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Infinite Graphs — A Survey

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Abstract

This expository article describes work which has been done on various problems involving infinite graphs, mentioning also a few unsolved problems or suggestions for future investigation.

1. Introduction

This is an expository article which originated as mimeographed notes issued in connection with a lecture given in 1966 at the N.A.T.O. Conference on Contemporary Methods of Discrete Mathematics directed by Professor F. Harary and Dr. B. Roy. The aim of this article is to survey a selection of topics in the theory of infinite graphs, an area of graph theory which perhaps receives at the present time less attention than it might deserve.

2. Definitions and Notation

The set of vertices of a graph G will be denoted by $V(G)$ and its set of edges will be denoted by $E(G)$. G is *finite* if $|V(G) \cup E(G)|$ is finite, *enumerable* if $|V(G) \cup E(G)| = \aleph_0$, *countable* if it is finite or enumerable, and *locally finite* if the degree of every vertex is finite (i.e. if each vertex is incident with only finitely many edges).

A *walk* in G is a finite, one-way infinite or two-way infinite sequence of one of the forms

$$v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_n, v_n \qquad (\textit{finite walk})$$

or

$$v_0, e_1, v_1, e_2, v_2, e_3, \ldots \qquad (\textit{one-way infinite walk})$$

or

$$\ldots, e_{-2}, v_{-2}, e_{-1}, v_{-1}, e_0, v_0, e_1, v_1, e_2, \ldots$$
$$(\textit{two-way infinite walk})$$

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where the v_i are vertices of G and each e_i is an edge joining the vertices immediately preceding and following it in the sequence.

A walk is a *trail* if no edge appears more than once in it, and is a *path* if no vertex appears more than once. A finite walk is *closed* if its first and last terms are the same. A closed walk with more than one term in which there are no repetitions apart from the first and last terms being the same is a *circuit*. A trail in G is *Eulerian* if every vertex and edge of G appears in it. A path or circuit in G is *Hamiltonian* if every vertex of G appears in it. A set of trails in G constitute a *decomposition* of G if every vertex of G appears in at least one of them and every edge of G appears in exactly one of them (so that the trails subdivide $E(G)$ into disjoint subsets).

Two vertices of a graph are *adjacent* if they are joined by an edge.

3. General Observations

It may be convenient to think of graph theory as being divided into four layers:

(I) the theory of finite graphs,

(II) the theory of locally finite graphs,

(III) the theory of countable (not necessarily locally finite) graphs,

(IV) the theory of graphs with no restrictions on cardinality.

The theory of infinite graphs appears at present to be in an even more incomplete state than the theory of finite graphs, in the sense that some of the work which has been done for finite graphs has either not been extended to infinite graphs or been extended only to *some* infinite graphs, e.g., locally finite ones. Of course, there are some problems in the theory of finite graphs (e.g., certain enumerative problems) which may not in any reasonable sense raise corresponding questions concerning infinite graphs. Other cases in which work done for finite graphs has not been extended to infinite graphs may be due either to difficulties involved in so extending it or to the fact that those concerned have (as is entirely permissible) no particular interest in extending their work to infinite graphs.

The degree of additional difficulty involved when we try to extend work done for finite graphs to infinite graphs varies from one problem to another. Extension to *enumerable* graphs is usually easiest when it can be done by means of König's "Unendlichkeitslemma" [13, Chapter VI].

Let $S_1, S_2, S_3, \ldots$ be an infinite sequence of disjoint non-empty finite sets and $\prec$ be a relation in $S_1 \cup S_2 \cup \cdots$ such that

$$\left.\begin{array}{c}\textit{whenever } n \textit{ is a positive integer and } x \in S_{n+1},\\ \textit{there exists a } y \in S_n \textit{ such that } y \prec x.\end{array}\right\} \qquad (1)$$

Then there exists an infinite sequence $x_1, x_2, x_3, \ldots$ such that $x_n \in S_n$ $(n = 1, 2, \ldots)$ and $x_1 \prec x_2 \prec x_3 \prec \cdots$. (The proof is quite an easy exercise.)

ILLUSTRATIVE APPLICATION. *Let k be a positive integer and G be an enumerable graph. If every finite subgraph of G is k-colorable, then so is G.* (We call G *k-colorable* if its vertices can be colored using at most k colors so that no two adjacent vertices have the same color.)

PROOF: Write $G = G_1 \cup G_2 \cup G_3 \cup \cdots$ where $G_1, G_2, \ldots$ are finite subgraphs and $G_1 \subset G_2 \subset \cdots$. Let $C_1, C_2, \ldots, C_k$ be k colors, and let S_n be the set of all colorings of the vertices of G_n using some or all of $C_1, \ldots, C_k$ in which no two adjacent vertices of G_n have the same color. For $x_i \in S_i$, $x_j \in S_j$, we define "$x_i \prec x_j$" to mean that $i \leqslant j$ and the coloring x_j of G_j induces (in the obvious sense) the coloring x_i of G_i. Since every coloring in S_{n+1} induces a coloring in S_n, condition (1) of König's lemma is satisfied and we may conclude that there exist colorings $x_1, x_2, \ldots$ belonging to $S_1, S_2, \ldots$ respectively such that $x_1 \prec x_2 \prec x_3 \prec \cdots$; and obviously these colorings may be combined to give an admissible k-coloring of G.

From this discussion, we can, for instance, deduce the theorem that every enumerable planar graph is 5-colorable from the corresponding theorem concerning finite planar graphs: for, if G is an enumerable planar graph, every finite subgraph of G is once again planar, and therefore 5-colorable, and therefore G is 5-colorable.

The following theorem, which is Lemma 1 of [30] and Theorem 7.1.3 of [27], can be used to prove that a general (not necessarily enumerable) infinite graph is k-colorable if all its finite subgraphs are k-colorable, k being a positive integer.

THEOREM A. *Let Y, Z be sets and let $\mathfrak{F}_Y$ denote the class of all finite subsets of Y. For each $A \in \mathfrak{F}_Y$ let f_A be a function from A into Z and, for each $y \in Y$, let the set $\{f_A(y) : y \in A \in \mathfrak{F}_Y\}$ be finite. Then there exists a function $f: Y \to Z$ such that, for every $A \in \mathfrak{F}_Y$, there exists a $B \in \mathfrak{F}_Y$ such that $A \subset B$ and $f \mid A = f_B \mid A$.*

(The notation $f \mid A$ indicates the function $g : A \to Z$ such that $g(y) = f(y)$ for every $y \in A$, and $f_B \mid A$ is similarly defined.)

To prove that an infinite (not necessarily enumerable) graph G is k-colorable if all its finite subgraphs are k-colorable let the set Y of

Theorem A be $V(G)$ and let $Z = \{C_1, ..., C_k\}$ be a set of k colors. If $A \in \mathfrak{F}_Y$, the k-colorability of every finite subgraph of G ensures that we can select a function $f_A : A \to Z$ such that no two elements of A which are adjacent in G have the same image under f_A. Assuming the Axiom of Choice we thus select an f_A for every $A \in \mathfrak{F}_Y$, and it is then not hard to see that a function $f: Y \to Z$ satisfying the condition of Theorem A gives a coloring of G using at most the colors $C_1, ..., C_k$ such that no two adjacent vertices of G receive the same color. This theorem concerning k-colorability of infinite graphs was first established by Erdös and de Bruijn in [7].

Theorem A is useful, in suitable cases, for extending known theorems concerning finite graphs to infinite graphs which are not necessarily enumerable, as the foregoing discussion may suggest.

4. Subtrees and Subforests

A graph is a *forest* if each of its components is a tree. A *subtree (subforest)* of a graph G is a subgraph of G which is a tree (forest), and a *spanning subtree (spanning subforest)* of G is a subtree (subforest) of G which includes all the vertices of G.

Theorem B. *Let G be a graph and k be a positive integer. Then G is the union of k subforests if and only if, for every non-empty finite subset X of $V(G)$, the number of edges of G which have both end-vertices in X is less than or equal to $k(|X| - 1)$.*

A proof of Theorem B for finite graphs was given in [22], and the truth of the theorem for enumerable graphs can be deduced from its truth for finite graphs precisely as in the proof of the 5-color theorem for enumerable planar graphs in Section 3. However, this is a case in which we need not confine ourselves to finite and enumerable graphs. Using Theorem A, one can show that G is the union of k subforests if each of its finite subgraphs is the union of k subforests (k being still finite): the proof follows much the same lines as that of Erdös and de Bruijn's theorem in Section 3, but Y has now to be taken to be $E(G)$. This makes it easy to deduce the truth of Theorem B for general infinite graphs from its truth for finite graphs.

In the theory of finite graphs, the following theorem of Tutte [42, 19] is closely related to Theorem B (indeed Theorem B for finite graphs and Theorem C are contained as special cases in a single more general theorem).

THEOREM C. *Let G be a finite graph and k be a positive integer. Then G has k edge-disjoint spanning subtrees if and only if, for every partition $\mathfrak{P}$ of $V(G)$, the number of edges of G which join vertices in different members of $\mathfrak{P}$ is greater than or equal to $k(|\mathfrak{P}| - 1)$.*

(By a *partition* of $V(G)$, we mean a set of disjoint non-empty subsets of $V(G)$ whose union is $V(G)$.)

There would seem to be a very good chance that the following extension of Theorem C to countable graphs may be true:

CONJECTURE A. *Let G be a countable graph and k be a positive integer. Then G has k edge-disjoint spanning subtrees if and only if, for every finite partition $\mathfrak{P}$ of $V(G)$, the number of edges of G which join vertices in different members of $\mathfrak{P}$ is greater than or equal to $k(|\mathfrak{P}| - 1)$.*

However, if we tried to prove this straightforwardly by using König's lemma as in the proof of the 5-color theorem for enumerable planar graphs in Section 3, we should encounter the following two stumbling blocks:

(i) It is not true that, if G satisfies the above condition involving partitions of $V(G)$, then so does every finite subgraph of G. (In Section 3, it *was* true that every finite subgraph of an enumerable planar graph was planar.)

(ii) It is not true that, if $G_n \subset G_{n+1}$, a set of k edge-disjoint spanning subtrees of G_{n+1} "induces" a set of k edge-disjoint spanning subtrees of G_n (so that it is not clear how we can satisfy Condition (1) of König's lemma).

As far as I am aware, Conjecture A remains unsettled at the present time. It might, however, be worth mentioning that the theory of matroids has recently proved very helpful in throwing additional light, as far as finite graphs are concerned, on Theorems B and C [3, 4, 5, 26], and some approach on these lines might be a good way of attempting to prove Conjecture A.

A slightly different extension of Theorem C to locally finite graphs was, however, obtained by Tutte [42]. Let $\mathfrak{C}(F)$ denote the set of components of a graph F. Define a *spanning semisubtree* of a graph G to be a spanning subforest F of G such that, for every non-empty proper subset $\mathfrak{S}$ of $\mathfrak{C}(F)$, infinitely many edges of G have one end-vertex in a member of $\mathfrak{S}$ and the other in a member of $\mathfrak{C}(F) - \mathfrak{S}$. Evidently a spanning semisubtree of a *finite* graph is the same thing as a spanning subtree of the graph. Tutte proved

THEOREM D. *Let G be a locally finite graph and k be a positive integer. Then G has k edge-disjoint spanning semisubtrees if and only if, for every* *finite partition* $\mathfrak{P}$ *of* $V(G)$, *the number of edges of* G *which join vertices belonging to different members of* $\mathfrak{P}$ *is greater than or equal to* $k(|\mathfrak{P}| - 1)$.

Once again, if one tries to deduce Theorem D from Theorem C by applying König's lemma precisely as in Section 3, one runs into difficulties of the kind already observed: nevertheless, Tutte's proof consisted essentially in applying König's lemma in a somewhat different way. Thus Theorem D illustrates two general points regarding the theory of infinite graphs:

(i) even where König's lemma cannot be applied quite as straightforwardly as in Section 3, it may sometimes happen that König's lemma plus a little ingenuity will extend a theorem from finite to countable graphs;

(ii) there may be more than one generalization to infinite graphs of a given theorem concerning finite graphs. (Both Conjecture A and Theorem D reduce to Theorem C when applied to finite graphs.)

The second of these points will be illustrated again by our discussion of Menger's Theorem in Section 5.

As far as I am aware, no extension of Theorem C to uncountable graphs, or of Theorem B or Theorem C to infinite values of k, has been published. (For countable graphs and $k = \aleph_0$, Theorem B extends trivially.)

Added in Proof. I understand that Professors P. Erdös and A. Hajnal have recently obtained a necessary and sufficient condition for a graph to be the union of k subforests when k is infinite.

5. Menger's Theorem

If A, B are disjoint subsets of $V(G)$, an *AB-walk* is a walk starting at a vertex in A and ending at a vertex in B; and a set S of vertices or edges of G will be said to *separate* A from B if every AB-walk includes an element of S.

Statement of Menger's Theorem. *Let* G *be a finite graph,* k *be a positive integer and* A, B *be disjoint subsets of* $V(G)$. *Then there exists a set of* k *disjoint AB-paths in* G *if and only if no set of less than* k *vertices separates* A *from* B.

To extend this theorem to infinite graphs and to infinite values of k is (as is well known) not difficult. However, Erdös [37, p. 159] has observed that Menger's Theorem can be restated in a different form. For, if G is finite and A, B are disjoint subsets of $V(G)$, we can select a subset X

of $V(G)$ such that X separates A from B but no set of less than $| X |$ vertices of G does so. Then, according to Menger's Theorem, there exists a set of $| X |$ disjoint AB-paths, and obviously each vertex in X must lie on exactly one of these paths. Thus we arrive at the following

RESTATEMENT OF MENGER'S THEOREM. *Let G be finite and A, B be disjoint subsets of $V(G)$. Then there exists a subset X of $V(G)$ which separates A from B and a set of disjoint AB-paths such that each vertex in X lies on exactly one of these paths.*

Erdös has suggested the problem of determining whether this version of Menger's Theorem extends to infinite graphs. I personally would predict that this problem should not be excessively difficult to settle.

A question of a somewhat similar type was asked by Dirac [37, pp. 158–159]. Let A, B be disjoint subsets of $V(G)$ and k be an infinite cardinal number such that no set of less than k edges of G separates A from B. It is well known that in these circumstances there exists a set of k edge-disjoint AB-paths, and Dirac asked whether these paths can always be so chosen that, whenever two of them have vertices in common, these vertices occur in the same order on each path. That this is true when k is finite is easily seen from Theorem C of [1]. However, in a lecture at the 1966 Colloquium on Graph Theory organized by the Bolyai János Mathematical Society of Hungary, B. Zelinka announced a proof that, perhaps somewhat unexpectedly, the answer to Dirac's question concerning infinite values of k is in the negative. (The proceedings of this Colloquium are scheduled to be published in due course.)

It should be added that a theorem similar to Menger's concerning *one-way infinite* paths in a locally finite graph appears in [8]; and various related matters are dealt with in other papers by the same author.

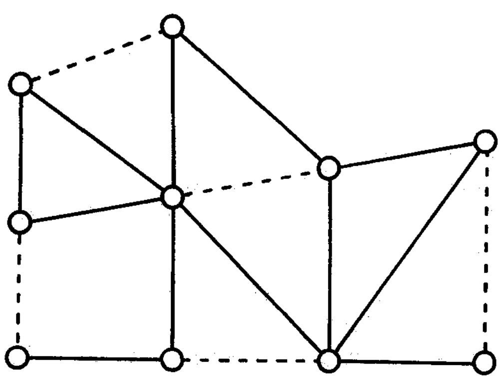

FIGURE 1

6. Factors

A *1-factor* (or *perfect matching*) of G is a subset F of $E(G)$ such that each vertex of G is incident with exactly one edge in F. (In Figure 1, the edges indicated by broken lines constitute a 1-factor of the graph.) Tutte [38] proved

THEOREM E. *A finite graph G has a 1-factor if and only if, for every subset S of $V(G)$, the number of components of $G - S$ which have an odd number of vertices is less than or equal to $|S|$.*

($G - S$ denotes the graph obtained from G by deleting all vertices in S and all edges incident with them.)

If we try to extend this theorem to countable graphs by applying König's lemma *simple-mindedly* as in Section 3, we run into the same kind of difficulties as are encountered in extending Theorem C; but this is another case in which König's lemma *plus* a little ingenuity achieves an extension of the theorem. In fact, Tutte [39] proved by this method

THEOREM F. *A locally finite graph G has a 1-factor if and only if, for every finite subset S of $V(G)$, the number of components of $G - S$ which have an odd finite number of vertices is less than or equal to $|S|$.*

It does not appear to be known whether anything at all closely related to Tutte's condition is necessary and sufficient for the existence of a 1-factor in a graph which is not locally finite. Although Kaluza [11] gives a condition on a general finite or infinite graph which is equivalent to its possessing a 1-factor, this condition is of an entirely different kind from Tutte's.

If f is a cardinal-number-valued function on the vertices of a graph G, an *f-factor* of G is a subset F of $E(G)$ such that every $v \in V(G)$ is incident with exactly $f(v)$ elements of F. Tutte has established a necessary and sufficient condition on G, f for G to possess an f-factor, provided that G is locally finite. His first proof [40] was a complicated one based on the use of "alternating paths," but subsequently [41] he noticed that a locally finite graph G has an f-factor if and only if a graph $\Gamma(G,f)$ constructed from G and f in a certain manner has a 1-factor, and he used this to give a short proof of the f-factor theorem for locally finite graphs. (In fact, Tutte confines attention to finite graphs in [41]; but his argument with very minor adjustments appears to work for infinite locally finite graphs as well.) Nothing much appears to be known about f-factors in graphs which are not locally finite: one can only observe that, in the case in which $f(v)$ is finite for every vertex v of the graph, this problem can be reduced to a 1-factor problem by a slight variant of Tutte's construction of $\Gamma(G,f)$.

7. Infinite Trees

A very difficult unsolved problem in graph theory is to prove or disprove

Kelly's Conjecture. *If G, G' are graphs with at least three vertices and there exists a one-to-one function ϕ from $V(G)$ onto $V(G')$ such that $G - v$ is isomorphic to $G' - \phi(v)$ for every $v \in V(G)$, then G is isomorphic to G'.*

($G - v$ denotes the graph obtained from G by deleting v and all edges incident with it.)

Kelly [12] proved this for finite trees. It might be interesting to try to prove it for infinite trees. Kelly's proof for finite trees depends heavily on having some notion of the "center" of a finite tree, and, since this notion does not apply to infinite trees, it seems likely that some quite different technique of proof might have to be devised.

Kruskal [14] proved that, if $T_1, T_2, \ldots$ is an infinite sequence of finite trees, then there exist i and j such that $i < j$ and T_i is homeomorphic to a subtree of T_j. (A subsequent but independent proof of this result was obtained by Tarkowski and briefly announced in [36].) The proofs discovered by Kruskal and Tarkowski were complicated ones, but a shorter proof was given by the present author in [21]. Finally, the theorem was extended to infinite trees in [23], by a very lengthy and complicated argument. However, it seems worth noting that, in the case of this particular problem, it is the passage from finite to infinite trees which is mainly responsible for the increase in difficulty, and the degree of this difficulty does not seem to be very much reduced by restricting attention to enumerable, or to locally finite, infinite trees. This may be contrasted with the situation in the next problem, which is very easy for countable graphs but undergoes a sharp rise in difficulty when uncountable graphs are considered.

8. Trail and Path Problems

In considering various problems about trails in infinite graphs, the problem strongly suggests itself of characterizing those graphs which are decomposable into closed trails (in the sense defined in Section 2). It is easy to conjecture what the answer might be. For each $X \subset V(G)$, the set of those edges of G which join vertices in X to vertices in $V(G) - X$ is called a *cincture* of G. The *order* of this cincture is the number of edges in it, and a cincture of odd finite order is an *odd* cincture. Since a closed trail crosses over between any subset X of $V(G)$ and the complementary subset $V(G) - X$ an even number of times, it includes an even number of the edges in any cincture of G, and thus, if G is decomposable into closed trails, every cincture of G can be partitioned into finite subsets of even order and so G has no odd cincture. This suggests the conjecture that a graph is decomposable into closed trails *if and only if* it has no odd cincture. What remains to be proved is that, if G has no odd cincture, then it is decomposable into closed trails. If $|E(G)|$ is finite, this turns out to be an immediate consequence of the classical and elementary Euler trail theorem [13, Chapter II]. If $|E(G)| = \aleph_0$, the proof is very easy. For it is easy to show that, in a graph with no odd cincture, every edge belongs to at least one closed trail. Thus we can certainly select a closed trail C_1 in G. Since G has no odd cincture and C_1 includes an even number of the edges in each cincture of G, it follows that $G - E(C_1)$ (the graph obtained by deleting from G the edges in C_1) also has no odd cincture. Repeating the argument, we can select a closed trail C_2 in $G - E(C_1)$ and then a closed trail C_3 in $G - E(C_1) - E(C_2)$ and so forth. If we enumerate the edges of G as $\lambda_1, \lambda_2, \ldots$ and impose on our choices the additional condition that C_n must include λ_n whenever λ_n is not already in one of $C_1, \ldots, C_{n-1}$, it is easy to see how we get a decomposition of G into closed trails.

The difficulty in the case $|E(G)| > \aleph_0$ can perhaps be indicated by considering what would happen if we tried to carry out a procedure like the above by choosing a *transfinite* sequence

$$C_1, C_2, C_3, \ldots, C_\omega, C_{\omega+1}, C_{\omega+2}, \ldots, C_{\omega 2}, \ldots$$

of closed trails from G. There is no difficulty about choosing the C_n ($n < \omega$), but, once we have chosen all of these, we might find that the graph

$$G - \bigcup_{n<\omega} E(C_n)$$

with which we are left has an odd cincture (and so is not decomposable into closed trails), since, although each C_n ($n < \omega$) uses up an even number of the edges in each cincture of G, there might be a cincture of infinite order in G from which the C_n ($n < \omega$) have between them extracted an infinite number of edges leaving an odd finite number behind. A solution of this problem for the case $|E(G)| > \aleph_0$ has in fact been obtained [17]; but, in view of the difficulties just mentioned, it is a much more complicated argument involving a careful examination of the structure of the graph. (An analogous theorem characterizing those *directed* graphs which are decomposable into closed *directed* trails can be proved in a similar manner.) It is thought that techniques like those used in this argument of [17] might prove useful in solving various other problems concerning uncountable graphs: conceivably, even, one might be able to arrive in this way at some sort of "master theorem" which simultaneously extends a whole class of theorems from countable to uncountable graphs.

Necessary and sufficient conditions for a finite graph to have an Eulerian trail are well known and easily proved: this is one of the oldest theorems of graph theory. The corresponding problem for infinite graphs is a little more difficult and complicated, but was solved in 1936 by Erdös, Gallai, and Vázsonyi (see [6]). They established

(i) necessary and sufficient conditions for an enumerable graph to possess a one-way infinite Eulerian trail and

(ii) necessary and sufficient conditions for an enumerable graph to possess a two-way infinite Eulerian trail.

To illustrate the ideas involved, let us look briefly at (ii). It is perhaps illuminating to embed this in the following more general problem: if k is a positive integer, what are necessary and sufficient conditions for a graph to be decomposable into k but not fewer two-way infinite trails? (If we answer this question, our answer when $k = 1$ will solve (ii).) To this end, let us call a graph *Eulerian* if it has no vertices of odd degree (i.e., if the degree of every vertex is even or infinite). Let us call a graph G *l-limited* (where l is a non-negative integer) if G is the union of l disjoint infinite subgraphs and a finite subgraph and is not the union of $l + 1$ disjoint infinite subgraphs and a finite subgraph. Call G *limited* if it is l-limited for some non-negative integer l. An l-limited graph looks rather like something with l infinite "wings" branching out of a finite "center": Figure 2 attempts to illustrate this symbolically for $l = 5$. Moreover a "wing" W of a limited Eulerian graph can be classified as *even* or *odd* according to whether we sever an even or odd number of edges when we make a slash across the graph which, roughly speaking, separates W from the rest of the graph (as indicated by the broken line in Figure 2). If k is a positive integer, a graph G is decomposable into k but not fewer two-way infinite trails if and only if G is enumerable, Eulerian, and limited and has no finite component and $\frac{1}{2}w_o + w_e = k$, where w_o denotes the number of odd wings of G and w_e denotes the number of even wings. A precise statement and a proof of this theorem are given in [20].

An analogous result for directed graphs is stated in [24], which also contains a treatment of the Euler trail problem for infinite directed

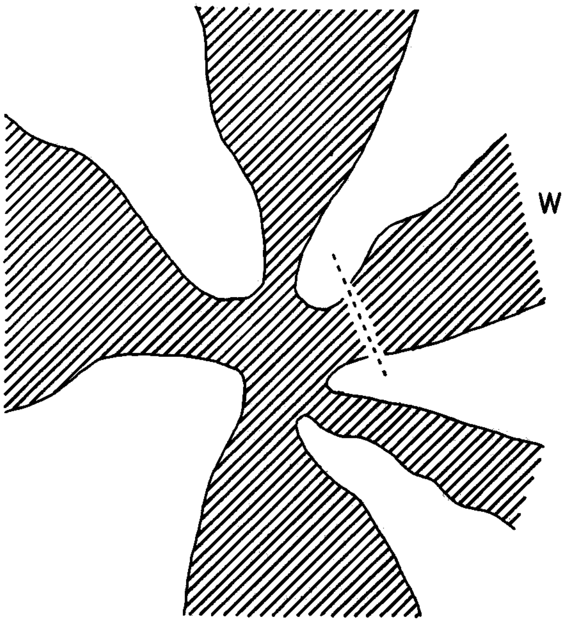

FIGURE 2

graphs. Attention may also be drawn to [32], which treats some further problems concerning decomposition of infinite graphs into trails.

Relatively little is known about the very difficult problem of characterizing finite graphs which possess Hamiltonian paths or circuits or infinite graphs which possess one-way infinite or two-way infinite Hamiltonian paths. One particular case may be worth mentioning. Define the *n-dimensional lattice-graph* L_n to be a graph whose vertices are the integer points of Euclidean n-dimensional space and in which two of these vertices are adjacent if and only if the Euclidean distance between them is 1, each pair of adjacent vertices being joined by one edge only. If $n \geqslant 2$, a simple but ingenious argument of Vázsonyi [43, 16], involving induction on n, shows that L_n has both a one-way infinite and a two-way infinite Hamiltonian path. A more complicated problem is to prove that L_n is decomposable into n two-way infinite Hamiltonian paths of itself: this was proved by Ringel [31] for the case where n is a power of 2 and by the author [18] for general values of n.

Vázsonyi [43] also proved that, for $n \geqslant 2$, the vertices of L_n can be traced out in both a one-way infinite and a two-way infinite sequence of knight's moves so that (in each case) each vertex is visited once and only once. The author [16] generalized this result by showing that a knight can for the purposes of this theorem be replaced by any "chess piece" whose set of allowable moves satisfies the following two conditions:

(i) every vertex of L_n can be reached from the origin in a finite sequence of moves by the piece,

(ii) whenever $\boldsymbol{v}$ is an allowable move of the piece, so is $-\boldsymbol{v}$. (We are here representing moves by vectors in an obvious way.)

The proof of this theorem is not unduly difficult, since it depends essentially on elementary properties of Abelian groups. ($V(L_n)$ can, after all, be regarded as a free Abelian group of rank n.) The question of what happens when we drop condition (ii) appears to be much more difficult, and I have not made any significant progress with it.

A theorem of Pósa [29, 25] concerning Hamiltonian circuits in finite graphs suggests by analogy that it might be interesting (although perhaps difficult) to attempt to determine whether a locally finite 1-limited graph in which no edge joins a vertex to itself and no two edges join the same pair of vertices must necessarily have a one-way and/or a two-way infinite Hamiltonian path if, for every positive integer k, the number of vertices of the graph with degree $\leqslant k$ is less than k (or if some such condition on the degrees of the vertices is satisfied). Mention may also be made of the work of Sekanina [33, 34] concerning the problem: if k is an integer $\geqslant 3$, which infinite graphs have the property that their vertices can be arranged (without repetitions) in an infinite sequence such that the distance between successive vertices in the sequence is always $\leqslant k$?

9. Infinite Graphs and Probability Theory

Some interesting probabilistic questions arise in connection with infinite graphs. For instance, suppose that G is a locally finite connected graph. Let a particle start at a vertex v and perform an infinite sequence of steps, each step being from a vertex to an adjacent one. On the first step, the particle moves with equal probability to any of the vertices adjacent to v. If the vertex to which it in fact moves is v', then the next step takes it with equal probability to any of the vertices adjacent to v'; and so forth. Let $p(v, G)$ denote the probability that the particle revisits v at some time during its motion. It is an easy exercise to show that either $p(v, G) = 1$ for every $v \in V(G)$ or $p(v, G) < 1$ for every $v \in V(G)$: in the former case, we call G *recurrent*. Pólya [28] proved that L_n is recurrent for $n = 1, 2$ and non-recurrent for $n \geqslant 3$. This suggests the problem of distinguishing in general between recurrent and non-recurrent locally finite connected graphs. Intuition (reinforced by Pólya's result) suggests that a graph is likely to be recurrent if and only if "it does not widen out too rapidly as it goes off to infinity," and, in [15], this idea is embodied in a precise theorem. It might be interesting to find some extension of this theorem to the more general problem of a particle moving in a locally finite connected *directed* graph in which the number of outgoing edges from any vertex is equal to the number of edges incoming to that vertex, the particle performing a random walk as before except that, when it is at any vertex, its next step takes it with equal probability along any of the *outgoing* edges from that vertex. However, I have been unable to make any progress toward such a generalization of the theorem of [15].

As a second example, let G be a countable graph and f be a function from $E(G)$ into the closed interval $[0, 1]$. Imagine each edge of G to be a light controlled by an electric circuit so that, when a switch is depressed, all the edges of G *should* light up. However, since all the circuits are defective, any edge e in fact only lights up with probability $f(e)$. Since the edges are all on independent circuits, the lighting up or not lighting up of some edges does not in any way affect the probabilities of other edges lighting up. In these circumstances, one may consider the probability that the subgraph S formed by the edges which light up (and their end-vertices) has an infinite component. According to the zero-one law [2, p. 102], this probability must be 0 or 1, and this suggests the problem of distinguishing between those pairs (G, f) for which the probability is 0 and those for which it is 1. Once again, one would intuitively expect the probability to be zero if and only if G (with each edge e weighted in some way according to the value of $f(e)$) "does not widen out too quickly as it goes off to infinity." However, it appears to be extremely difficult to embody this idea in any general theorem, and even special cases of the problem appear to present severe difficulty. For instance, if we consider the special case in which $f(e) = p$ for every $e \in E(G)$, then there will be a $p_0 \in [0, 1]$ such that probability of S having an infinite component is 1 whenever $p > p_0$ and 0 whenever $p < p_0$ (G being for the present fixed). We call p_0 the *critical probability* for the graph G. The critical probability for L_2 is of some interest to applied mathematicians, and some years ago it was conjectured to be $\frac{1}{2}$. Hammersley [9] proved that this critical probability is $\leqslant 0.646790$ and Harris [10] proved that it is $\geqslant \frac{1}{2}$. The exact evaluation of this critical probability remained apparently very intractable for some while, and although a mathematically non-rigorous argument in [35] arrives at the value $\frac{1}{2}$ for the critical probability for L_2 (and also at values for the critical probabilities for certain other graphs forming regular patterns in the plane), I understand that this result has still not been *proved* in the mathematical sense.[1] There is a

[1] I cannot claim expert familiarity with this topic, and have based some of these observations on discussions with Mr. J. M. Hammersley and a communication from Dr. M. F. Sykes, for both of which I am indebted.

fairly extensive literature on critical probability problems of this type, and we do not attempt to give a full bibliography here.

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