$$V(n, r) = b(n, n-r) \tag{19}$$

where $b(n, r)$ is the associated Stirling number of the second kind [9, p. 77]. Moreover, considering lines rhyming with no other line, we have (compare (11))

$$b_k(n) = \sum_{r=0}^{n} \binom{n}{r} v_k(n-r).$$

We may show, arguing as for (15), (17), that the number of spaced schemes in $\mathscr{B}_n$ is B_{n-1}, $n \geq 1$, as also follows by the approach in Section 3. Further, considering the lines, if any, rhyming with the final line, we may show that

$$B(n+1, s+1) = B(n, s+1) + (n-s)B(n, s), \tag{20}$$

$$V(n+1, s+1) = (n-s)V(n, s) + nV(n-1, s), \tag{21}$$

leading, via (14), (19), to familiar identities for the Stirling numbers $S(n, r)$ and $b(n, r)$ respectively.

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