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Factoring cardinal product graphs in polynomial time

Wilfried Imrich *

Department of Mathematics and Applied Geometry, Montanuniversität Leoben, A-8700 Leoben, Austria

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Abstract

In this paper a polynomial algorithm for the prime factorization of finite, connected nonbipartite graphs with respect to the cardinal product is presented. This algorithm also decomposes finite, connected graphs into their prime factors with respect to the strong product and provides the basis for a new proof of the uniqueness of the prime factorization of finite, connected nonbipartite graphs with respect to the cardinal product. Furthermore, some of the consequences of these results and several open problems are discussed. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

In [19] Ralph McKenzie investigated the cardinal product of relational structures, both finite and infinite. One of his results is that finite, nonbipartite connected graphs have unique prime factor decompositions (UPFD) with respect to the cardinal product in the class of undirected graphs with loops. This implies that finite, simple connected graphs have UPFD with respect to the strong product, a result which had independently also been obtained by Dörfler and Imrich [4].

Later Feigenbaum and Schäffer [7] presented a polynomial algorithm for the decomposition of connected simple graphs into their prime factors with respect to the strong product. This algorithm reduces the given graph to a thin one, finds certain Cartesian edges that form a product subgraph with respect to Cartesian multiplication and uses this information to construct the UPFD of the original graph with respect to the strong product.

The aim of this paper is a polynomial algorithm for the prime factorization of finite, connected nonbipartite graphs with respect to the cardinal product. As a special case this algorithm also allows to decompose finite, connected graphs with respect to the strong product of graphs. Moreover, it provides the basis for a new proof of the

* E-mail: imrich@unileoben.ac.at.

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uniqueness of the prime factor decomposition of finite, connected nonbipartite graphs with respect to the cardianl product. We also begin by reducing the given graph to a thin graph G, which we assume to be finite, nonbipartite and connected. Then an adaption of the marking algorithm of Feigenbaum and Schäffer [7] is used to produce a set of Cartesian pairs of vertices of G, not edges as in their case. These pairs form a connected graph which we call the Cartesian skeleton H of G. We show that it is invariant under automorphisms of G and that its prime factorization with respect to the Cartesian product is compatible (in a sense to be defined later) with any decomposition of G with respect to the cardinal product. This fact is then used to show that any two cardinal product decompositions of G have a common refinement, which implies unique prime factor decompositions of finite, nonbipartite, connected thin graphs with respect to the cardinal product. Moreover, this UPFD of G can then be found in polynomial time from the UPFD of H with respect to the Cartesian product.

Sections on the extraction of complete factors and on finding decompositions of graphs from the decomposition of their thin reductions complete the proofs for the uniqueness of the prime factorization and that it can be found in polynomial time.

The paper also includes remarks about the nonunicity of cardinal product decompositions and the relationship of the automorphism group of a graph with those of its factors. Several open problems are collected at the end.

2. Preliminaries

In this paper finite undirected graphs with or without loops and without multiple edges are considered. We shall denote this class of graphs by Γ_0 and the subclass of Γ_0 which contains only graphs without loops by Γ. The class Γ is also called the class of *simple graphs*.

If G is a graph, we shall write $V(G)$ or V for its vertex set and $E(G)$ or E for its edge set. $E(G)$ shall be considered as a set of unordered pairs $[x, y]$ of vertices of G. Frequently we shall also use the notation xy instead of $[x, y]$. If $[x,x]$ is an edge, we say x carries a loop. Considering G as $V(G) \cup E(G)$, we may sometimes write $x \in G$ for $x \in V(G)$ and $e \in G$ for $e \in E(G)$.

Although we shall only be concerned with finite graphs, we continue with a general definition of the cardinal product. It is valid for arbitrarily many, also infinitely many, factors.

Let G_ι, $\iota \in I$, be a set of graphs. Then the *cardinal product* $G = \prod_{\iota \in I} G_\iota$ is defined as follows:

(i) $V(G)$ is the Cartesian product of the vertex sets of the factors. In other words, $V(G)$ is the set of functions $x : \iota \mapsto x_\iota \in V(G_\iota)$ of I into $\bigcup_{\iota \in I} V(G_\iota)$.

(ii) $E(G)$ consists of all unordered pairs $[x, y]$ of distinct vertices of G such that $[x_\iota, y_\iota] \in E(G_\iota)$ for all $\iota \in I$.

The cardinal product is commutative and associative in an obvious way, having the one vertex graph with a loop as a unit. For two factors G and H we denote it by $G \times H$.

If $x \in \prod_{\iota \in I} G_\iota$ we call the x_ι the *coordinates* of x and note that every edge in a cardinal product of k graphs without loops connects endpoints that differ in all k coordinates.

It has first been shown by Weichsel [26], but can easily be shown directly, that the cardinal product of two graphs is connected if and only if both factors are connected and if at least one is not bipartite. To illustrate this, let P_n denote the *path of length* n, i.e. a path with n edges and $n+1$ vertices, and C_n the *cycle* on n vertices. Then $P_1 \times P_1$ consists of two disjoint edges, $P_1 \times C_{2k}$ of two copies of C_{2k} and $P_1 \times C_{2k+1}$ is $C_{2(2k+1)}$.

Note that a simple graph G is connected and nonbipartite if and only if its square is connected. This follows immediately from the definition of the *square* G^s of a graph G as a graph defined on $V(G)$ with the edge set

$$E(G^s) = \{xy \mid \exists z \text{ such that } xz \in E(G) \text{ and } zy \in E(G)\}.$$

Thus, K_n^s stands for the complete graph with loops at every vertex, unless $n = 1$. However, it will be a convenient abuse of language to use the notation K_1^s for the one vertex graph with a loop.

A graph is called *totally disconnected* if it has no edges (and thus also no loops). Clearly the cardinal product of totally disconnected graphs is totally disconnected.

We say two vertices $x, y \in V(G)$ are in relation R on $V(G)$ if every vertex z is either adjacent to both vertices x and y or to neither one of them, i.e. if $xz \in E(G)$ if and only if $yz \in E(G)$. Clearly R is an equivalence relation. Its equivalence classes span complete graphs with loops at every vertex or completely disconnected graphs. To indicate that R is defined on G we shall also use the notation R_G.

We call a graph G *R-thin* if R is the identity relation, i.e. if every vertex is an equivalence class with respect to R.

It is clear what is meant by G/R, but will nevertheless be treated in detail in Section 8. It is then easily seen that G/R is thin. Moreover, as Proposition 6 shows,

$$(G \times H)/R = (G/R) \times (H/R).$$

The case in which every vertex of the factors carries a loop deserves special attention, because it gives rise to the *strong product* of graphs in Γ. For $X \in \Gamma$ let $\mathscr{L}X$ be formed from X by adding a loop to every vertex of X. Conversely, for every $X \in \Gamma_0$ we let $\mathscr{N}X$ denote the graph formed from X by removing all loops. Then the strong product $Z = X \boxtimes Y$ of two graphs X and Y in Γ is defined by

$$X \boxtimes Y = \mathscr{N}(\mathscr{L}X \times \mathscr{L}Y).$$

For graphs in Γ, i.e. for simple graphs, it is then convenient to replace the relation R by a relation S defined as follows: Two vertices x, y of a simple graph X are in relation S, if they are equal or if they are adjacent and if every vertex z adjacent to x is also adjacent to y. For $X \in \Gamma$ we have $R_{\mathscr{L}X} = S_X$ and $(\mathscr{L}X)/R = \mathscr{L}(X/S)$. Furthermore,

$$Z/S = X/S \boxtimes Y/S.$$

We thus call a simple graph X *S-thin* if X/S is trivial.

Lemma 1. *Let G be a graph with n vertices and m edges. Then the complexity of finding R or S (if G is in Γ) is at most $O(n^3)$. This is also the complexity of checking whether a graph is thin and of constructing G/R (resp. G/S).*

Proof. Let $\{a,b\}$ be any pair of vertices in G. In order to check whether aRb (resp. aSb) holds it suffices to check the adjacency of every vertex x in G with a and b. This can be done in a total of $O(n^3)$ steps over all pairs of vertices.

Finding the equivalence classes of R (resp. S) in $V(G)$ amounts to finding the connected components of a graph on $V(G)$ whose edges are the pairs a,b of vertices with aRb (resp. aSb). As the complexity of finding connected components of a graph is no more than the number of edges, this can clearly be done in at most $O(n^2)$ steps.

If all classes consist of only one vertex, then G is thin.

For the construction of G/R (resp. G/S) the adjacencies of the equivalence classes with respect to R (resp. S) have to be checked. Clearly this can be done in $O(m)$ steps. □

We now introduce the Cartesian product. Let G,H be simple graphs. Then the *Cartesian product* $G \Box H$ is defined by

$$V(G \Box H) = V(G \times H),$$
$$E(G \Box H) = E(G \boxtimes H) \setminus E(G \times H).$$

Both the strong and the Cartesian product are commutative, associative and have K_1 as a unit. The strong and the Cartesian product of two graphs are connected if and only if both factors are connected. (For the strong product this also holds for the product of infinitely many factors.)

Examples of Cartesian products abound. The four-cycle is $K_2 \Box K_2$ and the cube $K_2 \Box K_2 \Box K_2$. More generally, the *n-dimensional hypercube* is the nth power of K_2 with respect to the Cartesian product. These graphs are also known as *binary Hamming graphs*, whereas Cartesian products of complete graphs are known as *Hamming graphs* [15]. Also, the *n-dimensional integer lattice* is the Cartesian product of n copies of the two-sided infinite path.

In the next section we shall summarize some of the known results about prime factor decompositions of graphs with respect to the three products introduced so far.

3. Prime factor decompositions

A graph is called *prime* with respect to any of the above products if it cannot be written as a product of two nontrivial graphs, i.e. of two graphs with at least two vertices each. Clearly any finite graph can be represented as a product of prime graphs. If two presentations of a graph G as a product of prime graphs are the same up to isomorphisms and the order of the factors, we say that G has unique prime factor decomposition (UPFD). For any of the three products considered there are graphs without UPFD. To see this, denote the disjoint union of graphs by $+$ and, for the time being, the n-th power of a graph with respect to the Cartesian product by G^n. Then it is not hard to see that the identity

$$(K_1 + K_2 + K_2^2)\Box(K_1 + K_2^3) = (K_1 + K_2^2 + K_2^4)\Box(K_1 + K_2)$$

holds and that both sides of the identity are products of prime graphs, no two of which are isomorphic. If one replaces $\Box$ by $\boxtimes$ and lets K_2^n denote powers with respect to the strong product the identity remains valid.

Clearly, this is a consequence of the fact that the subalgebra $\boldsymbol{Z}^+[x]$ composed of nonzero polynomials with positive coefficients in the polynomial ring $\boldsymbol{Z}[x]$ with integer coefficients over the indeterminate x does not have the unique prime factorization property. It seems to have first been exploited for the construction of finite reflexive structures without unique prime factor decomposition by Hashimoto and Nakayama [9].

If we are looking for a counterexample to the UPFD with respect to the cardinal product in Γ_0 we may replace $\Box$ by $\times$ and K_i by $\mathscr{L}K_i$ in the identity above and let $(\mathscr{L}K_2)^n$ denote the n-th power of $\mathscr{L}K_2$ with respect to the cardinal product.

This example does not work in the class of simple graphs, because K_1 is not a unit. However, there are counterexamples even in the class of connected simple graphs. Following an approach of Miller [20], let K denote K_2 with a loop added to one vertex. Then we have

$$K_3 \times (K_2 \times K) = K_2 \times (K_3 \times K). \tag{1}$$

Moreover, note that $P_3 = K_2 \times K$ and set $G = K_3 \times K$. Then Eq. (1) gives rise to

$$K_3 \times P_3 = K_2 \times G. \tag{2}$$

It is not hard to see that all factors in equation (2) are prime with respect to the cardinal product in the class of simple graphs and that we have thus found two distinct prime factorizations of a graph with respect to the cardinal product in Γ. Furthermore, $K_3 \times P_3$ is connected, whereas all the other graphs with distinct prime factorizations which we considered were disconnected.

This is explained by the fact that connected finite graphs have unique prime factorizations in the class of simple graphs with respect to the Cartesian and with respect to the strong product. Sabidussi [22] and Vizing [25] have independently shown this for the Cartesian product and McKenzie [19] and Dörfler and Imrich [4] for the strong one. In fact, McKenzie's result for the strong product is a special case of his investigation [19] of relational structures which imply that finite graphs in Γ_0 with connected square have UPFDs in Γ_0.

We also wish to mention that the first polynomial algorithm for the prime factorization of finite, connected graphs with respect to the Cartesian product was published by Feigenbaum et al. [6] and recall that two of these authors, Feigenbaum and Schäffer [7], have published the first polynomial algorithm for the prime factorization of finite connected graphs with respect to the strong product [7].

We continue with a more precise description of these results. For the Cartesian product, let $G = G_1\Box\cdots\Box G_k$. We consider the vertices x of G as k-tuples $(x_1,x_2,\ldots,x_k)$ where x_i is the ith coordinate or the *projection* of x into the ith factor under the mapping $p_i : G \to G_i$ defined by $p_i(x) = x_i$. Let $a \in V(G)$. Then the G_i*-copy* G_i^a is the subgraph of G induced by the vertex set $\{x \mid x_j = a_j \text{ for all } j \neq i\}$. It is easy to see that every G_i-copy is isomorphic to G_i and that every edge of G is in some G_i-copy. For brevity we may sometimes simply speak of i-copies.

These copies are of utmost importance since they are preserved by the automorphisms of G in case the G_i are connected prime graphs. Here an automorphism of G is a permutation ϕ of $V(G)$ that preserves adjacencies, i.e. $[x,y] \in E(G)$ if and only if $[\phi x, \phi y] \in E(G)$. The automorphisms form a group which we denote by Aut(G). As has been shown by Miller [21] and independently by Imrich [11] they can be characterized as follows:

Proposition 1. *Let ϕ be an automorphism of a finite connected graph G with prime factor decomposition $G_1\Box\cdots\Box G_k$ with respect to the Cartesian product. Then there exists a permutation π of $\{1,2,\ldots,k\}$ together with isomorphisms $\psi_i : G_i \to G_{\pi i}$ such that*

$$\phi(x_1,x_2,\ldots,x_k) = (\psi_{\pi^{-1}1}x_{\pi^{-1}1},\ \psi_{\pi^{-1}2}x_{\pi^{-1}2},\ \ldots,\ \psi_{\pi^{-1}k}x_{\pi^{-1}k}).$$

In other words, all G_i-copies are mapped into $G_{\pi i}$-copies. We say copies with respect to prime factors of connected finite graphs G are preserved by automorphisms of G.

In case π is the identity permutation, every ψ_i is an automorphism of G_i and we say ϕ is generated by automorphisms of the factors G_i. If all factors are pairwise nonisomorphic, these automorphisms already generate the full automorphism group of G. In case at least two prime factors G_r and G_s are isomorphic, let π be the transposition (r,s), ψ_r be an isomorphism of G_r onto G_s, ψ_s an isomorphism of G_s onto G_r and all the other ψ_i be the identity. Then we say the automorphism ϕ corresponding to the just defined π and ψ_i is a transposition of two isomorphic prime factors of G. Thus, the automorphisms group Aut(G) of a connected finite graph G with prime factor decomposition $G_1\Box\cdots\Box G_k$ with respect to the Cartesian product is generated by automorphisms and transpositions of the prime factors.

In case of the strong product $G = G_1 \boxtimes \cdots \boxtimes G_k$ we can define i-copies as before. Again they are isomorphic to the i-th factor. However, not every edge will be in an i-copy. Following a notation introduced in [7] we call edges in i-copies *Cartesian* (with respect to the given presentation) and the others *non-Cartesian*. Let $G = G_1 \boxtimes \cdots \boxtimes G_k$ be the prime factor decomposition of a finite connected graph G with respect to the strong product. Then every automorphism of G can be presented as in case of the Cartesian product if G is S-thin [4,7,19]. In other words, in this case every automorphism preserves Cartesian and non-Cartesian edges. It is a special case of the results about the cardinal product.

Let $G = G_1 \times \cdots \times G_k$ be a cardinal product. Again we can define i-copies, but the G_i^a are completely disconnected unless $p_j a$ carries a loop in G_j for all $j \neq i$. In this case though, the G_i^a are again isomorphic to G_i. In spite of these differences, the automorphisms of a *finite, connected, nonbipartite thin graph* G with prime factor decomposition

$$G = G_1 \times \cdots \times G_k$$

in Γ_0 can still be presented as in case of the Cartesian product. This is a consequence of the results of McKenzie [19] which can be found in Dörfler [3] and will also be proved here.

In the above cases we have a situation in which any representation of the given graph as a product of prime graphs leads to a coordinatization in which the number of coordinates is independent of the representation and where the number of coordinates in which two vertices x, y differ is also independent of the representation. We say that the *coordinatization is unique*.

4. Principal result and straightforward consequences

The main result of the paper is that nonbipartite, connected graphs can be decomposed into their prime factors with respect to the cardinal product in polynomial time. As a by-product we obtain a new proof for the uniqueness of the prime factorization of such graphs in Γ_0.

Theorem 1. *Let G be a finite, nonbipartite connected graph in Γ_0. Then G has unique prime factor decomposition with respect to the cardinal product in Γ_0. This decomposition can be determined in polynomial time in the number of vertices of G.*

This implies that nonbipartite, connected graphs in Γ have unique prime factorizations in Γ_0, whereas the factorization need not be unique in Γ as we have seen. However, with the observation that $G \times H \notin \Gamma$ if and only if both G and H are in $\Gamma_0 \setminus \Gamma$ it is not too hard to verify the following corollaries (for details see [3]):

Corollary 1. *Let $G = Q_1 \times Q_2 \times \cdots \times Q_k$ be the prime factor decomposition in Γ_0 of a nonbipartite, connected graph $G \in \Gamma$. Furthermore, let the graphs G_i be defined by $G_i = \prod_{j \in I_i} Q_j$, where the sets $I_1, I_2, \ldots, I_r$ form a partition of the index set $\{1, 2, \ldots k\}$. Then $G_1 \times G_2 \times \cdots \times G_r$ is a prime factor decomposition of G in Γ if and only if every set $\{Q_j \mid j \in I_i\}$ contains exactly one element in Γ.*

It is not difficult to find sequences of arbitrarily large nonbipartite, connected simple graphs G_k for which the number of prime factorizations in Γ is not bounded by a polynomial in $|V(G_k)|$. The next corollary shows that even the complexity of deciding whether a nonbipartite, connected simple graph has unique prime factorization in Γ is the same as that of isomorphism testing of graphs and thus most likely not polynomial.

Corollary 2. *Let $G = Q_1 \times Q_2 \times \cdots \times Q_k$ be the prime factor decomposition in Γ_0 of the nonbipartite, connected graph $G \in \Gamma$. Then G has unique prime factor decomposition with respect to the cardinal product in Γ if and only if one of the following conditions is satisfied:*

(i) *All Q_i, $i = 1, \ldots, k$, are in Γ.*
(ii) *Only one of the Q_i, $i = 1, \ldots, k$, is in $\Gamma_0 \setminus \Gamma$ and all the other factors are in Γ and pairwise isomorphic.*

The unique factorization property of nonbipartite, connected graphs in the class Γ_0 immediately implies a cancellation property: If $A \times B \cong A \times C$ and if A, B are both nonbipartite and connected, then $B \cong C$. Of course, this remains valid for nonbipartite, connected graphs in Γ.

If one restricts attention to connected graphs with loops at every vertex, then Theorem 1 yields the known result that finite, connected graphs have unique prime factor decompositions with respect to the strong product [4,19] and that this decomposition can be found in polynomial time [7].

Theorem 2. *Every finite, connected graph has a unique prime factor decomposition with respect to the strong product. This decomposition can be found in polynomial time.*

The assertions made in the preceding paragraph about automorphisms of R-thin (and S-thin) structures G are immediate consequences of three main properties of the Cartesian skeleton H, as defined in Section 7. These properties are (1) the invariance of H under automorphisms of G, (2) that the copies of the prime factors of G have the same vertex sets as the copies of certain, well defined factors of H and (3) of the structure of the automorphism groups of Cartesian products. One can therefore describe without difficulty the relationship between the structure of the automorphism group of a cardinal product and the structure of the groups of the factors.

For the Cartesian product this has first been done in [10] and has been extended to the cardinal product by Dörfler [3].

Below we list these results for the cardinal product. It is clear, how they apply to the strong one. We shall not use or refer to them in the sequel and assume that the reader is familiar with the notions used, but emphasize that we consider the automorphism group of graph G as a permutation group on the vertex set $V(G)$. Thus, the concepts of transitivity, regularity and primitivity are defined as for permutation groups.

Proposition 2. *The automorphism group of a finite, nonbipartite connected graph in Γ_0 is transitive if and only if the group of every prime factor of this graph is transitive.*

Proposition 2 remains valid for disconnected, nonbipartite graphs.

Proposition 3. *Let G be a finite, nonbipartite connected graph. Then the automorphism group of $G \in \Gamma_0$ is regular if and only if the prime factors of G are pairwise nonisomorphic and have regular groups.*

Proposition 4. *Let G be a finite, nonbipartite connected graph with prime factor decomposition $G_1 \times G_2 \times \cdots \times G_k$. Then $\mathrm{Aut}(G)$ is abelian if and only if every $\mathrm{Aut}(G_i)$ is abelian and if one of the following conditions holds:*

(i) *If G is R-thin, then all G_i with nontrivial group are pairwise nonisomorphic and to every factor G_r with trivial group there is at most one $G_s \cong G_r$, $r \neq s$.*

(ii) *If G is not R-thin, then all prime factors, except exactly one, are R-thin, pairwise nonisomorphic and have trivial groups.*

Proposition 5. *Let G be a finite, nonbipartite connected graph with primitive automorphism group. Then G is a complete graph with loops at every vertex or the power of a prime graph with primitive group.*

5. Cartesian pairs of vertices

Let G be a nonbipartite, connected R-thin graph which is a nontrivial cardinal product $G_1 \times G_2$. We call a pair $\{(x_1,x_2),(y_1,y_2)\}$ of vertices *Cartesian* with respect to the decomposition $G_1 \times G_2$, if either $x_1 = y_1$ or $x_2 = y_2$. In analogy to the approach followed by Feigenbaum and Schäffer [7] for the strong product we wish to identify as many pairs of vertices as possible which are Cartesian with respect to any decomposition of G as a cardinal product.

The starting point of our investigations are neighborhoods. For $x \in V(G)$, the *neighborhood* $N(x)$ of x is defined by

$$N(x) = \{y \mid y \in V(G),\ [x,y] \in E(G)\}.$$

If $G \in \Gamma$, then $N(x)$ is also called the *open neighborhood* of x, contrary to the closed one (often denoted by $\overline{N}(x)$, $\overline{N}(x) = N(x) \cup \{x\}$), which also contains x. For $G \in \Gamma_0$

it should be noted that $N(x)$ contains x whenever x carries a loop. In case G is a cardinal product $G = G_1 \times G_2$ we infer directly from the definition of the product that

$$N(x) = N_1(x_1) \times N_2(x_2),$$

where $N_i(x)$ denotes the neighborhood of x_i in G_i. Most importantly, we observe that this holds for any decomposition of G as a cardinal product.

We shall take care to select Cartesian pairs in $G = G_1 \times G_2$ such that the graph H they span is a connected graph with the same vertex set as G and that it has a representation $H = H_1 \Box H_2$ as a Cartesian product that is compatible with the cardinal product $G_1 \times G_2$ in the sense that the H_i-copies of H and the G_i-copies of G induce the same partitions of the the vertex sets $V(H) = V(G)$. To be more precise, for every $a \in V(G)$ and $i \in \{1,2\}$ we wish to have $V(H_i^a) = V(G_i^a)$.

We thus call a set F of pairs of distinct vertices of G *copy consistent* with respect to a decomposition $G_1 \times G_2$ of G, if F contains only Cartesian pairs and if for every pair $\{(u_1,u_2),(v_1,v_2)\}$ in F with $u_1 = v_1$ all pairs $\{(x_1,u_2),(x_1,v_2)\}$ for $x_1 \in V(G_1)$ are in F and, if $u_2 = v_2$, then $\{(u_1,x_2),(v_1,x_2)\} \in F$ for $x_2 \in V(G_2)$. The pairs $\{(x_1,u_2),(x_1,v_2)\}$, resp. $\{(u_1,x_2),(v_1,x_2)\}$, are called *copies* of $\{(u_1,u_2),(v_1,v_2)\}$.

For such a set F the vertex sets of the connected components of the graph H defined by $V(H) = V(G)$ and $E(H) = F$ span subproducts of $G_1 \times G_2$. We shall denote the vertex set of the connected component of H which contains x by $P(x)$ and the partition of $V(G)$ induced by the $P(x)$ will be denoted by $\mathscr{P}$.

Lemma 2. *Let G be a connected, nontrivial cardinal product $G = G_1 \times G_2$ of R-thin graphs, F be a copy consistent set of Cartesian pairs of vertices of G, and H be the graph with $V(H) = V(G)$ and $E(H) = F$. For every $x \in V(G)$ let $Q(x) = \{y \mid N(y) \subset N(x)\}$ and $P(x)$ denote the set of vertices in the connected component of H containing x. Furthermore, define*

$$\mathscr{I}(x) = \{N(y) \mid y \in Q(x) \setminus P(x)\}$$

and mark every pair $\{x,y\}$ if $N(y)$ is maximal in $\mathscr{I}(x)$. Then all marked pairs are Cartesian and satisfy the copy consistency property.

Proof. We show first that all marked pairs are Cartesian. Let $N(y)$ be maximal in $\mathscr{I}(x)$. Set $x = (x_1,x_2)$, $y = (y_1,y_2)$ and suppose that $\{x,y\}$ is not Cartesian. Then $x_1 \neq y_1$ and $x_2 \neq y_2$. Consider $y' = (y_1,x_2)$ and $y'' = (x_1,y_2)$.

If y' and y'' are in $P(x)$, then $y \in P(x)$ by the copy consistency of H. We can therefore assume without loss of generality that $y' \notin P(x)$. Since $N_1(y_1) \times N_2(y_2) \subset N_1(x_1) \times N_2(x_2)$ we have $N_1(y_1) \subseteq N_1(x_1)$ and, thus, $N_1(y_1) \subset N_1(x_1)$, since G_1 is thin. But then

$$N(y') = N_1(y_1) \times N_2(x_2) \subset N_1(x_1) \times N_2(x_2) = N(x)$$

and $y' \in Q(x) \setminus P(x)$. Therefore, $N(y') \subseteq N(y)$. As G is thin, we even have $N(y') \subset N(y)$ or, equivalently,

$$N_1(y_1) \times N_2(x_2) \subset N_1(y_1) \times N_2(y_2).$$

But then $N_2(x_2) \subset N_2(y_2)$, contrary to $y \in Q(x)$.

For the proof of the copy consistency assume now that the pair $\{u, v\}$ is marked, where u plays the role of x. We may assume without loss of generality that $\{u, v\}$ lies in a copy of G_1. Then $u = (u_1, u_2)$, $v = (v_1, u_2)$ and $N_1(v_1)$ must be strictly maximal in $\{N_1(w_1) \mid (w_1, u_2) \in Q(u) \setminus P(u)\}$.

Consider another copy $\{u', v'\}$ of this pair. Then $u' = (u_1, u_2')$ and $v' = (v_1, u_2')$. Can any vertex $v'' = (v_1'', v_2'')$ in $Q(u') \setminus P(u')$ prevent the pair $\{u', v'\}$ from being marked? If this were the case, this vertex v'' would have to satisfy $N(v') \subset N(v'') \subset N(u')$, or, equivalently, $N_1(v_1) \times N_2(u_2') \subset N_1(v_1'') \times N_2(v_2'') \subset N_1(u_1) \times N_2(u_2')$. But this is only possible if $N_2(v_2'') = N_2(u_2')$, whence $v_2'' = u_2'$ since G_2 is thin and $N_1(v_1) \subset N_1(v_1'') \subset N_1(u_1)$, contrary to the maximality of $N_1(v_1)$, unless $(v_1'', u_2) \in P(u)$. But then $v'' = (v_1'', u_2') \in P(u')$ by the copy consistency of H. □

Lemma 3. *Let G be a connected, nontrivial cardinal product $G_1 \times G_2$ of R-thin graphs, F be a copy consistent set of Cartesian pairs of vertices of G which is closed under applications of Lemma 2, and H be the graph with $V(H) = V(G)$ and $E(H) = F$. For every $x \in V(G)$ let, as in Lemma 2, $P(x)$ denote the set of vertices in the connected component of H containing x. Furthermore, set $I(x, y) = N(x) \cap N(y)$ and*

$$\mathscr{I}(x) = \{I(x, y) \mid y \notin P(x), I(x, y) \neq \emptyset\}.$$

Mark every pair $\{x, y\}$ if

(i) $I(x, y)$ is strictly maximal in $\mathscr{I}(x)$ or, if

(ii) $I(x, y)$ is nonstrictly maximal in $\mathscr{I}(x)$ and $N(z) \not\subset N(y)$ for all $z \notin P(x)$ with $I(x, z) = I(x, y)$.

Then all marked pairs are Cartesian and satisfy the copy consistency property.

Proof. We show first that all marked pairs are Cartesian. Let $I(x, y)$ be strictly or nonstrictly maximal in $\mathscr{I}(x)$. Set $x = (x_1, x_2)$, $y = (y_1, y_2)$ and suppose that $\{x, y\}$ is not Cartesian. Then $x_1 \neq y_1$ and $x_2 \neq y_2$. Consider $y' = (y_1, x_2)$ and $y'' = (x_1, y_2)$.

If y' and y'' are in $P(x)$, then $y \in P(x)$ by the copy consistency of H. We can therefore assume without loss of generality that $y' \notin P(x)$. Thus,

$$I(x, y') \subseteq I(x, y)$$

or, equivalently,

$$(N_1(x_1) \times N_2(x_2)) \cap (N_1(y_1) \times N_2(x_2)) \subseteq (N_1(x_1) \times N_2(x_2)) \cap (N_1(y_1) \times N_2(y_2)).$$

Hence,

$$(N_1(x_1)\cap N_1(y_1))\times N_2(x_2)\subseteq(N_1(x_1)\cap N_1(y_1))\times(N_2(x_2)\cap N_2(y_2))$$
$$\subseteq(N_1(x_1)\cap N_1(y_1))\times N_2(x_2).$$

Since $N_1(x_1)\cap N_1(y_1)$ must be nonempty this is only possible if equality holds in the above inequalities. In particular,

$$I(x,y')=I(x,y)$$

and $N_2(x_2)=N_2(x_2)\cap N_2(y_2)$. But then $N_2(x_2)\subseteq N_2(y_2)$ and, since G is thin,

$$N_2(x_2)\subset N_2(y_2).$$

Thus,

$$N(y')=N_1(y_1)\times N_2(x_2)\subset N_1(y_1)\times N_2(y_2)=N(y),$$

contrary to $N(z)\not\subset N(y)$ for all $z\notin P(x)$ with $I(x,z)=I(x,y)$.

Before we continue with the copy consistency property we wish to remark that $Q(x)$, as defined in Lemma 2, must be empty for every vertex x in G when H is closed under applications of Lemma 2.

To show that the set of marked pairs satisfies the copy consistency property, let $\{u,v\}$ be marked and let $\{u',v'\}$ be a copy of $\{u,v\}$. Clearly $v'\notin P(u')$ by the copy consistency property for H. Without loss of generality we can assume $u=(u_1,u_2)$, $v=(v_1,u_2)$ and $u'=(u_1,u_2')$, $v'=(v_1,u_2')$.

Suppose there is a $v''=(v_1'',v_2'')\notin P(u')$ such that $I(u',v')\subset I(u',v'')$ or $I(u',v')=I(u',v'')$ and there is no z with $I(u',v'')=I(u',z)$ and $N(z)\subset N(v'')$.

If v'' is in the same G_1-copy as $\{u',v'\}$, then $\{u,(v_1'',u_2)\}$ would be marked instead of $\{u,v\}$.

If v'' is not in the same G_1-copy as $\{u',v'\}$, then $v_2''\neq u_2'$ and from $I(u',v')\subseteq I(u',v'')$ we infer that $N_2(u_2')\subset N_2(v_2'')$. But then $N(u')\subset N((u_1,v_2''))$ and $(u_1,v_2'')\in P(u')$, because we assumed H to be closed under applications of Lemma 2.

Thus, we can assume that $v_1''\neq u_1$. Now it is easy to see from $I(u',v'')=I(u',(v_1'',u_2))$ that $(v_1'',u_2)\in P(u')$. But, if (u_1,v_2'') and (v_1'',u_2) are in $P(u')$ this is also the case for v'', contrary to assumption. □

6. The marking algorithm

The two lemmas in the preceding section are the basis for the following algorithm, which marks pairs of vertices as Cartesian. This algorithm is modelled after the marking algorithm of Feigenbaum and Schäffer [7] for the strong product. However, in their case all marked pairs of vertices are actually edges, whereas here marked pairs may or may not be edges. Moreover, we also need connectedness of the square of the graph G in question to ensure that the graph H, whose edges are the marked pairs and whose vertex set is $V(G)$, is connected. Note that H is constructed without reference to any product decomposition of G. As in case of the strong product the algorithm is polynomial in the number n of vertices of G.

Algorithm 1.
Input: *A nonbipartite, connected R-thin graph G.*
Output: *A set of marked pairs of vertices of G.*

```
Begin
  For Each x ∈ V(G)
    P(x) := {x}; Insert P(x) into 𝒫; Q(x) := {y | N(y) ⊂ N(x)}
  Next x
  M1: While ∃x ∈ V(G) for which Q(x) \ P(x) ≠ ∅
    For Each such x
      𝒥(x) := {N(y) | y ∈ Q(x) \ P(x)}
      If N(y) is maximal in 𝒥(x), Then mark {x, y}
    Next x
    If {x, y} has been marked, Then join P(x) and P(y) in 𝒫
  End M1
  Set all I(x, y) = N(x) ∩ N(y).
  M2: While |𝒫| > 1
    M3: For Each x ∈ V(G)
      𝒥(x) := {I(x, y) | y ∉ P(x), I(x, y) ≠ ∅}
      If 𝒥(x) ≠ ∅, Then
          For Each y ∉ P(x) with I(x, y) ≠ ∅
            If I(x, y) is strictly maximal in 𝒥(x), Then mark {x, y}
            If (I(x, y) is nonstrictly maximal in 𝒥(x)) And
              (N(z) ⊄ N(y) for each z ∉ P(x) with
              I(x, y) = I(x, z) ) Then
              mark {x, y}
            End If
          Next y
      End If
    End M3
    If {x, y} has been marked, Then join P(x) and P(y) in 𝒫
  End M2
End
```

Lemma 4. *All iterations of the loops* **M1** *and* **M2**, *except the ones in which the loops are terminated, reduce the size of* $\mathscr{P}$.

Proof. We begin with **M1**. If $Q(x) \setminus P(x)$ is empty this is the last iteration. Thus, let us assume $Q(x) \setminus P(x) \neq \emptyset$. Since at least one set in $\mathscr{J}(x)$ must be maximal, at least one pair $\{x, y\}$ is marked in this iteration. Since y does not belong to $P(x)$ at the beginning ot the iteration this reduces the number of equivalence classes in $\mathscr{P}$.

For the loop **M2** we can assume that $\mathscr{P} > 1$, otherwise this is the last iteration. Since G is nonbipartite and connected, its square is connected. Thus, there must exist a vertex x and a vertex y of distance two from x which is not in $P(x)$. In symbols, $I(x, y) \neq \emptyset$ and $y \notin P(x)$. This means that **M3** is executed.

If $\mathscr{I}(x)$ has a strictly maximal element some pair $\{x, y\}$ will be marked, which will reduce the size of $\mathscr{P}$ in this execution of **M2**.

If $\mathscr{I}(x)$ has no strictly maximal element there must exist a vertex y such that $I(x, y)$ is maximal and that among all vertices z with maximal $I(x, z)$ the vertex y has strictly minimal closed neighborhood, for otherwise there would be two vertices with the same closed neighborhood, which is not possible, because G is thin. Thus, the pair $\{x, y\}$ will be marked in **M3** and the size of $\mathscr{P}$ is reduced in this execution of **M2**. □

Concerning the complexity of this algorithm we wish to remark that **M2** is a While statement which contains three nested For Each statements, the last of which requiring a comparison of two sets. Thus, a rough bound for the complexity is $O(n^5)$.

Lemma 5. *Let G be a nonbipartite, connected R-thin graph. Then there exists a connected graph H defined on the same set of vertices as G whose edge set $E(H)$ consists of pairs of distinct vertices which are Cartesian with respect to any decomposition of G as a cardinal product and which is copy consistent with respect to any such decomposition. Moreover, H is invariant under automorphisms of G.*

Proof. Let H be the graph whose edges are the pairs marked by Algorithm 1. Then the assertion about the connectedness of H is equivalent to the statement that $\mathscr{P}$ has only one element when the algorithm terminates. But this is clear, because **M2** is executed while $|\mathscr{P}| > 1$.

We now recall that H is constructed by Algorithm 1 without reference to any product decomposition of G and that the decomposition $G_1 \times G_2$ of G in Lemma 2 and Lemma 3 was arbitrary. Thus, the edges of H are Cartesian with respect to any decomposition of G as a cardianal product and copy consistent with respect to any such decomposition.

The invariance of H under automorphisms of G follows from the observation that $\mathscr{P}$ and the set $Q(x)$ are invariant under $\mathrm{Aut}(G)$ when they are initially determined and that every iteration of the loops **M1** and **M2** marks sets of edges which are invariant under $\mathrm{Aut}(G)$, because the selection criteria (e.g. maximality) are invariant. □

7. Factoring *R*-thin graphs

In this section we show that every nonbipartite, connected R-thin graph has unique prime factor decomposition with respect to the cardinal product and that it can be found in polynomial time. The UPFD is an immediate consequence of the common refinement property, which we consider first.

Let the nonbipartite, connected R-thin graph G be a nontrivial cardinal product $G_1 \times G_2$ and let H be the graph of Lemma 5. Let H_1 be defined on $V(G_1)$ with the edge set

$$E(G_1) = \{[x_1, y_1] \mid \text{ for all pairs } \{(x_1,x_2),(y_1,y_2)\} \text{ marked by Algorithm 1}\}$$

and let H_2 be defined analogously. Then we clearly have $H = H_1 \square H_2$ by the copy consistency of the set of marked pairs.

We now recall that H is connected and has a unique prime factor decomposition

$$H = Q_1 \square Q_2 \square \cdots \square Q_k.$$

Thus there are disjoint index sets I_1, I_2 with $I_1 \cup I_2 = \{1,2,\ldots,k\}$ and $H_i = \prod_{j\in I_i}^{\square} Q_j$ for $i = 1,2$. (Here and in the sequel we use $\prod^{\square}$ for the Cartesian product to distinguish it from the cardinal one, for which we use $\prod$.) Because of the unique coordinatization for prime factor decompositions of connected graphs with respect to the Cartesian product we also have $H_i^x = (\prod_{j\in I_i}^{\square} Q_j)^x$ for $i = 1,2$ and for all $x \in V(G)$.

It is easy to see how this extends to decompositions of G into arbitrarily many factors G_i. Also, it will be convenient henceforth to call H the *Cartesian skeleton* of G, although it will in general not be a subgraph of G.

Lemma 6. *Let G be a nonbipartite, connected R-thin graph and let $A \times B$ and $C \times D$ be two decompositions of G with respect to the cardinal product. Then there exists a decomposition*

$$A_C \times A_D \times B_C \times B_D \tag{3}$$

of G such that $A = A_C \times A_D$, $B = B_C \times B_D$, $C = A_C \times B_C$ and $D = A_D \times B_D$.

We call the decomposition (3) *a common refinement of the decompositions $A \times B$ and $C \times D$ of G.*

Proof. Let $Q_1 \square Q_2 \square \cdots \square Q_k$ be the UPFD of the Cartesian skeleton H of G. Let I_A be the subset of the index set $\{1,2,\ldots,k\}$ with $V(A) = V(\prod_{i\in I_A}^{\square} Q_i)$ and let I_B, I_C and I_D be defined analogously. Furthermore, set

$$H_{A,C} = \prod_{i\in I_A\cap I_B}^{\square} Q_i$$

and let $H_{A,D}, H_{B,C}$ and $H_{B,D}$ be defined similarly. Then

$$H = H_{A,C} \square H_{A,D} \square H_{B,C} \square H_{B,D}$$

and it will be convenient to use only four coordinates (x_1,x_2,x_3,x_4) for every vertex $x \in V(G)$ henceforth. Of course it is possible that not all of the intersections

$I_A \cap I_C$, $I_A \cap I_D$, $I_B \cap I_C$ and $I_B \cap I_D$ are nonempty. Suppose $I_B \cap I_D = \emptyset$. Then $I_A \cap I_D \neq \emptyset$. If $I_A \cap I_C$ were empty, then $I_A = I_D$ and thus $I_B = I_C$, but then there would be nothing to prove. We can thus assume that all but possibly $I_B \cap I_D$ are nonempty and at least three of the four coordinates are nontrivial, i.e. there are at least two vertices which differ in the first, second and third coordinates, but it is possible that all vertices have the same fourth coordinate. Clearly, for $y = (y_1, y_2, y_3, y_4)$,

$$\begin{aligned}
V(A^y) &= \{(x_1,x_2,y_3,y_4) \mid x_1 \in V(H_{A,C}),\ x_2 \in V(H_{A,D})\},\\
V(B^y) &= \{(y_1,y_2,x_3,x_4) \mid x_3 \in V(H_{B,C}),\ x_4 \in V(H_{B,D})\},\\
V(C^y) &= \{(x_1,y_2,x_3,y_4) \mid x_1 \in V(H_{A,C}),\ x_3 \in V(H_{B,C})\},\\
V(D^y) &= \{(y_1,x_2,y_3,x_4) \mid x_2 \in V(H_{A,D}),\ x_4 \in V(H_{B,D})\}
\end{aligned}$$

are the vertex sets of the A-, B-, C- and D-copies of G. We now define A_C as $p_1(G)$, i.e. $V(A_C) = V(H_{A,C})$ and $[x_1, y_1] \in E(A_C)$ if there are vertices $x = (x_1,x_2,x_3,x_4)$, $y = (y_1, y_2, y_3, y_4)$ in G with $[x, y] \in E(G)$. It is clear what is meant by A_D, B_C and B_D. If all vertices have the same fourth coordinate, then B_D is the one vertex graph with a loop, i.e. the unit graph with respect to cardinal multiplication.

For the proof of the lemma it suffices to show that $A = A_C \times A_D$. Recall that A is obtained by projection of G onto the vertex set of A. We call this projection p_A and define p_B, p_C, p_D analogously. With our present coordinatization we thus have

$$\begin{aligned}
p_A(x_1,x_2,x_3,x_4) &= (x_1,x_2,-,-),\\
p_B(x_1,x_2,x_3,x_4) &= (-,-,x_3,x_4),\\
p_C(x_1,x_2,x_3,x_4) &= (x_1,-,x_3,-),\\
p_D(x_1,x_2,x_3,x_4) &= (-,x_2,-,x_4).
\end{aligned}$$

In order to show that $A = A_C \times A_D$ it suffices to prove that $[p_A x, p_A y] \in E(A)$ if and only if $[p_1 x, p_1 y] \in E(A_C)$ and $[p_2 x, p_2 y] \in E(A_D)$.

Suppose $[p_A x, p_A y] \in E(A)$. We can assume without loss of generality that x, y are chosen such that $[x, y] \in E(G)$. But then $[p_1 x, p_1 y] \in E(A_C)$ and $[p_2 x, p_2 y] \in E(A_D)$ by the definition of A_C and A_D.

On the other hand, suppose the edge $[(x_1,-,-,-),(y_1,-,-,-)]$ is in A_C and $[(-,x_2,-,-),(-,y_2,-,-)] \in E(A_D)$. Then there are vertices x', y', x'', y'' in G of the form

$$\begin{aligned}
x' &= (x_1,a_2',a_3',a_4'), & y' &= (y_1,b_2',b_3',b_4'),\\
x'' &= (a_1'',x_2,a_3'',a_4''), & y'' &= (b_1'',y_2,b_3'',b_4'')
\end{aligned}$$

with $[x', y'] \in E(G)$ and $[x'', y''] \in E(G)$. Thus, $[(x_1,-,a_3',-),(y_1,-,b_3',-)] \in E(C)$ and $[(-,x_2,-,a_4''),(-,y_2,-,b_4'')] \in E(D)$.

Since $G = C \times D$ this implies $[(x_1,x_2,a_3',a_4''),\ (y_1,y_2,b_3',b_4'')] \in E(G)$ and, hence, $[(x_1,x_2,-,-),\ (y_1,y_2,-,-)] \in E(A)$. □

Lemma 7. *Every finite, nonbipartite, connected R-thin graph G has unique prime factor decomposition with respect to the cardinal product.*

Proof. We proceed by induction with respect to the number of vertices, assumming that the lemma is true for all graphs with fewer vertices than G. Let

$$G_1 \times G_2 \times \cdots \times G_r = Q_1 \times Q_2 \times \cdots \times Q_s$$

be two prime factor decompositions of the nonbipartite, connected R-thin graph G. Then there are graphs B and D such that $G \cong G_1 \times B \cong Q_1 \times D$. We now invoke Lemma 6 for $A \cong G_1$ and $C \cong Q_1$. Then $G \cong A_C \times A_D \times B_C \times B_D$ and $G_1 \cong A_C \times A_D$.

Suppose $G_1 \cong A_C$. Then $Q_1 \cong A_C \times B_C \cong A_C \cong G_1$ and $A_D \cong K_1^s$, $B_C \cong K_1^s$ and $B \cong B_C \times B_D \cong B_D$. But then $D \cong A_D \times B_D \cong B_D \cong B$ and $r \cong s$ and $G_i \cong Q_i$ for $2 \leqslant i \leqslant r$.

On the other hand, let $A_C \cong K_1^s$. Then $G_1 \cong A_D$ and $Q_1 \cong B_C$. Furthermore, $G_2 \times G_2 \times \cdots \times G_r \cong B \cong B_C \times B_D \cong Q_1 \times B_D$ and $Q_2 \times Q_3 \times \cdots Q_s \cong D \cong A_D \times B_D \cong G_1 \times B_D$. By the induction hypothesis both B and D have UPFD. Thus, let us assume the notation to be chosen such that $Q_1 \cong G_2$. Then $D \cong Q_2 \times Q_3 \times \cdots \times Q_s \cong G_1 \times B_D \cong G_1 \times G_3 \times G_4 \times \cdots \times G_r$ and UPFD immediately follows from the induction hypothesis and

$$G \cong G_1 \times B \cong G_1 \times Q_1 \times B_D \cong Q_1 \times G_1 \times B_D \cong Q_1 \times D \cong G. \qquad \square$$

Lemma 8. *The prime factor decomposition of nonbipartite, connected R-thin graphs with respect to the cardinal product can be found in polynomial time.*

Proof. Let G be a nonbipartite, connected R-thin graph. By Lemma 5 the Cartesian skeleton H of G can be found in polynomial time. Also, the prime factor decomposition of H,

$$H = Q_1 \square Q_2 \square \cdots \square Q_k,$$

can be found in polynomial time [1,5,6,27]. Furthermore, let

$$G = G_1 \times G_2 \times \cdots \times G_r$$

be the UPFD of G. By Lemma 5 there is a partition of the index set $I = \{1, 2, \ldots, k\}$ into sets $I_1, I_2, \ldots I_r$ such that

$$V(G_i^x) = V\left(\left(\prod_{j \in I_i}^{\square} Q_j\right)^x\right)$$

for all $x \in V(G)$ and $1 \leqslant i \leqslant r$. Our problem is to find the I_i.

Suppose we take any subset J of I. We can then define graphs A and B by the projections $p_J(G)$ and $p_{I \setminus J}(G)$ onto the vertex sets $V(\prod_{i \in J}^{\square} Q_i)$ and $V(\prod_{i \in I \setminus J}^{\square} Q_i)$. If I is one of the sets I_i or a union of such sets, then $G = A \times B$.

Clearly, $k \leqslant \log_2(n)$ and thus I has at most $2^{\log_2(n)} = n$ subsets. Construction of the graphs A and B requires the projection of $m = |E(G)|$ edges into the coordinate sets and has complexity $O(m \log n)$. It is clear that $E(G) \subseteq E(A \times B)$. Thus, $G = A \times B$ if $|E(G)| = |E(A)| \cdot |E(B)|$ and we can find a minimal subset of I such that $G = A \times B$ in polynomial time. This A must be one of the G_i and $B = \prod_{1 \leqslant j \leqslant r, j \neq i} G_j$. After at most $\log_2 n$ repetitions of this procedure we have decomposed G into its prime factors. □

The invariance of the Cartesian skeleton under automorphisms is the basis of the following characterization of the automorphism group of a cardinal product of nonbipartite, connected R-thin graphs.

Theorem 3. *Let $G_1 \times G_2 \times \cdots \times G_r$ be the prime factor decomposition of an R-thin graph G with respect to the cardinal product and let ϕ be an automorphism of G. Then there exists a permutation π of $\{1, 2, \ldots, r\}$ together with isomorphisms $\psi_i : G_i \to G_{\pi i}$ such that*

$$\phi(x_1, x_2, \ldots, x_r) = (\psi_{\pi^{-1}1} x_{\pi^{-1}1}, \psi_{\pi^{-1}2} x_{\pi^{-1}2}, \ldots, \psi_{\pi^{-1}r} x_{\pi^{-1}r}).$$

Proof. Let H be the Cartesian skeleton of the G and $Q_1 \Box Q_2 \Box \cdots \Box Q_k$ be the prime factor decomposition of H with respect to the Cartesian product. Since H is preserved under automorphisms of G every automorphism of G induces one of H. Using the notation of the previous lemma, we see that the vertex sets of the G_i-copies in G are the vertex sets of the copies of Cartesian products $\prod^{\Box}_{j \in I_i} Q_j$. We set $H_i = \prod^{\Box}_{j \in I_i} Q_j$ and note that every automorphism of G has to respect this partition, since the prime factor decomposition of G is unique. Taking this into account, an application of Proposition 1 now shows that the above description of ϕ is true for isomorphisms ψ_i of H_i into H_{π_i}. Clearly, these isomorphisms are bijections of $V(G_i)$ onto $V(G_{\pi_i})$. By the definition of the cardinal product these bijections must be isomorphisms. □

8. Extraction of complete factors

Complete graphs with loops at every vertex can easily be factored into their prime constituents. For, let $p_1 \cdot p_2 \cdots p_k$ be the prime factorization of the natural number n. Then

$$K_n^s = K_{p_1}^s \times K_{p_2}^s \times \cdots \times K_{p_k}^s$$

is the prime factor decomposition of K_n^s with respect to the cardinal product. We now show that every nonbipartite, connected graph has a unique maximal complete graph with loops at every vertex as a factor. This factor may of course be trivial.

We begin with several observations about the relation R and first recall that two vertices u and v of a graph G are in relation R, if a vertex w is adjacent to u if and only if it is adjacent to v. Clearly R is an equivalence relation on $V(G)$ and every equivalence class induces a complete subgraph of G with loops at every vertex or it induces a completely disconnected graph. The graph G/R is obtained as the usual quotient graph. More precisely, its vertex set is

$$V(G/R) = \{D_i \mid D_i \text{ is an equivalence class of } R\}$$

and $D_iD_j \in E(G/S)$ whenever $uv \in E(G)$ for some $u \in V(D_i)$ and for some $v \in V(D_j)$.

Lemma 9. *Let G and H be graphs. Then*

$$V((G \times H)/R) = \{U \times W \mid U \in V(G/R), W \in V(H/R)\}.$$

Proof. Let U be a vertex of G/R and let W be a vertex of H/R. By the definition of the cardinal product the vertices of $U \times W$ belong to the same equivalence class of $V(G \times H)/R$. It remains to show that $U \times W$ is an equivalence class by itself. Let $(u,w) \in U \times W$ and assume that a vertex (a,b) belongs to the same equivalence class as the vertices in $U \times W$. Then any vertex $x \in V(G)$ that is adjacent to any vertex in U is also adjacent to u. Since we assume H to be nontrivial and connected, there must be a vertex $z \in V(H)$ which is adjacent to w. Hence, (x,z) is adjacent to (u,w) and thus also to (a,b). But then x must be adjacent to a by the definition of the product.

On the other hand, if $x \in V(G)$ is adjacent to a we can find a vertex y in H that is adjacent to b. Then (x,y) is adjacent to (a,b) and therefore to every vertex of $U \times W$. But then x must be adjacent to every vertex in U and so $a \in U$.

By the same argument b must be in W. This means that every vertex in the same equivalence class as the vertices of $U \times W$ already is in $U \times W$. □

Proposition 6. *Let G and H be graphs. Then $(G \times H)/R \cong G/R \times H/R$.*

Proof. Follows from Lemma 9 and the definition of the cardinal product. □

Lemma 10. *Let G be a nonbipartite, connected graph and let $k > 1$ divide $|D_i|$ for every $D_i \in V(G/R)$. Then there is a graph H such that $G \cong K_k^s \times H$. Conversely, if $G \cong K_k^s \times H$ for some $k > 1$, then k divides $|D_i|$ for any $D_i \in V(G)/R$.*

Proof. Let $V(G/R) = \{D_i \mid i \in I\}$. Let $\{D_i' \mid i \in I\}$ be a family of disjoint sets with $|D_i| = k|D_i'|$. Define a graph H with the vertex set $V(H) = \bigcup_{i \in I} D_i'$ and

$$E(H) = \{xy \mid x \in V(D_i'), y \in V(D_j') \text{ and } D_iD_j \in E(G/R)\}.$$

Then it is straightforward to see that $G \cong K_k^s \times H$.

For the converse suppose that $G \cong K_k^s \times H$ for some $k > 1$. Then Lemma 9 implies that $V(G/R) = \{V(K_k^s) \times U \mid U \in V(H/R)\}$. $\square$

This entrains the following proposition on graph isomorphisms and automorphisms.

Proposition 7. *Two graphs G and H are isomorphic if and only if the following two conditions are satisfied*:

(i) *There exists an isomorphism $\pi : G/R \rightarrow H/R$.*

(ii) $|D_i| = |\pi(D_i)|$, *for all $D_i \in V(G/R)$.*

Proof. Let $\varphi : G \rightarrow H$ be an isomorphism. Then $\varphi(D_i) \in V(H/R)$ for any $D_i \in V(G/R)$. If we define $\pi : G/R \rightarrow H/R$ by $\pi(D_i) = A \in V(H/R)$ for $\varphi(D_i) = A$, it follows immediately that π fulfills (i) and (ii).

Conversely, let (i) and (ii) be satisfied. For $D_i \in V(G/S)$ let $\varphi_i : D_i \rightarrow \pi(D_i)$ be a bijection. Then $\varphi : V(G) \rightarrow V(H)$ defined by $\varphi|D_i = \varphi_i$ gives us an isomorphism $G \rightarrow H$. $\square$

We shall also apply this proposition for the investigation of automorphisms, because if $G \cong H$, then conditions (i) and (ii) describe the relationship between Aut(G) and Aut(G/R).

Lemma 11. *Let $G \cong K_k^s \times H$ and let $G \cong K_k^s \times H'$. Then $H \cong H'$.*

Proof. If $X = K_m^s \times Y$ then $X/R \cong Y/R$ by Lemma 9. Then $V(X/R) = \{V(K_m^s) \times U \mid U \in V(Y/R)\}$ and the mapping $\pi : Y/R \rightarrow X/R$ defined by $\pi(U) = V(K_m^s) \times U$ is an isomorphism. It follows that $G/R \cong (K_k^s \times H)/R \cong H/R$ and similarly $G/R \cong (K_k^s \times H')/R \cong H'/R$. So $H/R \cong H'/R$. Moreover, an isomorphism from H/R onto H'/R can be chosen in such a way that condition (ii) of Proposition 7 is satisfied. Hence, by Proposition 7 we have $H \cong H'$. $\square$

Lemma 12. *Let G be a nonbipartite, connected graph with the decompositions $G \cong K_m^s \times H$ and $G \cong K_n^s \times H'$. If H and H' are not divisible by K_k^s for any $k > 1$, then $m = n$ and $H \cong H'$.*

Proof. The second statement of Lemma 10 implies that m divides the greatest common divisor d of the numbers $|D_i|$, where $D_i \in V(G/R)$. Since the (multi)-set $\{\frac{1}{m}|D_i|\}$ represents the sizes of the classes of H/R and as H is not divisible by K_k^s for $k > 1$, the first assertion of Lemma 10 implies that the greatest common divisor of $\{\frac{1}{m}|D_i|\}$ is equal to 1. Therefore $m = d$. Analogously we obtain $n = d$ and so $m = n$. The proof is completed by an application of Lemma 11. $\square$

We conclude the section with the observation that R, G/R and, hence, also the largest complete factor of G, can be found in polynomial time.

9. Factoring nonbipartite, connected graphs

Let $G_1 \times \cdots \times G_n$ and $G'_1 \times \cdots \times G'_m$ be prime factor decompositions of a nonbipartite, connected graph G. We may assume that $G_{r+1}, \ldots, G_n$ and $G'_{s+1}, \ldots, G'_m$ are complete graphs with loops at every vertex and that the other factors are not isomorphic to any K_k^s. Hence, $G_1 \times \cdots \times G_r$ and $G'_1 \times \cdots \times G'_s$ are not divisible by a nontrivial K_k^s and thus, by Lemma 12, we have

$$G_1 \times \cdots \times G_r \cong G'_1 \times \cdots \times G'_s,$$
$$G_{r+1} \times \cdots \times G_n \cong G'_{s+1} \times \cdots \times G'_m.$$

As $G_{r+1} \times \cdots \times G_n$ is a K_k^s and since such graphs have unique prime factor decomposition, the factors $G_{r+1}, \ldots, G_n$ coincide with the $G'_{s+1}, \ldots, G'_m$.

Let $P \cong G_1 \times \cdots \times G_r \cong G'_1 \times \cdots \times G'_s$, set $Q \cong P/R$, $H_i = G_i/R$ and $H'_i = G'_i/R$. Furthermore, let

$$Q \cong Q_1 \times Q_2 \times \cdots \times Q_k$$

be the prime factor decomposition of Q. We are then left with two problems:

(i) Given a nonbipartite, connected graph C and a decomposition $A' \times B'$ of C/R, find A, B with $A/R = A'$ and $B/R = B'$ such that $C = A \times B$.

(ii) Is it possible to partition the index set $\{1, 2, \ldots, k\}$ in two ways $I \cup J$ and $I' \cup J'$, such that $I \cap I'$ is nonempty, but that the copies of $\prod_{i \in I} Q_i$ in Q are the copies of some H_j, say H_1, and that the copies of $\prod_{i \in I'} Q_i$ in Q are the copies of some H'_i, say H'_1?

The next two lemmas show that the first problem has a very easy solution and that the answer to the question in the second problem is no.

As we shall see, we can then find the prime factor decomposition of P in polynomial time by checking all subsets of the index set $\{1, 2, \ldots, k\}$.

Lemma 13. *Suppose it is known that a given graph G which does not admit any K_k^s as a factor is a cardinal product graph $G_1 \times G_2$ and suppose the decomposition $G/R = G_1/R \times G_2/R$ is known. Then G_1 and G_2 can be easily determined.*

In fact, if $D(x_1, x_2)$ denotes the size of the R-equivalence class of G that is being mapped into $(x_1, x_2) \in G_1/R \times G_2/R$, then the size $D(x_1)$ of the equivalence class of G_1 being mapped into $x_1 \in G_1/R$ is $\gcd\{D(x_1, y) \mid y \in V(G_2)\}$. *Analogously for $D(x_2)$.*

Proof. It suffices to prove the assertion for $D(x_1)$. Let $(x_1, x_2) \in G_1/R \times G_2/R$, then $D(x_1, x_2) = D(x_1)D(x_2)$. Also, if G does not admit any K_k^s as a factor, then $\gcd\{D(y) \mid y \in V(G_2)\} = 1$, and thus, $D(x_1) = \gcd\{D(x_1)D(y) \mid y \in V(G_2)\}$. □

Lemma 14. *Let G be a nonbipartite, connected graph which does not admit any K_k^s as a factor and let $A \times B$ and $C \times D$ be two decompositions of G with respect to the* *cardinal product for which $A/R \times B/R$ and $C/R \times D/R$ are distinct decompositions of G/R. Then there exists a decomposition*

$$A_C \times A_D \times B_C \times B_D \tag{4}$$

of G such that $A = A_C \times A_D$, $B = B_C \times B_D$, $C = A_C \times B_C$ and $D = A_D \times B_D$.

Proof. Clearly the decompositions $A/R \times B/R$ and $C/R \times D/R$ of G/R have a common refinement. Suppose, $A'_C \times A'_D \times B'_C \times B'_D$ is this refinement, where $A/R = A'_C \times A'_D$, $B/R = B'_C \times B'_D$, $C/R = A'_C \times B'_C$ and $D/R = A'_D \times B'_D$. Furthermore, let (x, y, u, v) be the coordinatization corresponding to the decomposition $A'_C \times A'_D \times B'_C \times B'_D$. Then there are functions $a(x, y)$, $b(u, v)$, $c(x, u)$, $d(y, v)$ where $a(x, y)$ is the size of the R-class of A mapped into the vertex $(x, y) \in A/R$ and where the other functions are similarly defined.

Clearly $a(x, y)b(u, v) = c(x, u)d(y, v)$. We have to show that there exist functions $a_1(x), a_2(y), b_1(u)$ and $b_2(v)$ such that $a(x, y) = a_1(x)a_2(y)$, $b(u, v) = b_1(u)b_2(v)$, $c(x, u) = a_1(x)b_1(u)$ and $d(y, v) = a_2(y)b_2(v)$. Moreover, we have to show that $\gcd\{a_1(x) \mid x \in V(A'_C)\} = 1$ and that analogous properties hold for a_2, b_1 and b_2.

In order to prove this we first observe that

$$\frac{a(x, y)}{d(y, v)} = \frac{c(x, u)}{b(u, v)}.$$

Clearly this implies that both fractions depend only on x and v, i.e. they are independent of both y and u. But then $a(x, y)/d(y, v) = a(x, y_0)/d(y_0, v)$, and hence $a(x, y)/a(x, y_0) = d(y, v)/d(y_0, v)$. Again it is easy to see that both sides depend only on y and y_0, i.e. are equal to a function $f(y, y_0)$. Thus, $a(x, y)$ is representable as a product $a(x, y_0)f(y, y_0)$. Since y_0 was arbitrarily chosen but fixed, we write $a(x, y) = a_1(x)a_2(y)$. Similarly we obtain decompositons of b, c and d. We then have

$$a_1(x)a_2(y)b_1(u)b_2(v) = c_1(x)c_2(u)d_1(y)d_2(v).$$

Separating variables as above, it is easily seen that there must be constants k_i such that $k_1a_1(x) = c_1(x)$, $k_2a_2(y) = d_1(y)$, $k_3b(u) = c_2(u)$ and $k_4b_2(v) = d_2(v)$. Also $k_1k_2k_3k_4 = 1$.

From the above construction it is clear that the a_i, b_i, c_i and d_i can be chosen as rational functions. We wish to chose them such that they are integer functions. Consider

$$\frac{\text{Denominator}(a_1(x))}{\text{Numerator}(a_1(x))} \cdot \frac{\text{Denominator}(a_2(y))}{\text{Numerator}(a_2(y))} = a(x, y).$$

Assuming all fractions to be cancelled, it is clear that the numerator of $a_2(y)$ divides the denominator of $a_1(x)$ and the numerator of $a_1(x)$ divides the denominator of $a_2(y)$, and this for arbitrary x and y. But then the least common multiple n_1 of the numerators of $a_1(x)$ also divides the denominator of every $a_2(y)$ and the least common multiple

n_2 of the numerators of the $a_2(y)$ divides the denominator of every $a_1(x)$. We can thus write the $a_1(x)$ as a fraction $a_1'(x)/n_1$ and every $a_2(y)$ as $a_2'(y)/n_2$. Then

$$\frac{a_1'(x)}{n_1} \cdot \frac{a_2'(y)}{n_2} = \frac{a_1'(x)}{n_2} \cdot \frac{a_2'(y)}{n_1} = a(x,y).$$

Since n_2 divides every $a_1'(x)$ and n_1 divides every $a_2'(y)$, we replace $a_1(x)$ by $a_1'(x)/n_2$ and $a_2(y)$ by $a_2'(x)/n_1$ to get an integer factorization of $a(x,y)$. Analogously we proceed for b, c and d. It is then clear that all k_i must be 1.

Finally, $\gcd\{a_1(x) \mid x \in V(A_C')\} = 1$, for otherwise $\gcd\{a(x,y) \mid (x,y) \in V(A)\}$ would not be one and A would admit a K_k^s as a factor. Similarly we argue for a_2, b_1 and b_2. □

Now sufficiently many preliminary results have been assembled for the proof of the main theorem of this paper.

Proof of Theorem 1. Let Q be a finite, nonbipartite connected graph and $G_1 \times G_2 \times \cdots \times G_n$ a prime factorization of Q, where the notation is chosen such that $G_{r+1}, G_{r+2}, \ldots, G_n$ are those factors (if any) which are isomorphic to a K_k^s and that $|G_{r+1}| \leqslant |G_{r+2}| \leqslant \cdots \leqslant |G_n|$. By Lemma 12 the factors $G_{r+1}, G_{r+2}, \ldots, G_n$ are uniquely determined and also the graph G with $Q = G \times G_{r+1} \times G_{r+2} \times \cdots \times G_n$.

It remains to be shown that nonbipartite, connected K_k^s-free graphs G have unique prime factorization. Let $G_1 \times G_2 \times \cdots \times G_r$ be a prime factorization of G and $Q_1 \times Q_2 \times \cdots \times Q_k$ be the unique prime factor decomposition of G/R ensured by Lemma 7. Clearly there is a partition $\mathscr{I} = \{I_1, I_2, \ldots, I_r\}$ of the index set $I = \{1, 2, \ldots, k\}$ such that $G_i/R = \prod_{j \in I_i} Q_j$.

Let $G_1' \times G_2' \times \cdots \times G_s'$ be any other prime factorization of G. Again, there is a partition $\mathscr{I}' = \{I_1', I_2', \ldots, I_s'\}$ of I such that $G_i'/R = \prod_{j \in I_i'} Q_j$. If the partitions $\mathscr{I}$ and $\mathscr{I}'$ are not equal, then Lemma 14 shows that not all G_i or G_i' can be prime. Thus, we have proved the uniqueness of the prime factorization.

As in the proof of Theorem 3 we can show that all automorphisms of G correspond to automorphisms of G/R that are induced by automorphisms of the G_i/R or by transpositions of isomorphic factors G_i/R and G_j/R. Thus, $\mathrm{Aut}(G/R)$ can be described in this fashion and all automorphisms of G can then be found by application of Proposition 7.

For the proof of the assertion about the complexity, let us recall that we can find the unique prime factorization of a given nonbipartite, connected graph $Q \in \Gamma$ by taking the following steps:

(i) Determination of R and G/R.
(ii) Representation of Q in the form $G \times K_t^s$, where G is K_k^s-free.
(iii) Prime factorization of K_t^s, i.e. of t.
(iv) Construction of the Cartesian skeleton of G/R and of the prime factor decomposition $Q_1 \times Q_2 \times \cdots \times Q_k$ of G/R.

(v) Determination of all minimal subsets J of $I=\{1,2,\ldots,k\}$ such that there are graphs A and B with $G=A\times B$, $A/R=\prod_{i\in J}Q_i$ and $B=\prod_{j\in I\setminus J}Q_j$.

This step can be executed by repeated applications of Lemma 13. Moreover, A must be prime by the minimality of J and Lemma 14.

Clearly the complexity of all these steps is polynomial. □

10. Conclusion and open problems

The above results provide a rather satisfactory description of the cardinal product of finite, nonbipartite connected graphs in Γ_0 and in Γ. Nothing has been said though about bipartite graphs or disconnected ones. Oriented graphs have not been treated and the infinite case has been completely neglected. Of the multitude of open directions of research we list a few below. Most of them seem to be accessible with known methods and may provide interesting and surprising answers.

(a) How do results on the cardinal product of nonbipartite, connected graphs extend to bipartite connected graphs and to disconnected ones? For steps in this direction see [18,23].

(b) How do the results of this paper extend to oriented graphs? In fact, this extension appears to be rather natural as McKenzie's investigations [19] actually pertain to oriented graphs and contain some results in this direction, also for infinite graphs. In the finite case there is no appearent reason why the complexity of decomposing oriented graphs with respect to the cardinal product should be higher than that of decomposing unoriented ones if all factors are connected.

(c) In this paper we have not stated the overall complexity of our decomposition algorithm as it depends on the efficient implementation of its main steps. In any case, it is still open whether there exists a prime factorization procedure with respect to the cardinal product of complexity $O(mn)$ or even $O(m\log n)$, where n is the number of vertices of the graph to be factored and m its number of edges.

(d) Graphs with much structure often have interesting reconstruction properties. For the reconstruction of Cartesian products there exist particularly strong results [17]. For other products compare [2]. Can cardinal or strong product graphs be efficiently reconstructed?

(e) As has been shown by Graham and Winkler [8], every connected graph G has a natural isometric embedding into a Cartesian product associated with this graph. Is there a natural way of representing graphs as partial cardinal products?

Note that every graph can be isometrically embedded into a strong product of paths (see [24] or [28]).

(f) In case of the Cartesian product the investigations of the structure of finite graphs in the search of polynomial algorithms for their factorization has brought new insight and shorter proofs for related results pertaining to infinite graphs, see e.g. [13]. How is the situation with respect to the cardinal product?

(g) There is one additional associative product of simple graphs defined on the Cartesian product of the vertex sets of the factors for which prime factorizations have not been sufficiently investigated. In this product two vertices (x_1,x_2) and (y_1,y_2) are defined as adjacent if both x_1,x_2 and y_1,y_2 are nonadjacent in the appropriate factor or if, as in case of the strong product, $[x_1,x_2]$ and $[y_1,y_2]$ are edges. For partial results we refer to [12] and for classifications of asociative products to [14,16]. It has been called *equivalence product* in [12]. Which graphs have unique prime factor decomposition with respect to the equivalence product and can this decomposition be found in polynomial time?

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