Quotient Star Bodies, Intersection Bodies, and Star Duality

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INTRODUCTION

Various topological constructions have their metric counterparts; for example, metric product corresponds to topological product (see [6]) and metric inverse limit corresponds to topological inverse limit (see [9, 10]). The main subject of this paper is a metric counterpart of the topological notion of quotient space (see [2] or [3]). We restrict our consideration to a compact metric space \( A \) and a very simple equivalence relation, with only one nontrivial equivalence class \( C \subseteq A \). Then, evidently, the natural quotient map \( p: A \to A/C \) (defined by \( p(x) = [x] \)) satisfies the condition

\[
p(C) \text{ is a singleton and } p(A \setminus C) \text{ is a topological embedding. (0.1)}
\]

Topologists used to look at quotient space “up to a homeomorphism”; i.e., more precisely, they work with the category Top of topological spaces and continuous maps. For geometers, it is important to choose a “good” representative of the topological type of \( A/C \); this representative will be the image of \( A \) under a map \( p_c \) satisfying (0.1), with some nice geometric properties.

We shall keep in mind the following physical interpretation. Let \( A \) be a soft body with a stone \( C \) inside. When the stone is cut out, the hole shrinks to a point and the remaining part \( A \setminus C \) changes its geometric shape. The shape of the resulting body \( A/C \) depends not only on geometric properties of \( A \) and \( C \), but also on physical properties of \( A \) and forces acting on \( A \).

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This interpretation justifies further restrictions. We assume that $A$ is a subset of $\mathbb{R}^n$, $0 \in C$, and $A \setminus C$ is homeomorphic to $A \setminus \{0\}$. Then $A/C$ is homeomorphic to a subset of $\mathbb{R}^n$ and so we can look for a map $p_C$ satisfying (0.1), with values in $\mathbb{R}^n$. Moreover, if $p_C: \mathbb{R}^n \to \mathbb{R}^n$ satisfying (0.1) is independent of $A$, then it is reasonable to consider properties of $A$ invariant under $p_C$.

Following [3], we shall refer to $p_C$ as a *quotient map* (an *identification* in terminology of [2]).

The following theorem follows directly from [2, Theorem 7.7, p. 17 and Theorem 4.3, p. 126].

**Theorem 0.1.** Let $\mathcal{C}$ be a nonempty family of compact subsets of $\mathbb{R}^n$ with $0 \in C$ and $\mathbb{R}^n \setminus C$ homeomorphic to $\mathbb{R}^n \setminus \{0\}$ for every $C \in \mathcal{C}$, and let $p_C$ be a quotient map with $p_C(C) = 0$. For every map $f: \mathbb{R}^n \to \mathbb{R}^n$ preserving $\mathcal{C}$, with $f(0) = 0$, and for every $C \in \mathcal{C}$, let

$$\hat{f}(x) := p_{f(C)}f^{-1}(x) \quad \text{for every } x \in \mathbb{R}^n.$$  

Then

(i) the formula (0.2) defines a map $\hat{f}: \mathbb{R}^n \to \mathbb{R}^n$ such that $\hat{f}(0) = 0$ and the diagram

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^n \\
\downarrow{p_C} & & \downarrow{p_{f(C)}} \\
\mathbb{R}^n & \xrightarrow{\hat{f}} & \mathbb{R}^n
\end{array}$$

commutes, and

(ii) if $f$ is a homeomorphism, then so is $\hat{f}$.

We shall work with the class of star bodies in $\mathbb{R}^n$ with nonnegative and continuous radial maps. Section 1 concerns star bodies and star maps. In Section 2 we define radial quotient maps and study corresponding quotient bodies. In particular, we prove that the operation $(A,C) \mapsto A/C$ preserves the class of intersection bodies of star bodies (Corollary 2.8).

Section 3 deals with the category $\text{St}^n$ of star bodies and star maps, and its subcategory $\text{St}^+_n$ with star bodies whose radial functions are positive. We define and study a duality on this subcategory: a functor from $\text{St}^+_n$ to itself which is an involution (compare [11]).

We use the following notation: For affine independent points $x_0, \ldots, x_k$ in $\mathbb{R}^n$, the simplex with vertices $x_0, \ldots, x_k$ is denoted by $\Delta(x_0, \ldots, x_k)$; in particular, if $a, b$ are distinct points, then $\Delta(a, b)$ is the segment with endpoints $a, b$.
As usual, $B^n$ and $S^{n-1}$ are the unit ball and the unit sphere, and $\kappa_n$ is the volume of $B^n$. We denote by $\sigma$ the spherical Lebesgue measure on $S^{n-1}$. For any $u \in S^{n-1}$, the (linear) hyperplane orthogonal to $u$ is denoted by $u^\perp$. For any convex body $A \subset R^n$, its support function is $h(A, \cdot)$, supporting hyperplane is $H(A, \cdot)$, and $b(A)$ is the mean width of $A$. The class of convex bodies in $R^n$ is $K_n^0$.

For basic notions of the category theory, see [8].

1. STAR BODIES AND STAR MAPS

Since in the literature there are various notions of a star body (see [4, p. 18; [12], p. 416), we start from definitions: Let $A$ be a nonempty compact subset of $R^n$ and $a \in A$. Then, $A$ is a body if and only if $A = \text{cl int } A$, and $A$ is star shaped at $a$ if and only if $\Delta(a, x) \subset A$ for every $x \in A$.

For convenience, we shall assume that $a = 0$. The radial function $\varrho_A: R^n \setminus \{0\} \to R$ of a set $A$ which is star shaped at 0 is defined by the formula

$$\varrho_A(x) := \sup \{ \lambda \geq 0 \mid \lambda x \in A \}.$$  

(Sometimes, we write $\varrho(A, x)$ instead of $\varrho_A(x)$.) Obviously, $\varrho_A$ is homogeneous of degree $-1$.

A set $A \subset R^n$ will be called a star body whenever $A$ is a body which is star shaped at 0 and whose radial function restricted to $S^{n-1}$ is continuous.

Let us notice that this notion of star body is less general than that of Gardner [4], since we assume that $0 \in A$ and thus $\varrho_A \geq 0$. Moreover, Gardner assumes only continuity of the restriction of radial function to its support. On the other hand, our notion is more general than that of Schneider [12], since we do not require $\varrho_A > 0$; that is, we allow 0 to belong to $\partial A$.

Let $\mathcal{S}_0^n$ be the class of all the star bodies in $R^n$ and let

$$\mathcal{S}_1^n = \mathcal{S}_0^n \cup \{0\}.$$  

We are looking for a possibly large group of transformation which preserve $\mathcal{S}_0^n$ and $\mathcal{S}_1^n$.

**Definition 1.1.** A map $f: R^n \to R^n$ is a star map if and only if $f$ is a positively homogeneous homeomorphism.

Of course, the set of all the star maps of $R^n$ is a group of transformations. We shall denote it by $G\text{S}(n)$ (general star maps).

The following is evident.
PROPOSITION 1.2. If $f \in G S(n)$ and $A \in \mathcal{S}_0^n$, then for every $x \in \mathbb{R}^n$,
\[ \varphi_{f(A)}(f(x)) = \varphi_A(x). \]

The next statement is a direct consequence of Proposition 1.2.

PROPOSITION 1.3. The classes $\mathcal{S}_0^n$ and $\mathcal{S}_1^n$ are invariant under star maps.

Let us prove the following.

PROPOSITION 1.4. Every two star bodies $A_1, A_2$ with 0 in the interior are star equivalent; i.e., there exists $g \in G S(n)$ such that $g(A_1) = A_2$.

Proof. Let $A_1, A_2 \in \mathcal{S}_0^n$. We define $g: \mathbb{R}^n \to \mathbb{R}^n$ by the formula
\[ g(x) = \begin{cases} \frac{\varphi_{A_2}(x)}{\varphi_{A_1}(x)} \cdot x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \quad (1.1) \]

It is clear that $g \in G S(n)$.

Let us show that $g(A_1) = A_2$. Indeed, $g$ is a homeomorphism and maps $\text{bd } A_1$ onto $\text{bd } A_2$:
\[ x \in \text{bd } A_1 \iff \varphi_{A_1}(x) = 1 \iff g(x) = \varphi_{A_2}(x) \cdot x \]
\[ \iff \varphi_{A_1}(g(x)) = 1 \iff g(x) \in \text{bd } A_2. \]

Evidently, $G L(n)$ is a subgroup of $G S(n)$. As a consequence of Proposition 1.4, we obtain the following corollary:

COROLLARY 1.5. $G L(n)$ is a proper subgroup of $G S(n)$.

Let us notice that the group $G S(n)$ is generated by its two subgroups,
\[ \mathcal{S}S(n) := \{ f \in G S(n) \mid f(S^{n-1}) = S^{n-1} \} \quad (1.2) \]
and
\[ \mathcal{R}S(n) := \{ f \in G S(n) \mid \exists \lambda: \mathbb{R}^n \to \mathbb{R}_+, f(x) = \lambda(x) \cdot x \text{ for every } x \}. \quad (1.3) \]

The first subgroup consists of spherical star maps, the star extensions of homeomorphisms of $S^{n-1}$ onto itself; the second subgroup consists of radial star maps, the star extensions of central projections of $S^{n-1}$.

PROPOSITION 1.6. For every $f \in G S(n)$ there exists $g \in \mathcal{R}S(n)$ and $h \in \mathcal{S}S(n)$ such that $f = gh$. This decomposition is unique.
Proof. Let \( f \in \text{GS}(n) \) and let \( A = f(B^n) \). Then \( A \) is a star body with \( 0 \in \text{int} A \), and thus, by Proposition 1.4, there exists \( g \in \text{GS}(n) \) such that \( g(S^{n-1}) = \text{bd} A \). The map \( g \) is defined by (1.1) for \( A_1 = B^n \) and \( A_2 = A \), whence \( g \in \text{RS}(n) \).

Let \( h := g^{-1}f \).

Then \( h \) is a star map preserving \( S^{n-1} \), and \( f = gh \).

To prove the uniqueness of the decomposition of \( f \), suppose that

\[
\begin{align*}
  f &= g_1h_1 = g_2h_2
\end{align*}
\]

for some radial star maps \( g_1, g_2 \) and spherical star maps \( h_1, h_2 \). Then \( g_2^{-1}g_1 = h_2h_1^{-1} \), where the left side belongs to \( \text{RS}(n) \) and the right side belongs to \( \text{SS}(n) \). Since these two subgroups have only the identity map in common, it follows that \( g_1 = g_2 \) and \( h_1 = h_2 \).

Every map \( f: R^n \rightarrow R^n \) preserving \( S^n \) induces the map \( f_*: S^n \rightarrow S^n \) defined by

\[
  f_*(A) = f(A).
\]

According to [4], the radial metric \( \delta \) in \( S^n \) is defined by the formula

\[
  \delta(A_1, A_2) = \sup_{u \in S^{n-1}} \left| q_{A_1}(u) - q_{A_2}(u) \right|.
\]

Theorem 1.7. If \( f \in \text{GS}(n) \), then \( f_* \) is a Lipschitz map with respect to the radial metric \( \delta \). If, moreover, \( f \in \text{SS}(n) \), then \( f_* \) is an isometry.

Proof. Let \( A_1, A_2 \in S^n \). Then, by Proposition 1.2,

\[
\begin{align*}
  \delta(f_*(A_1), f_*(A_2)) &= \sup_{u \in S^{n-1}} \left| q_{f(A_1)}(u) - q_{f(A_2)}(u) \right| \\
  &= \sup_{u \in S^{n-1}} \left| q_{A_1}(f^{-1}(u)) - q_{A_2}(f^{-1}(u)) \right| \\
  &= \sup_{u \in S^{n-1}} \frac{1}{\|f^{-1}(u)\|} \left| q_{A_1} \left( \frac{f^{-1}(u)}{\|f^{-1}(u)\|} \right) - q_{A_2} \left( \frac{f^{-1}(u)}{\|f^{-1}(u)\|} \right) \right| \\
  &\leq \sup_{u \in S^{n-1}} \frac{1}{\|f^{-1}(u)\|} \cdot \delta(A_1, A_2),
\end{align*}
\]

where \( \sup(1/\|f^{-1}(u)\|) < \infty \), because \( f^{-1} \) is a bijection, \( f(0) = 0 \), and \( S^{n-1} \) is compact.

The second assertion is obvious.
2. RADIAL QUOTIENT STAR BODIES

We shall consider the following family of maps of $\mathbb{R}^n$ onto itself.

**Definition 2.1.** Let $C \in \mathcal{K}^n$. We define $p_C: \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$p_C(x) := \begin{cases} 0, & \text{if } x \in C, \\ \left(1 - q_C(x)\right) \cdot x, & \text{if } x \in \mathbb{R}^n \setminus \text{int } C. \end{cases}$$

**Proposition 2.2.** For every $C \in \mathcal{K}^n$:

(i) $p_C$ is a quotient map;

(ii) if $A$ is a star body containing $C$ and $A \neq C$, then $p_C(A) \in \mathcal{K}_0^n$.

In view of Proposition 2.2, we shall refer to $p_C$ as a radial quotient map and to $p_C(A)$ as a radial quotient star body of $A$. We shall use the symbol $A / C$ also for $p_C(A)$.

Let us note the following.

**Proposition 2.3.** If $A, C \in \mathcal{K}^n$ and $C \subset A$, then

$$q(A / C, x) = q(A, x) - q(C, x)$$

for every $x \in \mathbb{R}^n \setminus \{0\}$.

**Proof.** It suffices to prove the assertion for $q|S^{n-1}$. Let $u \in S^{n-1}$ and $x_0 \in \partial A \cap \text{pos } u$. Then

$$q(p_C(A), u) = \left\|p_C(x_0)\right\| = (1 - q_C(x_0)) \cdot \|x_0\|$$

$$= \|x_0\| - q_C\left(\frac{x_0}{\|x_0\|}\right) = q_A(u) - q_C(u).$$

The following statement is a direct consequence of Proposition 2.3.

**Corollary 2.4.** The function $\Phi: \{(A, C) \in \mathcal{K}_1^n \times \mathcal{K}_1^n | C \subset A\} \to \mathcal{K}^n$ defined by

$$\Phi(A, C) = A / C$$

is continuous with respect to the radial metric $\delta$.

For a given $C \in \mathcal{K}_1^n$, let

$$\mathcal{K}_0^n := \{A \in \mathcal{K}^n | A \supset C\}. \tag{2.2}$$

**Theorem 2.5.** For every $C \in \mathcal{K}^n$, there exists a homotopy

$$(\psi_t: \mathbb{R}^n \to \mathbb{R}^n)_{t \in [0, 1]}$$
such that $\psi_0 = \text{id}_{\mathbb{R}^n}$, $\psi_1 = p_C$, and for every $t \in [0,1]$, the induced map $(\psi_t)_*: \mathcal{S}_C^m \to \mathcal{S}_1^m$ is an isometry with respect to $\delta$.

**Proof.** We define $\psi_t: \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$
\psi_t(x) = \begin{cases} (1-t)x, & \text{if } x \in C, \\ (1-tq_C(x))x, & \text{if } x \notin C. \end{cases}
$$

Evidently, $\psi: \mathbb{R}^n \times [0,1] \ni (x,t) \mapsto \psi_t(x) \in \mathbb{R}^n$ is continuous,

$\psi(x,0) = x$, and $\psi(x,1) = p_C(x)$.

Let $t \in [0,1]$ and $A_1, A_2 \in \mathcal{S}_C^n$. We shall show that

$$
\delta(A_1, A_2) = \delta(\psi_t(A_1), \psi_t(A_2)).
$$

If $u \in S^{n-1}$, $i \in \{1,2\}$, and $a_i \in \text{bd } A_i \cap \text{pos } u$, then $\varrho(A_i,u) = \|a_i\|$ and

$$
\varrho(\psi_t(A_i), u) = \|\psi_t(a_i)\| = \begin{cases} (1-t)\|a_i\|, & \text{if } a_i \in C, \\ (1-tq_C(a_i))\|a_i\|, & \text{if } a_i \notin C. \end{cases}
$$

Since $(1-tq_C(a_i))\|a_i\| = \|a_i\| - tq_C(u)$ and $a_1, a_2 \in C$ implies $a_1 = a_2$, by easy calculation it follows that

$$
\varrho(\psi_t(A_1), u) - \varrho(\psi_t(A_2), u) = \begin{cases} 0, & \text{if } a_1, a_2 \in C, \\ \|a_1\| - \|a_2\|, & \text{otherwise}. \end{cases}
$$

Thus,

$$
\delta(\psi_t(a_1), \psi_t(a_2)) = \sup_{u \in S^{n-1}} \varrho(A_1,u) - \varrho(A_2,u) = \delta(A_1, A_2),
$$

which proves (2.3). □

Let us observe that for every star map $f$ the induced map $f_*$ is a homomorphism with respect to the operation $\Phi$ defined by (2.1):

**PROPOSITION 2.6.** For every $f \in \text{G}(n)$ and $A, C \in \mathcal{S}_C^n$ with $C \subset A$,

$$
f(A/C) = f(A)/f(C).
$$

**Proof.** Since $f$ is positively homogeneous and $\varrho_{f(C)}(f(x)) = \varrho_C(x)$, by Definition 2.1 it follows that

$$
fp_C(x) = p_{f(C)}(f(x)) \quad \text{for every } x \in \mathbb{R}^n.
$$

□

The operation $\Phi$ is also invariant under the function $I_1: \mathcal{S}_1^n \to \mathcal{S}_1^n$, the
first of $I_1, \ldots, I_{n-1}$, which are defined as follows: For every $u \in S^{n-1}$,

$$\varrho(I_k A, u) := \tilde{V}_{k,n-1}(A \cap u^\perp),$$

where

$$\tilde{V}_{k,n-1}(A \cap u^\perp) = \frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \varrho(A,v)^k \, d\sigma(v)$$

for any $k \in \{1, \ldots, n - 1\}$ (see [4] or [7]). If $A \in \mathcal{S}_0^n$, then $I_k A$ is called the intersection body of order $k$ of the star body $A$.

**Theorem 2.7.** For every $A, C \in \mathcal{S}_1^n$ with $C \subset A$,

$$I_1(A/C) = I_1A/I_1C.$$

**Proof.** By Proposition 2.3, for every affine subspace $E$ in $\mathbb{R}^n$,

$$(A/C) \cap E = (A \cap E)/(C \cap E). \quad (2.4)$$

Let $u \in S^{n-1}$. Using Proposition 2.3 combined with (2.4) for $E = u^\perp$, we obtain

$$\varrho(I_1(A/C), u) = \varrho(I_1A/I_1C, u).$$

As was noticed in [4, Note 8.3, p. 305], for any $k \in \{1, \ldots, n - 1\}$, a set $A$ is the intersection body of order $k$ of a star body if and only if $A$ is the intersection body of a star body. Thus Theorem 2.7 yields the following corollary:

**Corollary 2.8.** Let $A, C \in \mathcal{S}_1^n$ and $C \subset A$. If $A$ and $C$ are the intersection bodies of some star bodies, then so is $A/C$.

Corollary 2.8 can be useful in geometric tomography, where the intersection bodies play an essential role. See, for instance, Theorem 8.2.8 of [4], which gives an answer to the question whether

$$V_{n-1}(A_1 \cap u^\perp) \leq V_{n-1}(A_2 \cap u^\perp) \quad \text{for every } u \in S^{n-1}$$

implies $V_{n}(A_1) \leq V_{n}(A_2)$. According to that theorem, the answer is positive when $A_1$ is the intersection body of a star body.
3. CATEGORICAL APPROACH. DUALITY OF
STAR BODIES

Let $\text{St}^n$ be the category whose objects are star bodies in $\mathbb{R}^n$:

$$\text{Ob St}^n = \mathcal{P}_0^n,$$

and for every $A_1, A_2 \in \mathcal{P}_0^n$ the set of morphisms, $\text{St}^n(A_1, A_2)$, consists of star maps sending $A_1$ to $A_2$. Furthermore, let $\text{St}^n_\sharp$ be the category with the class of objects

$$\text{Ob St}^n_\sharp = \{(A, C) \in \mathcal{P}_0^n \times \mathcal{P}_0^n | C \subset A \neq C\}$$

and with morphisms being suitable star maps. The function $\Phi: \text{Ob St}^n_\sharp \rightarrow \mathcal{P}_0^n$ defined by (2.1) can be extended to a functor:

**Proposition 3.1.** The formulae

$$\Phi(A, C) := A/C \quad \text{for } A, C \in \mathcal{P}_0^n$$

and

$$\Phi(f) := \hat{f} \quad \text{for } f \in \text{St}^n_\sharp((A_1, C_1), (A_2, C_2))$$

[see (0.2)] define a covariant functor $\Phi: \text{St}^n_\sharp \rightarrow \text{St}^n$, such that any morphism and its image under $\Phi$ are restrictions of the same star map.

**Proof.** In view of Theorem 0.1, it suffices to show that $\hat{f} = f$ for every $f \in \mathcal{G}(n)$. It is easy to check that, for every $x \in \mathbb{R}^n \setminus \{0\}$,

$$p_{C}^{-1}(x) = (1 + q_C(x)) \cdot x$$

and

$$p_{C}(\beta x) = (\beta - q_C(x)) \cdot x$$

for every $\beta \in \mathbb{R}$. Thus,

$$\hat{f}(x) = p_{(C)}p_{C}^{-1}(x) = p_{(C)}f((1 + q_C(x)) \cdot x)$$

$$= p_{(C)}((1 + q_C(x))f(x)) = (1 + q_C(x) - q_{(C)}(f(x)))f(x)$$

$$= f(x).$$

Given a category $\mathcal{C}$ of star bodies, it is natural to look for a duality in this category, that is, a functor from $\mathcal{C}$ to itself being an involution (compare [11]). In what follows we shall restrict our consideration to star
bodies with positive radial function (i.e., with the origin in the interior). Let \( \text{St}_n^+ \) be the corresponding full subcategory of \( \text{St}_n^+ \).

Let \( i \) be the inversion of the one-point compactification \( \mathbb{R}^n \) of \( \mathbb{R}^n \), with respect to \( S^{n-1} \):

\[
i(x) := \frac{x}{\|x\|^2} \quad \text{for} \quad x \in \mathbb{R}^n \setminus \{0\}.
\]  

(3.1)

**Definition 3.2.** For every object \( A \) of \( \text{St}_n^+ \),

\[
A^o = \text{cl}(\mathbb{R}^n \setminus i(A));
\]

for every \( f \in \text{GS}(n) \),

\[
f^o := if(\mathbb{R}^n).
\]

Let us note the following.

**Proposition 3.3.** Let \( A \in \mathcal{S}_0^o \) and \( 0 \in \text{int} \ A \). Then \( A^o \in \mathcal{S}_0^o \) and for every \( u \in S^{n-1} \),

\[
\varrho(A^o, u) = \frac{1}{\varrho(A, u)}.
\]

**Proof.** By Definition 3.2, \( A^o \) is a star body and

\[
\varrho(A^o, u) = \|a\|
\]

where \( a \in \text{bd}(\mathbb{R}^n \setminus i(A)) \cap \text{pos} \ u \). Since \( \text{bd}(\mathbb{R}^n \setminus A) = i(\text{bd} A) \), it follows that \( i(a) \in \text{bd} A \cap \text{pos} \ u \), and thus

\[
\|i(a)\| = \varrho(A, u).
\]

By (3.1), this completes the proof. \( \blacksquare \)

**Theorem 3.4.** The pair of functions \( A \mapsto A^o, \ f \mapsto f^o \) is a covariant functor from \( \text{St}_n^+ \) to itself such that

\[
A^{o o} = A \quad \text{and} \quad f^{o o} = f
\]

(3.2)

for every \( A \) and \( f \).

Moreover,

\[
A \subset B \quad \Rightarrow \quad B^o \subset A^o.
\]

(3.3)

**Proof.** By Proposition 3.3, if \( A \) is a star body with 0 in the interior, then so is \( A^o \). Let \( f \in \text{GS}(n) \) and \( f(A) = B \). Then

\[
f^o \in \text{GS}(n),
\]

(3.4)
because $f^\circ$ is a homeomorphism of $\mathbb{R}^n$ and for every positive $\alpha$,

$$f^\circ(\alpha x) = if(i(\alpha x)) = f\left(\frac{1}{\alpha}i(x)\right) = \alpha if(i(x)) = \alpha f^\circ(x).$$

Moreover,

$$f^\circ(A^o) = B^o,$$  \hspace{1cm} (3.5)

because

$$f^\circ(A^o) = if(i(cl(\mathbb{R}^n \setminus i(A)))) = cl(\mathbb{R}^n \setminus if(A)) = (f(A))^\circ.$$  

By (3.4) and (3.5), we have a covariant functor.

Furthermore,

$$A^{oo} = cl(\mathbb{R}^n \setminus i(A^o)) = cl(\mathbb{R}^n \setminus i(cl(\mathbb{R}^n \setminus i(A))))$$

$$= R^n \setminus i(int cl(\mathbb{R}^n \setminus i(A))) = A,$$

since $f$ and $i$ are homeomorphisms, $0 \in int A$, and $i$ is an involution.

Evidently, $f^{oo} = f$. This proves (3.2). By similar arguments, (3.3) holds.

By Theorem 3.4, the pair of functions $A \mapsto A^o, f \mapsto f^\circ$ is a duality. We shall refer to this functor as the star duality.

For a convex body $A$ with the origin in the interior, there is a simple connection between polar dual $A^*$ and star dual $A^o$.

**Theorem 3.5.** If $A \in \mathcal{K}_0^n$ and $0 \in int A$, then

$$\tilde{V}_1(A^o) = \frac{\kappa_n}{2} \cdot b(A^*).$$

**Proof.** By Proposition 3.4 and Remark 1.7.7 in [12],

$$g(A^o, u) = h(A^*, u) \quad \text{for every } u \in S^{n-1}. \hspace{1cm} (3.6)$$

Hence, by the formula (1.7.2) in [12],

$$\tilde{V}_1(A^o) = \frac{1}{n} \int_{S^{n-1}} g(A^o, u) \, d\sigma(u) = \frac{1}{n} \int_{S^{n-1}} h(A^*, u) \, d\sigma(u)$$

$$= \frac{\kappa_n}{2} \cdot b(A^*).$$

Generally, star dual of a convex body is different from its polar dual; see Fig. 1.
THEOREM 3.6. Let \( n \geq 2 \). For every \( A \in \mathcal{H}_0^n \) with \( 0 \in \text{int} \ A \), the following are equivalent:

(i) \( A^o = A^* \);

(ii) \( A \) is a centered ball.

Proof. (ii) \( \Rightarrow \) (i) is obvious.

Assume (i). Then, by (3.6),

\[
\varrho(A^*, u) = h(A^*, u) \quad \text{for every } u \in S^{n-1}
\]

and thus

\[
\text{pos } u \cap \text{bd } A^* \subset H(A^*, u).
\]

As a consequence, \( \text{bd } A^* \) is of the class \( C^1 \).

Since \( E \cap A^o = (E \cap A)^o \) for every two-dimensional linear subspace \( E \), without any loss of generality we may assume that \( n = 2 \).

Let \( r : \mathbb{R} \to \text{bd } A^* \) be the parametrization of \( \text{bd } A^* \) in polar coordinates,

\[
r(t) = \varrho(t) \cdot u(t),
\]

where \( \varrho(t) = \varrho(A^*, u(t)) \) and \( u(t) = (\cos t, \sin t) \). Then \( \langle r(t), r'(t) \rangle = 0 \) for every \( t \); i.e., \( \varrho' = 0 \). Hence \( \varrho' = 0 \) and thus \( \varrho_{A^*} \) is constant, and so is \( \varrho_A \). This completes the proof. \( \blacksquare \)

The following statement is an analogue of the conjecture (7.4.33) in [12], concerning the polar duality of convex bodies.

THEOREM 3.7. Let \( A \) be a star body in \( \mathbb{R}^n \) with \( 0 \in \text{int } A \). For any \( p, q > 0 \) with \( 1/p + 1/q = 1 \),

\[
\widetilde{V}_p(A^o) \cdot \widetilde{V}_q(A)^{p-1} \geq \kappa^n_p.
\] \hspace{1cm} (3.7)
In particular,
\[ \tilde{V}_2(A^\circ) \cdot \tilde{V}_2(A) \geq \kappa_n^2. \] (3.8)

The equality in (3.7) or (3.8) holds if and only if \( A \) is a centered ball.

Proof. Since \( \varrho_{A^\circ} \cdot \varrho_A = 1 \) (see Proposition 3.3), by the Hölder inequality,
\[
n \kappa_n = \sigma(S_n) = \int_{S_n} \varrho_A \cdot \varrho_A \ d\sigma \\
\leq \left( \int_{S_n} (\varrho_A)^p \ d\sigma \right)^{1/p} \cdot \left( \int_{S_n} (\varrho_A)^q \ d\sigma \right)^{1/q} \\
= (nV_p(A^\circ))^{1/p} \cdot (n\tilde{V}_q(A))^1/q = nV_p(A^\circ)^{1/p} \cdot \tilde{V}_q(A)^{(p-1)/p}.
\]
This implies (3.7) (and 3.8).

The equality holds if and only if there exists \( \alpha > 0 \) such that \( \varrho_{A^\circ} = \alpha \varrho_A \).

This happens if and only if \( A \) is a centered ball. \[ \blacksquare \]

Finally, let us return to the notion of intersection body of order \( k \). It has been defined only for \( k > 0 \) (see [4, Note 8.3]), but its definition can be automatically extended over \( k < 0 \). The following statement is evident:

3.8. For every star body \( A \) with \( 0 \in \text{int } A \),
\[ I_{-k} A = I_k(A^\circ). \]

There is a relationship between star duality and the operation \( I_k \):

**Theorem 3.9.** Let \( |k| \in \{1, \ldots, n-1\} \), \( A \in \mathcal{S}_0^n \), and \( 0 \in \text{int } A \). Then
\[ \kappa_{n-1}^2 \cdot (I_k A)^\circ \subset I_k(A^\circ) \] (3.9)
and equality holds if and only if \( A \) is a centered ball.

Proof. Let \( X := I_k A \) and \( Y := I_k(A^\circ) \). Then (3.9) is equivalent to the condition
\[ \kappa_{n-1}^2 \varrho(X^\circ, u) \leq \varrho(Y, u) \quad \text{for every } u \in S_n^{n-1}, \]
which, in view of Proposition 3.3, can be rewritten as
\[ \kappa_{n-1}^2 \leq \varrho_X \cdot \varrho_Y. \] (3.10)
Using the definition of $I_k$, the Hölder inequality, and Proposition 3.3, we obtain

$$
\varrho_X(u) \cdot \varrho_Y(u) = \frac{1}{(n-1)^2} \int_{S^{n-1} \cap u^\perp} \varrho_A(v)^k \, d\sigma(v) \cdot \int_{S^{n-1} \cap u^\perp} \varrho_B(v)^k \, d\sigma(v)
$$

$$
\geq \frac{1}{(n-1)^2} \left( \int_{S^{n-1} \cap u^\perp} 1 \, d\sigma(v) \right)^2 = \kappa_{n-1}^2.
$$

Thus (3.10) is satisfied.

The equality in (3.9) holds if and only if it holds in the Hölder inequality for the functions $(\varrho_A)^{k/2}$ and $(\varrho_B)^{k/2}$; hence, by Proposition 3.3, it holds if and only if $\varrho_A$ is constant. This completes the proof.

Let us observe that, in view of Proposition 3.3, if $I_k$ is replaced by the proportional function $J_k := (\kappa_{n-1})^{-1}I_k$, then condition (3.9) can be expressed in the more elegant form

$$
(J_k A) \circ J_k (A^\circ).
$$

4. FINAL REMARKS

A. The Classification Problem

To every $A \in \mathcal{S}_0^n$ we assign the subset $S_A$ of the unit sphere:

$$
S_A = \{ u \in S^{n-1} | \varrho_A(u) > 0 \}.
$$

Theorem 4.1. Let $A_1, A_2 \in \mathcal{S}_0^n$. If there exists $f \in G S(n)$ such that $f(A_1) = A_2$, then $S_{A_1}$ is homeomorphic to $S_{A_2}$.

Proof. If $S_{A_1} = S^{n-1}$, then also $S_{A_2} = S^{n-1}$ and the assertion is trivial. Let $S^{n-1} \setminus S_{A_1} \neq \emptyset$ and let $g_i: S_{A_i} \to \text{bd} A_i \setminus \{0\}$ be the central projection for $i = 1, 2$:

$$
g_i(u) = \varrho_{A_i}(u) \cdot u.
$$

Clearly, $g_1$ and $g_2$ are homeomorphisms. Thus, setting

$$
h(u) = (g_2)^{-1}g_1(u) \quad \text{for} \ u \in S_{A_1},
$$

we obtain a homeomorphism $h: S_{A_1} \to S_{A_2}$, as required. ☐
The following problem is open.

**Problem 1.** Is the existence of a homeomorphism of \( S_{A_1} \) onto \( S_{A_2} \) sufficient for \( A_1, A_2 \) to be star equivalent?

### B. Topological Properties of Star Bodies

While, evidently, star bodies in the sense of [12] are homeomorphic to \( B^n \), the elements of \( S_0^n \) are retracts of \( B^n \) and thus they are absolute retracts (see [1]):

**Proposition 4.2.** If \( A \in S_0^n \), then \( A \) is an absolute retract.

**Proof.** Of course, without any loss of generality we may assume that \( A \subset B^n \). Let us define \( r: B^n \to A \) by the formula

\[
r(x) = \begin{cases} 
  xq_A(x), & \text{if } x \in B^n \setminus A, \\
  x, & \text{if } x \in A.
\end{cases}
\]

Then \( r \) is continuous, because

\[
xq_A(x) = x \quad \text{for } x \in \text{bd } A \setminus \{0\},
\]

and

\[
xq_A(x) = uq_A(u) \quad \text{for } u = \frac{x}{\|x\|},
\]

where \( q_A(u) \to 0 \) if \( x \to 0 \in \text{bd } A \).

Thus, \( r \) is a retraction, since \( r|A = \text{id} \). This completes the proof.

### C. Convex Quotient Bodies

It is easy to see that, for convex bodies \( A, C \) with \( 0 \in C \subset A \), the quotient star body \( A/C \) generally need not be convex, but sometimes is convex; for instance, homothetic convex bodies have convex quotient bodies:

**Proposition 4.3.** Let \( A, C \in S_0^n \) and \( C \subset A \). If \( A = \lambda C \) for some \( \lambda > 1 \), then \( A/C \in S_0^n \).

**Proof.** Let \( f(x) = \lambda x \) for every \( x \in \mathbb{R}^n \). Then \( f \in GS(n) \); thus, by Proposition 1.2,

\[
q_A(\lambda x) = q_{f,C}(f(x)) = q_C(x)
\]

for every \( x \neq 0 \). Since \( q_A \) is homogeneous of degree \( -1 \), it follows that

\[
(1/\lambda)q_A(\lambda x) = q_C(x).
\]

This, together with Proposition 2.3, implies

\[
q_{A/C}(x) = (\lambda - 1)q_C(x).
\]
Thus, $\varphi_{A/C} = \varphi_{(A-1)C}$ and consequently $A/C = (\lambda - 1)C$. Hence $A/C \in \mathcal{K}_0^\ast$.

**Problem 2.** What can be said about the pairs $(A, C)$ in $\mathcal{K}_0^\ast$ with convex $A/C$?

A contribution to this problem will be the subject of a separate paper, by Tomasz Żukowski.

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