Bifurcation with a two-dimensional kernel

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Abstract

We present a theorem ensuring the existence of local solution branches for one-parameter bifurcation problems in which the linearization at the trivial solution possesses a two-dimensional kernel. In particular, we provide a straightforward “test” that is sufficient for the existence of local solution continua. We demonstrate our abstract theorem with several concrete examples for second-order systems of elliptic partial differential equations with symmetry.

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1. Introduction

Techniques of bifurcation theory are generally of three types—degree-theoretic methods, analytic methods and critical-point methods for gradient systems. The first has the advantage of yielding solution branches (continua), but it is only applicable in problems for which the linearization has an odd crossing number [7], II.4. Analytic methods, based upon the implicit function theorem, are most successful in problems with one-dimensional kernels, typically leading to the existence of solution curves. Critical-point

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methods are applicable to problems with finite-dimensional kernels (even or odd), but have the disadvantage of generally giving only discrete solutions clustering at the bifurcation point, e.g., [3], [7], II.7.

In this work, we present a theorem ensuring the existence of solution branches for one-parameter bifurcation problems with two-dimensional kernels. We were motivated in part by the recent work [1], in which sufficient conditions insuring the existence of branches in gradient bifurcation problems are presented. We focus exclusively on problems with two-dimensional kernels, possibly not having a gradient structure, for which the odd crossing condition typically fails. Analytical methods to obtain bifurcating curves of solutions for problems with two-dimensional kernels, irrespective of gradient structure, are given, e.g., in [7], I.19. But the verification of those hypotheses is generally difficult. Here we require relatively simple hypotheses, the verification of which yields the existence of bifurcating continua. When applied to gradient systems, our hypotheses simplify even more. The results are then closely related to [1], but our hypotheses are slightly more general—albeit restricted to problems having two-dimensional kernels.

It is well known that bifurcation problems with high-dimensional kernels arise “generically” due to symmetries, the exploitation of which in “static” problems typically delivers one-dimensional bifurcation problems. This is the usual setting of the so-called equivariant branching lemma, cf. [2,4]. We provide several concrete examples associated with systems of two coupled second-order elliptic partial differential equations. In each case we end up with bifurcation problems having two-dimensional kernels, i.e., the examples are not “generic” according to the usual rules of equivariant bifurcation theory. This requires some explanation. In specific examples it is common for the linearized problem to possess more symmetry than the full nonlinear system. For example, this is often due in part to the large symmetry group of the Laplace operator. In any case, this can lead to “non-generic” problems. Each of our examples are of this type, viz., the linearized problem is “isotropic”, whereas the nonlinear problem has much less symmetry. Indeed, in the first example, the nonlinearity has no symmetry at all. In the second example, the nonlinearity is equivariant under the product group $C_4 \times D_4$, where $C_4$ denotes the cyclic symmetry group of a square in $\mathbb{R}^2$, and the dihedral group $D_4$ is its complete symmetry group. Here we use well-known group-theoretic methods, yielding three two-dimensional bifurcation problems.

The outline of the work is as follows: In Section 2 we present our bifurcation theorem, giving the existence of local bifurcating continua of solutions. The hypotheses, involving only the “first two” non-vanishing terms in the Taylor expansion of the two-dimensional bifurcation function, are simple and direct. Later in the section we sharpen the theorem by distinguishing between the cases when the order of the second non-vanishing term is even and odd, respectively. We also specialize our theorem to the case of potential operators. In Section 3, we provide several examples coming from systems of elliptic partial differential equations. Motivated by some of the examples in Section 3, we consider abstract two-dimensional problems equivariant under the cyclic group $C_n, n \geq 3$, in Section 4. Our results here are quite general but specialize to those of Ikeda et al. [6] in the case of non-degenerate potential operators.
2. Bifurcation theorems with a two-dimensional kernel

We consider a class of nonlinear problems

\[ F(x, \lambda) = 0, \quad (2.1) \]

where \( F : U \times V \to Z, \) \( 0 \in U \subset X, \lambda_0 \in V \subset \mathbb{R}, \) \( X, Z \) are real Banach spaces, and \( U, V \) are open sets. We assume that

\[
\begin{align*}
F(0, \lambda) &= 0 \quad \text{for all } \lambda \in V, \\
F &\in C^m(U \times V, Z) \quad \text{for some } m \geq 2, \\
\dim N(D_x F(0, \lambda_0)) &= \text{codim } R(D_x F(0, \lambda_0)) = 2.
\end{align*} \tag{2.2}
\]

In particular, \( D_x F(0, \lambda_0) \) is a Fredholm operator of index zero. Here \( N \) denotes the kernel and \( R \) denotes the range. The number \( m \) above is sufficiently large, insuring that the derivatives employed in the sequel exist. We choose complementing spaces such that

\[
\begin{align*}
X &= N(D_x F(0, \lambda_0)) \oplus X_0, \\
Z &= R(D_x F(0, \lambda_0)) \oplus Z_0,
\end{align*} \tag{2.3}
\]

which define continuous projections

\[
\begin{align*}
P : X &\to N \text{ along } X_0 (N = N(D_x F(0, \lambda_0))), \\
Q : Z &\to Z_0 \text{ along } R (R = R(D_x F(0, \lambda_0))).
\end{align*} \tag{2.4}
\]

The well-known method of Lyapunov–Schmidt then reduces problem (2.1) to a two-dimensional bifurcation problem:

\[ \Phi(v, \lambda) = 0 \quad \text{for } (v, \lambda) \in \tilde{U} \times \tilde{V} \subset N \times \mathbb{R}, \quad (2.5) \]

where

\[
\Phi(v, \lambda) = QF(v + \psi(v, \lambda), \lambda) \quad \text{and} \quad (I - Q)F(v + \psi(v, \lambda), \lambda) = 0
\]

for all \((v, \lambda) \in \tilde{U} \times \tilde{V} \subset N \times \mathbb{R}, \)

\[ \Phi(v, \lambda) = (\lambda - \lambda_0)\Phi_{11}v + \Phi_{0k}(v) + R(v, \lambda), \quad \text{where} \]

\[ \Phi_{11}v = QD^2_x F(0, \lambda_0)v \quad \text{is linear in } v \quad \text{and} \]

\[ \Phi_{0k}(v) = \frac{1}{k!}D^k_v F(0, \lambda_0)[v, \ldots, v] \text{ is } k\text{-linear and symmetric in } v. \]

\[ \tag{2.7} \]

cf., e.g., [7], I.2. Note that \( \Phi : \tilde{U} \times \tilde{V} \to Z_0, \) \( 0 \in \tilde{U}, \lambda_0 \in \tilde{V}, \) and \( \Phi(0, \lambda) = 0 \) for all \( \lambda \in \tilde{V}. \) From the Taylor expansion of \( \Phi \) around \((0, \lambda_0)\) we specify

\[
\begin{align*}
\Phi(v, \lambda) &= (\lambda - \lambda_0)\Phi_{11}v + \Phi_{0k}(v) + R(v, \lambda), \quad \text{where} \quad \\
\Phi_{11}v &= QD^2_x F(0, \lambda_0)v \quad \text{is linear in } v \quad \text{and} \quad \\
\Phi_{0k}(v) &= \frac{1}{k!}D^k_v F(0, \lambda_0)[v, \ldots, v] \text{ is } k\text{-linear and symmetric in } v.
\end{align*} \tag{2.7} \]
Here, $R(v, \lambda)$ is the remainder containing all terms of order 0 in $\lambda - \lambda_0$ and of order higher than $k$ in $v$, of order 1 in $\lambda - \lambda_0$ and of order higher than 1 in $v$, and of order higher than 1 in $\lambda - \lambda_0$. Recall that $k \geq 2$ and note that $\Phi_{01} = D_v \Phi(0, \lambda_0) = 0$; cf. [7], I.2. If

$$(I - Q)D^j_x F(0, \lambda_0)[v, \ldots, v] = 0 \quad \text{for } j = 1, \ldots, k - 1$$

and for all $v \in N$, then

$$\Phi_{0k}(v) = \frac{1}{k!} Q D^k_x F(0, \lambda_0)[v, \ldots, v].$$

If (2.8)$_{1,2}$ is not satisfied, then the expression for $\Phi_{0k}$ is more complicated (cf. [7], I.16 for the calculations of the Taylor expansion of $\Phi$). Here we assume that $\Phi_{0k}$ is the first non-vanishing derivative (2.7)$_3$. If the derivatives (2.7)$_3$ vanish for all $k \in \mathbb{N}$ and if there is no flat remainder $R(v, \lambda_0)$, then $\Phi(v, \lambda_0) = 0$ for all $v \in \tilde{U} \subset N$, which implies “vertical” bifurcation.

Before analyzing (2.5) we endow the spaces $N$ and $Z_0$ with a scalar product and an equivalent norm as follows: Let

\[
\{\tilde{v}_1, \tilde{v}_2\} \text{ be a basis in } N \text{ or } Z_0, \\
v = x_1 \tilde{v}_1 + x_2 \tilde{v}_2, \ w = y_1 \tilde{v}_1 + y_2 \tilde{v}_2, \ \text{then} \\
\langle v, w \rangle = x_1 y_1 + x_2 y_2 \text{ and } \|v\|^2 = \langle v, v \rangle.
\]

Next we find it convenient to make the substitutions

$$v = s \tilde{v} \text{ with } \|\tilde{v}\| = 1, \ \lambda = \lambda_0 + s^{k-1} \tilde{\lambda}$$

in (2.7)$_1$, yielding

$$\tilde{\Phi}(\tilde{v}, \tilde{\lambda}, s) = s^k (\tilde{\lambda} \Phi_{11} \tilde{v} + \Phi_{0k}(\tilde{v}) + R_1(\tilde{v}, \tilde{\lambda}, s)), \ for \ s \in (-\delta, \delta),$$

where $R_1(\tilde{v}, \tilde{\lambda}, 0) = 0$.

From [7], I.19, we quote the following conditions to solve $\Phi(\tilde{v}, \tilde{\lambda}, s) = 0$ for $s \neq 0$: If

$$\tilde{\lambda}_0 \Phi_{11} \tilde{v}_0 + \Phi_{0k}(\tilde{v}_0) = 0, \ \|\tilde{v}_0\| = 1, \ \tilde{\lambda}_0 \in \mathbb{R}, \ and$$

$$\tilde{\lambda}_0 \Phi_{11} h + \frac{1}{(k-1)!} D^k_v \Phi(0, \lambda_0)[\tilde{v}_0, \ldots, \tilde{v}_0, h] + \mu \Phi_{11} \tilde{v}_0 = 0, \ \langle \tilde{v}_0, h \rangle = 0,$$

$$\Leftrightarrow h = 0 \text{ and } \mu = 0, \ for \ h \in N, \ \mu \in \mathbb{R},$$

then the Implicit Function Theorem yields a non-trivial curve of solutions of (2.5) through $(0, \tilde{\lambda}_0)$ of the form

$$\{(s \tilde{v}(s), \tilde{\lambda}_0 + s^{k-1} \tilde{\lambda}(s))|s \in (-\delta, \delta)\},$$

where $\tilde{v}(0) = \tilde{v}_0$ and $\tilde{\lambda}(0) = \tilde{\lambda}_0$.
Under our assumption (2.14) below, the solvability (2.11) is a necessary condition for bifurcation from \((0, \lambda_0)\), cf. [7], Remark I.19. If \(\Phi_{11}\) and \(\Phi_{0k}\) have a potential (with respect to the scalar product in \(N = Z_0\)), i.e., if the leading part of the bifurcation function (2.7) has a variational structure, then the necessary condition (2.11) is easily verified provided \(\Phi_{11}\) is positive (or negative) definite: The maximum and the minimum of the potential of \(\Phi_{0k}\) on the ellipse \(\{v \in N | \langle v, \Phi_{11}v \rangle = 1 \text{ (or } -1)\}\) solve (2.11) after a suitable rescaling. The sufficient condition (2.11), for bifurcation, however, is not guaranteed by a variational structure.

Various methods for solving (2.5) non-trivially via the Implicit Function Theorem are given in [7], I.19—in particular, see Remark I.19.1. However, those hypotheses are difficult to verify in practice. We now present our first basic result, the hypotheses of which are simpler to verify. Note that (2.14) precludes an odd crossing number and therefore the use of degree theory.

**Theorem 2.1.** Let \(R_{\pi/2}\) denote the rotation

\[
R_{\pi/2}v = -x_2 \hat{v}_1 + x_1 \hat{v}_2;
\]

observe \(\langle v, R_{\pi/2}v \rangle = 0\) for all \(v \in Z_0\).

Assume that

\[
\Phi_{11} = Q D_x^2 F_1(0, \lambda_0) : N \to Z_0
\]

is an isomorphism,

and that there exist \(\tilde{v}_1, \tilde{v}_2 \in N\) with \(\|\tilde{v}_1\| = \|\tilde{v}_2\| = 1\) such that

\[
\langle \Phi_{0k}(\tilde{v}_1), R_{\pi/2}\Phi_{11}\tilde{v}_1 \rangle < 0,
\]

\[
\langle \Phi_{0k}(\tilde{v}_2), R_{\pi/2}\Phi_{11}\tilde{v}_2 \rangle > 0.
\]

Then there exists a local continuum \(C \subset X \times \mathbb{R}\) of non-trivial solutions of (2.1) through \((0, \lambda_0)\), and \(C \setminus \{(0, \lambda_0)\}\) consists of at least two components.

**Proof.** We solve \(\Phi(v, \lambda) = 0\) near \((0, \lambda_0)\) in \(N \times \mathbb{R}\). With the substitutions (2.10) we obtain (2.10). Defining

\[
\Psi(\tilde{v}, \tilde{\lambda}) = \tilde{\lambda}\Phi_{11}\tilde{v} + \Phi_{0k}(\tilde{v}), \text{ then, by (2.14),}
\]

\[
\tilde{\Phi}(\tilde{v}, \tilde{\lambda}, s) = 0 \text{ for } s \neq 0 \iff
\]

\[
f_1(\tilde{v}, \tilde{\lambda}, s) \equiv \langle \Psi(\tilde{v}, \tilde{\lambda}) + R_1(\tilde{v}, \tilde{\lambda}, s), \Phi_{11}\tilde{v} \rangle = 0\]

and

\[
f_2(\tilde{v}, \tilde{\lambda}, s) \equiv \langle \Psi(\tilde{v}, \tilde{\lambda}) + R_1(\tilde{v}, \tilde{\lambda}, s), R_{\pi/2}\Phi_{11}\tilde{v} \rangle = 0.
\]
For every \( \tilde{v}_0 \in N \) with \( \|\tilde{v}_0\| = 1 \) and for

\[
\tilde{\lambda}_0 = -\frac{\langle \Phi_{0k}(\tilde{v}_0), \Phi_{11}\tilde{v}_0 \rangle}{\|\Phi_{11}\tilde{v}_0\|^2}
\]

we obtain

\[
f_1(\tilde{v}_0, \tilde{\lambda}_0, 0) = \langle \Psi(\tilde{v}_0, \tilde{\lambda}_0), \Phi_{11}\tilde{v}_0 \rangle = 0 \quad \text{and} \quad D_{\tilde{\lambda}}^2 f_1(\tilde{v}_0, \tilde{\lambda}_0, 0) = \|\Phi_{11}\tilde{v}_0\|^2 \neq 0.
\]  

(2.17)

The Implicit Function Theorem gives the existence of a continuous function

\[
\tilde{\lambda} = \tilde{\lambda}(\tilde{v}, s), \quad \text{where} \quad \tilde{\lambda}(\tilde{v}_0, 0) = \tilde{\lambda}_0, \quad \text{so that}
\]

\[
f_1(\tilde{v}, \tilde{\lambda}(\tilde{v}, s), s) = 0 \quad \text{for} \quad \|\tilde{v} - \tilde{v}_0\| < \delta_2, \quad s \in (-\delta_2, \delta_2).
\]  

(2.18)

This can be done for all \( \tilde{v}_0 \in S^1 = \{ \tilde{v} \in N \| \tilde{v} \| = 1 \} \subset N \). Clearly \( \delta_2 \) depends on \( \tilde{v}_0 \), but by compactness of \( S^1 \), we can find a uniform \( \delta_3 > 0 \) such that

\[
f_1(\tilde{v}, \tilde{\lambda}(\tilde{v}, s), s) = 0 \quad \text{for all} \quad \tilde{v} \in S^1, \quad s \in (-\delta_3, \delta_3).
\]  

(2.19)

In order to solve

\[
g(\tilde{v}, s) \equiv f_2(\tilde{v}, \tilde{\lambda}(\tilde{v}, s), s) = 0
\]

for \( \tilde{v} \in S^1 \) with \( s \) near 0,

we observe that \( g \) is continuous,

\[
g(\tilde{v}, 0) = \langle \Phi_{0k}(\tilde{v}), R_{\pi/2}\Phi_{11}\tilde{v} \rangle, \quad \text{and by (2.15)}
\]

\[
g(\tilde{v}_1, 0) < 0 \quad \text{and} \quad g(\tilde{v}_2, 0) > 0.
\]  

(2.21)

By (2.21)1 the function \( g(\cdot, 0) \) is a \((k + 1)\)-linear mapping from \( N \) into \( \mathbb{R} \), i.e., it is a homogeneous polynomial of two variables of order \( k + 1 \). By (2.21)2 it has a zero on \( S^1 \), but its non-degeneracy for the application of the Implicit Function Theorem to solve (2.20) is hard to verify, in general.

Denote by \([\tilde{v}_1, \tilde{v}_2]\) the segment on \( S^1 \) with a counterclockwise orientation. By continuity, there is some \( \delta_3 \geq \delta_4 > 0 \) such that

\[
g(\tilde{v}_1, s) \leq d_1 < 0 \quad \text{and} \quad g(\tilde{v}_2, s) \geq d_2 > 0
\]

for all \( s \in [-\delta_4, \delta_4] \).

(2.22)

By the mean value theorem, the set

\[
\tilde{S} = \{(\tilde{v}, s) \in [\tilde{v}_1, \tilde{v}_2, ] \times [-\delta_4, \delta_4] | g(\tilde{v}, s) = 0\}
\]  

(2.23)
is not empty. We claim that $\tilde{S}$ contains a continuum $\tilde{C}$ that connects the “bottom” $B = [\tilde{v}_1, \tilde{v}_2] \times \{-\delta_4\}$ and the “top” $T = [\tilde{v}_1, \tilde{v}_2] \times \{\delta_4\}$. To see this, let $\tilde{C}^-$ denote the component of $B$ in $\tilde{S} \cup B$. Assume that $\tilde{C}^- \cap T = \emptyset$.

Let $W$ be an open neighborhood of $\tilde{C}^-$ in $N \times \mathbb{R}$ such that $W \cap T = \emptyset$. Define $K = W \cap (\tilde{S} \cup B)$. Then $K$ is compact, $\tilde{C}^- \subset K$, and $\partial W \cap \tilde{C}^- = \emptyset$. By the so-called Whyburn–Lemma [8] there exist compact subsets $K_1, K_2 \subset K$ satisfying

$$K_1 \cap K_2 = \emptyset, K_1 \cup K_2 = K, \quad \tilde{C}^- \subset K_1, \partial W \cap \tilde{S} \subset K_2.$$ (2.24)

Choose an open neighborhood $W_1$ of $K_1$ such that

$$\tilde{C}^- \subset K_1 \subset W_1 \subset W, \quad \bar{W}_1 \cap K_2 = \emptyset, \quad \bar{W}_1 \cap T = \emptyset, \quad \text{and} \quad \partial W_1 \cap \tilde{S} = \emptyset.$$ (2.25)

The connected set $L = \partial W_1 \cap ([\tilde{v}_1, \tilde{v}_2] \times \{-\delta_4, \delta_4\})$ (which can be taken to be a continuous curve w.l.o.g.) connects two points $(\tilde{v}_1, s_1)$ and $(\tilde{v}_2, s_2)$ with $s_1, s_2 \in (-\delta_4, \delta_4)$ on the sides of the “rectangle” $[\tilde{v}_1, \tilde{v}_2] \times [-\delta_4, \delta_4]$. Obviously $L$ is completely contained in that rectangle. But then by (2.25)$_2$, $L \cap \tilde{S} = \emptyset$, which, in view of (2.22), contradicts the mean value theorem.

Therefore $\tilde{C}^- \cap T \neq \emptyset$, which proves that $\tilde{S}$ contains a continuum $\tilde{C}$ that connects the bottom $B$ and the top $T$. The set

$$C = \{(v, \lambda) | v = s\tilde{v}, \lambda = \lambda_0 + s^{k-1} \tilde{\lambda}(\tilde{v}, s), (\tilde{v}, s) \in \tilde{C}\}$$ (2.26)

is then a continuum of solutions of $\Phi(v, \lambda) = 0$ that contains $(0, \lambda_0)$. Furthermore, $C \setminus \{(0, \lambda_0)\}$ consists only of non-trivial solutions and is not connected. Since the Lyapunov–Schmidt reduction preserves connectedness of (local) solutions, Theorem 2.1 is proved. □

**Remark 2.2.** (i) Assumption (2.15) can be weakened as follows: if

$$(\tilde{\Phi}(\tilde{v}, \lambda(\tilde{v}, s), s), R_{\pi/2} \Phi_{11} \tilde{v}) = s^{\tilde{k}} g_{\tilde{k}+1}(\tilde{v}) + o(s^{\tilde{k}}),$$

with some $\tilde{k} > k$, and if (2.15) is replaced by

$$g_{\tilde{k}+1}(\tilde{v}_1) < 0, \quad g_{\tilde{k}+1}(\tilde{v}_2) > 0,$$

then the same proof gives the statement of Theorem 2.1. Note that in this case $g_{\tilde{k}+1}(\tilde{v})$ is possibly no longer a homogeneous polynomial of order $\tilde{k} + 1$ as it is for $\tilde{k} = k$, where $g_{\tilde{k}+1}(\tilde{v}) = g(\tilde{v}, 0)$, cf. (2.21).
(ii) If $\Phi(v, \lambda) = (\lambda - \lambda_0)\Phi_{11} v + \Phi_{0k}(v) + R(v)$, where the remainder $R$ does not depend on $\lambda$, then

$$g_{k+1}(v) = \langle \Phi_{0k}(v), R_{\pi/2}\Phi_{11} v \rangle, \quad \Phi_{0k}(v) = \frac{1}{k!} D_v^k \Phi(0, \lambda_0)[\tilde{v}, \ldots, \tilde{v}].$$

We now sharpen Theorem 2.1, distinguishing between the cases when $k$ is odd and even:

**Corollary 2.3.** If $k$ is odd, then under the hypotheses of Theorem 2.1 there exist at least two local continua $C \subset X \times \mathbb{R}$ of non-trivial solutions of (2.1) through $(0, \lambda_0)$, and each $C \setminus \{(0, \lambda_0)\}$ consist of at least two components, respectively.

If $k$ is even then assumption (2.15) can be reduced to $\langle \Phi_{0k} (v_1), R_{\pi/2}\Phi_{11} \tilde{v}_1 \rangle \neq 0$ for some $\tilde{v}_1 \in S^1$ (or equivalently $\Phi_{0k}(v_1)$ and $\Phi_{11}\tilde{v}_1$ are linearly independent), and the conclusion of Theorem 2.1 holds true.

**Proof.** Assume that $k$ is odd. Then the $(k+1)$-linear form $\langle \Phi_{0k} (v), R_{\pi/2}\Phi_{11} \tilde{v} \rangle$ is even, so that the assumption (2.15) is true also for the antipodal vectors $-\tilde{v}_1, -\tilde{v}_2$. Therefore the proof of Theorem 2.1 applies to the 4 segments $[\tilde{v}_1, \tilde{v}_2], [\tilde{v}_2, -\tilde{v}_1], [-\tilde{v}_1, -\tilde{v}_2]$, and $[-\tilde{v}_2, \tilde{v}_1]$ on $S^1$ yielding 4 continua of the form (2.26) arising in each of the 4 segments. However, two antipodal segments, respectively, provide the same continuum by the following symmetries: By definitions (2.10) and (2.16) we obtain for odd $k$

$$\tilde{\Phi}(-\tilde{v}, \tilde{\lambda}, -s) = \tilde{\Phi}(\tilde{v}, \tilde{\lambda}, s),$$

$$f_1(-\tilde{v}, \tilde{\lambda}, -s) = f_1(\tilde{v}, \tilde{\lambda}, s) \quad \text{whence}$$

$$\tilde{\lambda}(-\tilde{v}, -s) = \tilde{\lambda}(\tilde{v}, s) \quad \text{by (2.19), and}$$

$$g(-\tilde{v}, -s) = g(\tilde{v}, s) \quad \text{cf. (2.20)}.$$ (2.27)

The zeros of $g$ exist in antipodal pairs $\{(\tilde{v}, s), (-\tilde{v}, -s)\}$ which, by (2.27)$_3$, yield only one continuum $C$ of the form (2.26) for $s \in [-\delta_4, \delta_4]$.

Assume now that $k$ is even. Then the $(k+1)$-linear form $\langle \Phi_{0k} (v), R_{\pi/2}\Phi_{11} \tilde{v} \rangle$ is odd, so that assumption (2.15) is satisfied by an antipodal pair $\tilde{v}_1, \tilde{v}_2 = -\tilde{v}_1$, provided the form does not vanish on $S^1$. The two antipodal segments $[\tilde{v}_1, -\tilde{v}_1], [-\tilde{v}_1, \tilde{v}_1]$ on $S^1$ provide the same continuum by the following symmetries: By definitions (2.10) and (2.16) we obtain for even $k$

$$\tilde{\Phi}(-\tilde{v}, \tilde{\lambda}, -s) = \tilde{\Phi}(\tilde{v}, \tilde{\lambda}, s),$$

$$f_1(-\tilde{v}, -\tilde{\lambda}, -s) = -f_1(\tilde{v}, \tilde{\lambda}, s) \quad \text{whence}$$

$$\tilde{\lambda}(-\tilde{v}, -s) = -\tilde{\lambda}(\tilde{v}, s) \quad \text{by (2.19), and}$$

$$g(-\tilde{v}, -s) = -g(\tilde{v}, s) \quad \text{cf. (2.20).}$$ (2.28)

The antipodal pairs $\{(\tilde{v}, s), (-\tilde{v}, -s)\}$ of zeros of $g$ give, by (2.28)$_3$, only one continuum $C$ of the form (2.26) for $s \in [-\delta_4, \delta_4]$. □
The following corollary gives an alternative condition that is more convenient for the applications in Sections 3 and 4.

**Corollary 2.4.** Assume (2.14) and that there exists a \( \tilde{v}_1 \in N \) with \( \| \tilde{v}_1 \| = 1 \) such that

\[
\Phi_{0k}(\tilde{v}_1) \text{ and } \Phi_{11}\tilde{v}_1 \text{ are linearly independent.}  
\]  

(2.29)

Let \( w(t) \) for \( t \in [0, 2\pi] \) be a parametrization of \( S^1 \subset N \). If

\[
\int_0^{2\pi} \langle \Phi_{0k}(w(t)), R_{\pi/2}\Phi_{11}w(t) \rangle \, dt = 0,  
\]  

(2.30)

then the statements of Theorem 2.1 and of Corollary 2.3 hold.

**Proof.** It is trivial that (2.29) and (2.30) imply (2.15): By (2.29) the integrand of (2.30) does not vanish identically, and therefore (2.30) implies (2.15). \( \square \)

Next, we show that a modification of Corollary 2.4 is applicable to a potential operator. Assume that \( X \subset Z \) is continuously embedded and that \( \langle \cdot, \cdot \rangle \) is a continuous scalar product on \( Z \). Then \( F(\cdot, \lambda): U \to Z, U \subset X \), is a potential operator, if there exists a function \( f \in C^1(U \times V, \mathbb{R}) \) such that

\[
D_x f(x, \lambda)h = \langle F(x, \lambda), h \rangle \text{ for all } (x, \lambda) \in U \times V, h \in X. 
\]  

(2.31)

The function \( f(\cdot, \lambda) \) is the potential of \( F(\cdot, \lambda) \) with respect to the scalar product \( \langle \cdot, \cdot \rangle \). If \( f \in C^2(U \times V, \mathbb{R}) \) then \( F \in C^1(U \times V, Z) \), and \( D_x F(x, \lambda) \) is symmetric with respect to \( \langle \cdot, \cdot \rangle \) for all \( (x, \lambda) \in (U \times V) \), cf. [7], I.3. Therefore we may assume:

\[
Z = R(D_x F(0, \lambda_0)) \oplus N(D_x F(0, \lambda_0)), \text{ where} \\
R \text{ and } N \text{ are orthogonal with respect to} \\
\text{the scalar product } \langle \cdot, \cdot \rangle \text{ on } Z, \\
Q: Z \to N \text{ along } R, \\
P: X \to N \text{ along } R \cap X; \text{i.e., } P = Q|_X. 
\]  

(2.32)

As shown in [7], I.3., e.g., the bifurcation function \( \Phi(\cdot, \lambda) \) is also a potential operator and

\[
\varphi(v, \lambda) = f(v + \psi(v, \lambda), \lambda) \text{ (see (2.6))} \\
\text{is the potential of } \Phi \text{ with respect to } \langle \cdot, \cdot \rangle \text{ on } N; \text{i.e.,} \\
D_v \varphi(v, \lambda)h = \langle \Phi(v, \lambda), h \rangle \text{ for all } (v, \lambda) \in \bar{U} \times \bar{V}, h \in N. 
\]  

(2.33)

The following theorem is essentially due to [1].
Theorem 2.5. Assume that $F(\cdot, \lambda)$ is a potential operator in the sense of (2.31) and that
\[ \Phi_{11} = QD^2_{x\lambda} F(0, \lambda_0) : N \to N \]
is positive (or negative) definite. (2.34)

If there exists a $\tilde{v}_1 \in N$ with $\|\tilde{v}_1\| = 1$ such that
\[ \Phi_{0k}(\tilde{v}_1) \text{ and } \Phi_{11}\tilde{v}_1 \text{ are linearly independent}, \] (2.35)
then the statements of Theorem 2.1 and Corollary 2.3 hold.

Proof. We use the scalar product $\langle \cdot, \cdot \rangle$ of $\mathbb{Z}$ restricted to $N = \mathbb{Z}_0$. If $\{\hat{v}_1, \hat{v}_2\}$ is an orthonormal basis in $N$ with respect to $\langle \cdot, \cdot \rangle$, then $\langle \cdot, \cdot \rangle$ is the same as the scalar product defined in (2.9). Let $E_1 = \{\tilde{v} \in N | \tilde{v}, \Phi_{11}\tilde{v} = 1 \text{ (or } -1)\}$ and let $w(t)$ for $t \in [0, 2\pi]$ be a parametrization of the ellipse $E_1$. Then, by the symmetry of $\Phi_{11}$, $\frac{d}{dt} \langle w(t), \Phi_{11} w(t) \rangle = 2\langle \dot{w}(t), \Phi_{11} w(t) \rangle = 0$ or $\dot{w}(t) = z(t)R_{\pi/2}\Phi_{11} w(t)$ for some $2\pi$-periodic $z(t) \neq 0$. Then
\[
\int_0^{2\pi} z(t)\langle \Phi_{0k}(w(t)), R_{\pi/2}\Phi_{11} w(t) \rangle dt \\
= \int_0^{2\pi} \frac{1}{k!} \langle D^k_v\Phi(0, \lambda_0)(w(t)), \dot{w}(t) \rangle dt \quad (\cdot = \frac{d}{dt}) \\
= \int_0^{2\pi} \frac{1}{k!} D^{k+1}_v\Phi(0, \lambda_0)[w(t), \ldots, w(t), \dot{w}(t)] dt \quad \text{(by (2.33))} \\
= \int_0^{2\pi} \frac{1}{(k + 1)!} \frac{d}{dt} D^{k+1}_v\Phi(0, \lambda_0)[w(t), \ldots, w(t)] dt = 0 \text{ by periodicity.} \]
(2.36)

Since $z(t) \neq 0$ for $t \in [0, 2\pi]$, we can draw the same conclusion from (2.35), (2.36) as from (2.29) and (2.30), cf. the proof of Corollary 2.4. \[\Box\]

Condition (2.35) implies that the $(k+1)$-linear functional
\[ D^{k+1}_v\Phi(0, \lambda_0) : N \to \mathbb{R} \]
is not constant on $S^1 \subset N$. (2.37)

Here $\phi$ is the potential of $\Phi$ given in (2.33). If (2.8) is satisfied, then
\[ D^{k+1}_v\phi(0, \lambda_0)[v, \ldots, v] = D^{k+1}_x f(0, \lambda_0)[v, \ldots, v] \text{ for } v \in N, \]
where $f$ is the potential of $F$ defined in (2.31).
3. Examples

We consider the system

\[
\begin{align*}
\Delta u + f(u, \lambda) &= 0 \text{ in } \Omega \subset \mathbb{R}^n, \\
u &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

(3.1)

where \( u = (u_1, u_2) \), \( u(x) = u(x_1, \ldots, x_n) \), \( \Delta u = (\Delta u_1, \Delta u_2) \), and \( \Delta \) denotes the scalar Laplacian. The domain \( \Omega \subset \mathbb{R}^n \) is bounded and the boundary \( \partial \Omega \) is smooth. We assume that the vector field

\[
f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \text{ is in } C^{k+1}(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2),
\]

(3.2)

\[
f(0, \lambda) = 0 \quad \text{for all } \lambda \in \mathbb{R}, \text{ and }
\]

\[
D_uf(0, \lambda) = \lambda I \quad \text{where } I \in \mathbb{R}^{2 \times 2} \text{ denotes the identity matrix.}
\]

As expounded in [7], III.1,

\[
\Lambda + \lambda I : C^{2,\alpha}(\overline{\Omega}) \cap \{ u | u|_{\partial \Omega} = 0 \} \to C^{\alpha}(\overline{\Omega})
\]

(3.3)

is a Fredholm operator of index zero.

The same holds true for

\[
\Lambda + \lambda I : X \to Z, \text{ where }
\]

(3.4)

\[
X = (C^{2,\alpha}(\overline{\Omega}) \cap \{ u | u|_{\partial \Omega} = 0 \})^2, \quad Z = (C^{\alpha}(\overline{\Omega}))^2.
\]

Then

\[
F(u, \lambda)(x) \equiv \Delta u(x) + f(u(x, \lambda))
\]

defines a mapping

\[
F : X \times \mathbb{R} \to Z \text{ of class } C^k(X \times \mathbb{R}, Z).
\]

(3.5)

Since \( D_uf(0, \lambda_0) = \Lambda + \lambda I \), cf. (3.2)\), we find a two-dimensional kernel corresponding to the first eigenvalue \( \lambda_0 \) of \( \Lambda \); i.e.,

\[
N(D_uf(0, \lambda_0)) \text{ is spanned by }
\]

\[
\hat{v}_1 = (v_0, 0) \text{ and } \hat{v}_2 = (0, v_0), \text{ where }
\]

(3.6)

\[
v_0 \text{ is the first positive eigenfunction of the principal eigenvalue } \lambda_0 \text{ of } -\Delta \text{ over } \Omega \text{ subject to homogeneous Dirichlet boundary conditions.}
\]
Furthermore, as expounded in [7], III.1,

\[ Z = R(D_u F(\mathbf{0}, \lambda_0)) \oplus N(D_u F(\mathbf{0}, \lambda_0)) \]

defining orthogonal projections with respect to the scalar product \( \langle \cdot, \cdot \rangle_0 \) in \( (L^2(\Omega))^2 \), namely

\[ \langle u, v \rangle_0 = \int_\Omega (u_1 v_1 + u_2 v_2) \, dx, \quad (3.7) \]

\[ Q : Z \to N \text{ along } R, \quad P : X \to N \text{ along } R \cap X, \quad P = Q|_X, \]

\[ Q u = \langle u, \hat{v}_1 \rangle_0 \hat{v}_1 + \langle u, \hat{v}_2 \rangle_0 \hat{v}_2, \quad \text{provided } \int_\Omega v_0^k \, dx = 1. \]

Assuming

\[ f(u, \lambda) = \lambda u + f_{0k}(u) + R(u, \lambda) \]

for some homogeneous polynomial \( f_{0k} : \mathbb{R}^2 \to \mathbb{R}^2 \) of order \( k \geq 2 \),

we obtain, by (2.8) and using (2.9), (2.13) for the basis \( \{\hat{v}_1, \hat{v}_2\} \) of \( N \),

\[ \Phi_{0k}(v) = Q f_{0k}(v) \quad \text{for } v = w_1 \hat{v}_1 + w_2 \hat{v}_2 \in N, \]

\[ \langle \Phi_{0k}(v), R_{\pi/2} v \rangle \]

\[ = -w_2 \int_\Omega f_{0k}^1(w_1 v_0, w_2 v_0) v_0 \, dx + w_1 \int_\Omega f_{0k}^2(w_1 v_0, w_2 v_0) v_0 \, dx \]

\[ = (-f_{0k}^1(w_1, w_2) w_2 + f_{0k}^2(w_1, w_2) w_1) \int_\Omega v_0^{k+1} \, dx \]

\[ = f_{0k}(w) \cdot R_{\pi/2} w \int_\Omega v_0^{k+1} \, dx, \quad w = (w_1, w_2). \quad (3.9) \]

In (3.9) \( \cdot \) is the Euclidean scalar product in \( \mathbb{R}^2 \) and \( R_{\pi/2} \) is a rotation about \( \pi/2 \) in \( \mathbb{R}^2 \). Since \( v_0 > 0 \) in \( \Omega \), our crucial condition (2.15) is satisfied in the following cases:

If \( k \) is odd, assume the existence of

\[ \mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^2 \text{ with } \|\mathbf{u}_1\| = \|\mathbf{u}_2\| = 1 \text{ and} \]

\[ f_{0k}(\mathbf{u}_1) \cdot R_{\pi/2} \mathbf{u}_1 < 0, \]

\[ f_{0k}(\mathbf{u}_2) \cdot R_{\pi/2} \mathbf{u}_2 < 0. \quad (3.10) \]

If \( k \) is even, assume the existence of

an \( \mathbf{u} \in \mathbb{R}^2 \) with \( \|\mathbf{u}\| = 1 \) and

\[ f_{0k}(\mathbf{u}) \cdot R_{\pi/2} \mathbf{u} \neq 0. \]

If \( k \) is odd, then \( f_{0k}(u) \cdot R_{\pi/2} u \) is a homogeneous scalar polynomial for \( u \in \mathbb{R}^2 \) of even order \( k + 1 \). Choosing the parametrization \( w(t) = (\cos t, \sin t) \) of \( S^1 \subset \mathbb{R}^2 \), this polynomial is a sum of terms \( a_{m\ell} \cos^m t \sin^\ell t \) with \( m, \ell \in \mathbb{N} \cup \{0\} \) and \( m + \ell = k + 1 \).

If both \( m \) and \( \ell \) are even and \( a_{m\ell} \neq 0 \), then the term adds a non-zero contribution
to the integral \( \int_0^{2\pi} f_{0k}(w(t)) \cdot R_{\pi/2} w(t) \, dt \). On the other hand, if both \( m \) and \( \ell \) are odd, then its contribution is zero. Thus for \( k \) odd, it suffices to find a \( \tilde{u} \in S^1 \) with \( f_{0k}(\tilde{u}) \cdot R_{\pi/2} \tilde{u} \neq 0 \) (as in the case of \( k \) even) and to verify in addition that the above integral vanishes. Then Corollary 2.4 applies. Simple examples are

\[
\begin{align*}
\left( \begin{array}{c} x \\ y \\ \beta \\ \delta \end{array} \right) \quad \text{with the only condition } (x, \beta, y, \delta) \neq (0, 0, 0, 0) \text{ for even } k, \\
\text{and the additional condition } \beta = \gamma \text{ for odd } k, \\
f_{02}(u) &= \left( \begin{array}{c} u_1^2 - u_2^2 \\ u_1 u_2 \end{array} \right). 
\end{align*}
\]

The last example has the peculiarity that the two zeros of \( \langle \Phi_{02}(\tilde{v}), R_{\pi/2} \tilde{v} \rangle = g(\tilde{v}, 0) = w_1^3 \int_\Omega v_0^3 \, dx \) for \( \tilde{v} = w_1 \hat{v}_1 + w_2 \hat{v}_2 \in S^1 \subset N \), i.e., \( w_1^2 + w_2^2 = 1 \), are degenerate, and the Implicit Function Theorem is not applicable to solve (2.20).

The application of Theorem 2.1 and of Corollary 2.3 provides one or two continua bifurcating from the trivial solution line \( \{ (0, \lambda) | \lambda \in \mathbb{R} \} \) at \( (0, \lambda_0) \) for all cases (3.10), (3.11) depending on whether \( k \) is even or odd, respectively.

We remark that the computations (3.9) are also valid for any other simple eigenvalue \( \lambda_0 \) of \(-\Delta\) with an eigenfunction \( v_0 \) satisfying homogeneous Dirichlet boundary conditions. Assumptions (3.10) then imply condition (2.15) provided that \( \int_\Omega v_0^{k+1} \, dx \neq 0 \). This is always true if \( k \) is odd.

Of course the higher eigenvalues of \(-\Delta\) (subject to homogeneous Dirichlet boundary conditions) need not be simple, in which case the dimension of the kernel of the linearization (3.4) is greater than two. In particular, this occurs when the domain \( \Omega \) has symmetries, as can be seen in our next example: Namely, we now study the system (3.1) over the square \( \Omega = (-\pi, \pi) \times (-\pi, \pi) \subset \mathbb{R}^2 \). We assume that the vector field \( f(\cdot, \lambda) \) has the symmetry

\[
\begin{align*}
\left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) = R_{\pi/2} \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right), \\
\end{align*}
\]

where \( R_{\pi/2} = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) is a rotation about \( \pi/2 \) in \( \mathbb{R}^2 \).

For a functional analytic setting of (3.1) over the square \( \Omega \), we define

\[
X_D = \{ u : \mathbb{R}^2 \to \mathbb{R} | u(x_1, x_2) = u(x_1 + 4\pi, x_2) = u(x_1, x_2 + 4\pi), \\
u(\pi + x_1, x_2) = -u(\pi, x_1, x_2), u(x_1, \pi + x_2) = -u(x_1, \pi - x_2) \}. 
\]

The subscript “\( D \)” refers to the fact that all functions \( u \in C(\mathbb{R}^2) \cap X_D \) satisfy homogeneous Dirichlet boundary conditions \( u = 0 \) on \( \partial \Omega \).
As expounded in [7], III.1,

\[
\Delta + \lambda I : C^{2,\varphi}(\mathbb{R}^2) \cap X_D \to C^{\varphi}(\mathbb{R}^2) \cap X_D
\]

is a Fredholm operator of index zero.

(3.14)

The same holds true for

\[
\Delta + \lambda I : X \to Z, \quad \text{where} \quad X = (C^{2,\varphi}(\mathbb{R}^2) \cap X_D)^2, \quad Z = (C^{\varphi}(\mathbb{R}^2) \cap X_D)^2.
\]

(3.15)

By virtue of (3.12), we find

\[
f(\cdot, \lambda) : X^2 \to X^2, \quad \text{and} \quad F(u, \lambda)(x_1, x_2) = \Delta u(x_1, x_2) + f(u(x_1, x_2), \lambda)
\]

defines a mapping

\[
F : X \times \mathbb{R} \to Z \quad \text{of class} \quad C^k(X \times \mathbb{R}, Z).
\]

(3.16)

Then

\[
D_u F(0, \lambda) = \Delta + \lambda I : X \to Z, \quad \text{and for} \quad \lambda_0 = \frac{5}{4} \quad \text{(the second eigenvalue of} \quad -\Delta),
\]

\[
N(D_u F(0, \lambda_0)) \quad \text{is spanned by}
\]

\[
\hat{v}_1 = \frac{1}{2\pi} \sqrt{2} (\cos \frac{1}{2} x_1 \sin x_2, -\sin x_1 \cos \frac{1}{2} x_2),
\]

\[
\hat{v}_2 = \frac{1}{2\pi} \sqrt{2} (\sin x_1 \cos \frac{1}{2} x_2, \cos \frac{1}{2} x_1 \sin x_2),
\]

\[
\hat{v}_3 = \frac{1}{2\pi} \sqrt{2} (\cos \frac{1}{2} x_1 \sin x_2, \sin x_1 \cos \frac{1}{2} x_2),
\]

\[
\hat{v}_4 = \frac{1}{2\pi} \sqrt{2} (\sin x_1 \cos \frac{1}{2} x_2, -\cos \frac{1}{2} x_1 \sin x_2).
\]

(3.17)

The kernel provides an orthogonal Lyapunov–Schmidt decomposition as in (3.7) with

\[
Q u = \sum_{j=1}^{4} \langle u, \hat{v}_j \rangle \hat{v}_j; \quad \text{cf.} \quad [7], \quad \text{III.1. Note that} \quad \{\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4\} \quad \text{is an orthonormal basis in} \quad N \quad \text{with respect to} \quad \langle \cdot, \cdot \rangle_0. \quad \text{The size of the kernel is a consequence of the symmetries of our problem, which we now discuss.}
\]

Let \(C_4 \cong \mathbb{Z}_4\) be the cyclic group generated by \(R_{\pi/2}\) and let \(D_4 \subset O(2)\) be the complete symmetry group of the square. Then the group \(C_4 \times D_4\) has a representation \(\gamma\) on the function space \(Z\) defined by

\[
\gamma(R, S)u(x) = Ru(S^T x) \quad \text{for all} \quad u \in Z \quad \text{("T" denotes transposed)},
\]

and for \((R, S) \in C_4 \times D_4\).

(3.18)

A straightforward calculation shows that (3.12) implies the equivariance

\[
F(\gamma(R, S)u, \lambda) = \gamma(R, S)F(u, \lambda) \quad \text{for all} \quad (u, \lambda) \in X \times \mathbb{R},
\]

\[
Q \gamma(R, S)u = \gamma(R, S)Q u \quad \text{for} \quad u \in Z, \quad (R, S) \in C_4 \times D_4.
\]

(3.19)
Accordingly, the reduced mapping (2.5) is also equivariant
\[ \Phi(\gamma(R, S)v, \lambda) = \gamma(R, S)\Phi(v, \lambda) \text{ for all } (v, \lambda) \in N \times \mathbb{R} \]
near \((0, \lambda_0), (R, S) \in C_4 \times D_4\). (3.20)

Let \(\tilde{\gamma}\) denote the restriction \(\gamma(\cdot)|_N\). Recall that the isotropy subgroup of \(\tilde{\gamma}\) at \(v\) is the largest subgroup that fixes \(v \in N\). An isotropy subgroup \(\Sigma\) is maximal if there is no proper isotropy subgroup containing \(\Sigma\). It is not hard to show that each of the following non-equivalent subgroups (up to conjugacy) of \(C_4 \times D_4\) correspond to maximal isotropy subgroups:

\[
\begin{align*}
\Sigma_1 &= \{(I, I), (I, E_1), (-I, -E_1), (-I, -I)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \\
\Sigma_2 &= \{(I, I), (I, E_2), (-I, -E_2), (-I, -I)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \\
\Sigma_3 &= \{(R, R)|R \in C_4\} \cong C_4,
\end{align*}
\]

(3.21)

where \(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, E_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \) and \(E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). Note that all three subgroups \(\Sigma_j, j = 1, 2, 3\), act as \(\mathbb{Z}_2\) on \(N\) since the action of \((-I, -I) = (R_\pi, R_\pi)\) is trivial on \(N\). Given an isotropy subgroup \(\Sigma\), recall that the fixed-point space of \(\Sigma\) acting on \(N\) is the linear space \(\text{Fix}_N(\Sigma) = \{v \in N | \tilde{\gamma}(g)v = v \text{ for all } g \in \Sigma\}\). As we show below, each of the fixed-point subspaces of \(\Sigma_j, j = 1, 2, 3\), (acting on \(N\)) are two-dimensional. One of the benefits of equivariance (3.20) is that fixed-point spaces are invariant under the nonlinear mapping \(\Phi(\cdot, \lambda)\). However, the two-dimensionality of \(\text{Fix}_N(\Sigma_j), j = 1, 2, 3\), precludes the use of the standard equivariant branching lemma [2,4], i.e., we cannot reduce further to one-dimensional bifurcation problems via symmetry (or any other means).

We now apply Theorem 2.1 and its corollaries to the analysis of the reduced problems associated with \(\Sigma_j, j = 1, 2, 3\), cf. (3.21).

First note that by assumption (3.2)3, the bifurcation function \(\Phi\) is of the form (2.7), where \(\Phi_{11} v = QD_{u,0}F(0, \lambda_0)v = v\) for all \(v \in N\). For each \(\Sigma_j\), we have

\[
\begin{align*}
\text{Fix}_X(\Sigma_j) &= \{u \in X | \gamma(g)u = u \text{ for all } g \in \Sigma_j\}, \\
\text{Fix}_Z(\Sigma_j) &= \text{defined analogously,} \\
F(\cdot, \lambda) : \text{Fix}_X(\Sigma_j) &\rightarrow \text{Fix}_Z(\Sigma_j), \text{ and for } \lambda \text{ near } \lambda_0, \\
\Phi(\cdot, \lambda) : \text{Fix}_N(\Sigma_j) &\rightarrow \text{Fix}_N(\Sigma_j), \text{ locally near } 0, \text{ where} \\
\text{Fix}_N(\Sigma_j) &= N \cap \text{Fix}_X(\Sigma_j).
\end{align*}
\]

(3.22)

We begin with \(\Sigma_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2\): Define

\[
\begin{align*}
\phi_1 &= \frac{1}{\sqrt{2}}(\hat{v}_1 + \hat{v}_3) = (\eta, 0), \\
\phi_2 &= \frac{1}{\sqrt{2}}(\hat{v}_2 - \hat{v}_4) = (0, \eta), \text{ where} \\
\eta(x_1, x_2) &= \frac{1}{\pi} \cos \frac{1}{2} x_1 \sin x_2.
\end{align*}
\]

(3.23)
Then
\[ \gamma(I, E_1)\varphi_x = \varphi_x, \ x = 1, 2, \text{ and} \]
\[ N \cap \text{Fix}_X(\Sigma_1) = \text{span}\{\varphi_1, \varphi_2\} = \text{Fix}_N(\Sigma_1). \]  \tag{3.24}

In order to apply Theorem 2.1 to \( \Phi \) in \( \text{Fix}_N(\Sigma_1) \times \mathbb{R} \), we need to verify (2.15). Since \( \{\varphi_1, \varphi_2\} \) is an orthonormal basis for \( \text{Fix}_N(\Sigma_1) \), the scalar products \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle_0 \), defined in (2.9) and (3.7), respectively, are the same on \( N \). Moreover, since \( R_{\pi/2}\varphi_1 = \varphi_2 \) and \( R_{\pi/2}\varphi_2 = -\varphi_1 \), with rotation \( R_{\pi/2} \) given by (3.12), the abstract rotation \( R_{\pi/2} \) defined in (2.13) and \( R_{\pi/2} \) on \( \text{Fix}_N(\Sigma_1) \) are the same. Similar to the setup in (3.7)–(3.9), we choose \( Q : Z \to \text{Fix}_N(\Sigma_1) \) along \( R \) via
\[ Qu = \langle u, \varphi_1 \rangle_0 \varphi_1 + \langle u, \varphi_2 \rangle_0 \varphi_2. \]  \tag{3.25}

Writing \( v = w_1\varphi_1 + w_2\varphi_2 \in \text{Fix}_N(\Sigma_1) \), we find
\[ \langle \Phi_{0k}(v), R_{\pi/2}v \rangle = \langle Qf_{0k}(v), R_{\pi/2}v \rangle \]
\[ = f_{0k}(w) \cdot R_{\pi/2}w \int_{\Omega} \eta^{k+1} \, dx. \]  \tag{3.26}

Now \( F(\cdot, \lambda) \) is odd, by virtue of (3.12), and thus, \( \Phi(\cdot, \lambda) \) and the order \( k \) are also odd. Accordingly, the integral above in (3.26) does not vanish, and we may verify condition (2.15) for \( f_{0k}(w) \cdot R_{\pi/2}w, \ w = (w_1, w_2) \in \mathbb{R}^2 \). We find it convenient to first discuss the two other cases, corresponding to \( \Sigma_2 \) and \( \Sigma_3 \), before providing the details involved in that verification.

Next we consider \( \Sigma_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \): Define
\[
\begin{align*}
\psi_1 &\equiv \frac{1}{\sqrt{2}}(\hat{v}_1 + \hat{v}_4) = (\xi, -\bar{\xi}), \\
\psi_2 &\equiv \frac{1}{\sqrt{2}}(\hat{v}_2 + \hat{v}_3) = (\xi, \bar{\xi}), \text{ where} \\
\xi &\equiv \frac{1}{2\pi}(\cos \frac{1}{2}x_1 \sin x_2 + \sin x_1 \cos \frac{1}{2}x_2).
\end{align*}
\]  \tag{3.27}

Then
\[ \gamma(I, E_2)\psi_x = \psi_x, \ x = 1, 2, \text{ and} \]
\[ N \cap \text{Fix}_X(\Sigma_2) = \text{span}\{\psi_1, \psi_2\} = \text{Fix}_N(\Sigma_2). \]  \tag{3.28}

From here the treatment is nearly identical to that above for \( \Sigma_1 \). Again, \( R_{\pi/2} \) (3.12) on \( \text{Fix}_N(\Sigma_2) \) and \( R_{\pi/2} \) (2.13) are the same, and \( Q : Z \to \text{Fix}_N(\Sigma_2) \) along \( R \) is defined by
\[ Qu = \langle u, \psi_1 \rangle_0 \psi_1 + \langle u, \psi_2 \rangle_0 \psi_2. \]  \tag{3.29}
Then writing \( \mathbf{v} = w_1 \psi_1 + w_2 \psi_2 \in \text{Fix}_N(\Sigma_2) \), we find

\[
\langle \Phi_{0k}(\mathbf{v}), R_{\pi/2}\mathbf{v} \rangle = \langle Qf_{0k}(\mathbf{v}), R_{\pi/2}\mathbf{v} \rangle \\
= f_{0k}(T\mathbf{v}) \cdot R_{\pi/2}T\mathbf{v} \int_\Omega \xi^{k+1} dx,
\]

(3.30)

where \( T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \) and \( \mathbf{w} = (w_1, w_2) \). Arguing as before for the case \( \Sigma_1 \), we find again that \( k \) is necessarily odd, and the integral in (3.30) does not vanish. Since \( T \) is invertible and commutes with \( R_{\pi/2} \), we see again that the verification of (2.15) reduces to that of \( f_{0k}(\mathbf{w}) \cdot R_{\pi/2}\mathbf{w} \) for \( \mathbf{w} = (w_1, w_2) \in \mathbb{R}^2 \).

For \( \Sigma_3 \cong C_4 \) we have

\[
\gamma_R(\mathbf{R}, R)\hat{\psi}_\alpha = \hat{\psi}_\alpha, \quad \alpha = 1, 2, \quad \text{for all } R \in C_4, \text{ and } \\
N \cap \text{Fix}_X(\Sigma_3) = \text{span}\{\hat{\psi}_1, \hat{\psi}_2\} = \text{Fix}_N(\Sigma_3).
\]

(3.31)

In this case, we find that \( R_{\pi/2} \) (3.12) is the matrix of \( R_{\pi/2} \) (2.13) with respect to the basis \( \{\hat{\psi}_1, \hat{\psi}_2\} \) in \( \text{Fix}_N(\Sigma_3) \), and \( Q : Z \rightarrow \text{Fix}_X(\Sigma_3) \) is defined analogously to (3.25) and (3.29). However, unlike our previous examples, calculations like (3.9), (3.26) and (3.30) are more complicated and less illuminating here. Accordingly, we represent \( \Phi_{0k} : N \cap \text{Fix}_X(\Sigma_3) \rightarrow N \cap \text{Fix}_X(\Sigma_3) \) directly in terms of the basis \( \{\hat{\psi}_1, \hat{\psi}_2\} \):

\[
g_k(w_1, w_2) = \begin{pmatrix} \langle \Phi_{0k}(w_1\hat{\psi}_1 + w_2\hat{\psi}_2), \hat{\psi}_1 \rangle_0 \\
\langle \Phi_{0k}(w_1\hat{\psi}_1 + w_2\hat{\psi}_2), \hat{\psi}_2 \rangle_0 \end{pmatrix}, \quad \mathbf{w} = (w_1, w_2),
\]

(3.32)

\[
g_k : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ is a homogeneous polynomial of order } k; \text{ i.e., } g_k(\mu\mathbf{w}) = \mu^k g_k(\mathbf{w}) \text{ for } \mu \in \mathbb{R}.
\]

Furthermore, \( g_k \) is equivariant with respect to the rotation \( R_{\pi/2} \) in \( \mathbb{R}^2 \):

\[
g_k(R_{\pi/2}\mathbf{w}) = \begin{pmatrix} \langle \Phi_{0k}(-w_2\hat{\psi}_1 + w_1\hat{\psi}_2), \hat{\psi}_1 \rangle_0 \\
\langle \Phi_{0k}(-w_2\hat{\psi}_1 + w_1\hat{\psi}_2), \hat{\psi}_2 \rangle_0 \end{pmatrix} = \begin{pmatrix} \langle \Phi_{0k}(R_{\pi/2}(w_1\hat{\psi}_1 + w_2\hat{\psi}_2)), \hat{\psi}_1 \rangle_0 \\
\langle \Phi_{0k}(R_{\pi/2}(w_1\hat{\psi}_1 + w_2\hat{\psi}_2)), \hat{\psi}_2 \rangle_0 \end{pmatrix}
\]

by \( R_{\pi/2}\hat{\psi}_1 = \hat{\psi}_2, \quad R_{\pi/2}\hat{\psi}_2 = -\hat{\psi}_1 \),

\[
= \begin{pmatrix} \langle R_{\pi/2}\Phi_{0k}(w_1\hat{\psi}_1 + w_2\hat{\psi}_2), \hat{\psi}_1 \rangle_0 \\
\langle R_{\pi/2}\Phi_{0k}(w_1\hat{\psi}_1 + w_2\hat{\psi}_2), \hat{\psi}_2 \rangle_0 \end{pmatrix}
\]

by (3.20),

\[
= \begin{pmatrix} \langle \Phi_{0k}(w_1\hat{\psi}_1 + w_2\hat{\psi}_2), R_{\pi/2}^T\hat{\psi}_1 \rangle_0 \\
\langle \Phi_{0k}(w_1\hat{\psi}_1 + w_2\hat{\psi}_2), R_{\pi/2}^T\hat{\psi}_2 \rangle_0 \end{pmatrix}
\]
We assume that

\[ = R_{\pi/2}g_k(w) \quad \text{for} \quad w \in \mathbb{R}^2 \]

by \( R_{\pi/2}^T \hat{v}_1 = -\hat{v}_2, \) \( R_{\pi/2}^T \hat{v}_2 = \hat{v}_1. \) (3.33)

We pause here to observe that in each of the previous cases \( \Sigma_1 \) and \( \Sigma_2, \) we may identify \( g_k(w) \equiv f_0k(w). \) Moreover, by virtue of (3.12), we see that the homogeneous polynomial \( f_0k(w) \) is also \( C_4 \)-equivariant, as in (3.33). (These facts are not surprising. The normalizer of \( \Sigma_j \) acts as \( C_4 \) on \( \text{Fix}_N(\Sigma_j) \) for \( j = 1, 2, 3. \))

Thus, in each of the three cases, the equivariant homogeneous polynomial \( g_k \) of order \( k \) has a special form. Identifying

\[
\begin{align*}
\mathbf{w} &= (w_1, w_2) \quad \text{and} \quad z = w_1 + iw_2, \\
g_k &= (g_k^1, g_k^2) \quad \text{and} \quad h_k = g_k^1 + ig_k^2 \\
\text{we quote from [2], Chapter 5:} \\
h_k(z) \text{ is a finite sum of terms} \\
ak_1z^1 + \ldots + \ldots + ak_mz^m + bk_1(z) + \ldots + bk_m(z) + x_1z^{\ell_1} + \ldots + x_1z^{\ell_1},
\end{align*}
\]

(3.34)

where \( a_k, b_k \in \mathbb{C}, r_j, m_j, \ell_j \in \mathbb{N} \cup \{0\} \) for \( j = 1, 2, \ldots, 3. \)

The last condition means, in particular, that \( k \) is odd. For \( \mathbf{v} = w_1\hat{v}_1 + w_2\hat{v}_2 \) with \( \|\mathbf{v}\|^2_0 = w_1^2 + w_2^2 = |z|^2 = 1, \) or \( z = e^{i\theta} \) with \( \theta \in [0, 2\pi], \) we obtain in view of (3.32) and (3.34):

\[
\langle \Phi_0k(\mathbf{v}), R_{\pi/2}\mathbf{v} \rangle_0 = g_k(w) \cdot R_{\pi/2}w = -g_k^1(w)w_2 + g_k^2(w)w_1
\]

\[
= \text{Im}[h_k(z)\bar{z}] \quad (\text{Im = imaginary part})
\]

\[
= \sum \text{Im}[a_k e^{4(m_1-\ell_1)i\theta} + b_k e^{4(m_2-\ell_2-1)i\theta}].
\]

(3.35)

We assume that

\[
\sum_{m_1=\ell_1, m_2=\ell_2+1} \text{Im}[a_k + b_k] = 0 \quad \text{and} \quad \text{Im}[h_k(z)\bar{z}] \neq 0. \quad (3.36)
\]

Conditions (3.36) imply that \( h_k \) does not contain a term \( i\beta_k|z|^{2m}z, \) where \( \beta_k \in \mathbb{R} \) and \( 2m + 1 = k, \) and that \( h_k \) does not consist of a single term \( \alpha_k|z|^{2m}z \) for \( \alpha_k \in \mathbb{R}, \) respectively. By (3.36) we see that

\[
\int_0^{\pi/2} \sum \text{Im}[a_k e^{4(m_1-\ell_1)i\theta} + b_k e^{4(m_2-\ell_2-1)i\theta}] d\theta = 0,
\]

(3.37)

and by (3.36) the integrand in (3.37) does not vanish. Since it is periodic in \( \theta \) with period \( \pi/2, \) (3.37) implies that there exist at least 4 positive maxima and 4 negative
minima of the integrand in $[0, 2\pi)$. These, in turn, provide at least 8 pairs of vectors \( \tilde{v}_1, \tilde{v}_2 \) on \( S^1 \) for which (2.15), or (2.15) with opposite signs, hold true.

The application of Theorem 2.1 and of Corollary 2.3 then gives the following: Since \( F(\cdot, \lambda) \) is odd by (3.12), \( \Phi(\cdot, \lambda) \) and the order \( k \) are clearly odd. Thus, by virtue of (2.27), the 8 segments consist of 4 antipodal pairs yielding four different local continuas \( C \subset \text{Fix}_X(\Sigma_j) \times \mathbb{R} \) of \( F(u, \lambda) = 0 \) through \( (0, \lambda_0) = (0, \frac{\pi}{4}) \), \( j = 1, 2, 3 \) — for a total of twelve non-equivalent continuas of bifurcating solutions.

We show how to apply our general results to a concrete case. We consider

\[
f(z, \lambda) = \lambda z + b_03 \bar{z}^3 + R(z, \lambda)
\]

in complex notation, or in real coordinates,

\[
f(u, \lambda) = \lambda u + \begin{pmatrix} z_{03} & -\beta_{03} \\ \beta_{03} & z_{03} \end{pmatrix} \begin{pmatrix} u_1^3 - 3u_1u_2^2 \\ u_2^3 - 3u_1^2u_2 \end{pmatrix} + R(u, \lambda),
\]

where \( b_{03} = z_{03} + i\beta_{03} \neq 0 \), or

\[
f(u, \lambda) = \lambda u + f_{03}(u) + R(u, \lambda).
\]

In this case we have

\[
\Phi_{03}(v) = Qf_{03}(v) \quad \text{for } v \in N, \text{ cf. (2.8)},
\]

\[
g_3(w) \equiv f_{03}(w) \text{ for } \Sigma_1 \text{ and } \Sigma_2,
\]

\[
g_3(w) \equiv \begin{pmatrix} \langle f_{03}(w_1\hat{v}_1 + w_2\hat{v}_2), \hat{v}_1 \rangle_0 \\ \langle f_{03}(w_1\hat{v}_1 + w_2\hat{v}_2), \hat{v}_2 \rangle_0 \end{pmatrix} \text{ for } \Sigma_3,
\]

\( g_3 \) does not vanish identically in each case.

By (3.34) any equivariant homogeneous polynomial of order \( k = 3 \) in complex notation is necessarily of the form

\[
h_3(z) = a_3|z|^2z + b_3\bar{z}^3 \quad \text{for some } a_3, b_3 \in \mathbb{C}.
\]

We show that \( a_3 = 0 \) if \( h_3 \) represents \( g_3 \) in the third case. Therefore \( b_3 \neq 0 \) by (3.39)_4. The complex representation (3.38)_4 shows the invariance

\[
f_{03}(e^{2\pi i/3}z) = f_{03}(z), \text{ or in real coordinates},
\]

\[
f_{03}(R_{2\pi/3}u) = f_{03}(u), \text{ where}
\]

\( R_{2\pi/3} \) is a rotation about \( 2\pi/3 \) in \( \mathbb{R}^2 \).

Using \( R_{\pi/2}\hat{v}_1 = \hat{v}_2 \) and \( R_{\pi/2}\hat{v}_2 = -\hat{v}_1 \) we obtain, after a similar calculation as in (3.33), the invariance of \( g_3 \), i.e.,

\[
g_3(R_{2\pi/3}w) = g_3(w) \quad \text{for } w \in \mathbb{R}^2.
\]
This, in turn, implies for the complex notation (3.40) that
\begin{equation}
\begin{aligned}
h_3(e^{2\pi i/3}z) = h_3(z) & \text{ or, for } |z| = 1, \\
a_3e^{2\pi i/3}z + b_3\bar{z}^3 = a_3z + b_3\bar{z}^3 & \text{ whence } a_3 = 0.
\end{aligned}
\end{equation}

Since \( m_2 = \ell_2 = 0 \), condition (3.36) is satisfied for \( b_3 \neq 0 \), which, in turn, is true if \( b_{03} \neq 0 \). Thus Theorem 2.1 and Corollary 2.3 are applicable to problem (3.1) over the square where \( f \) is given in (3.38) for any \( b_{03} \neq 0 \). They provide four different bifurcating continua in \( \text{Fix}_X(\Sigma_j) \times \mathbb{R}, j = 1, 2, 3 \), respectively. Since the different symmetries separate the continua (\( \text{Fix}_X(\Sigma_j) \cap \text{Fix}_X(\Sigma_j) = \{0\} \) for \( j \neq k \)) we have altogether at least twelve different bifurcating continua.

A closer look at example (3.38) reveals the following: In complex notation the scalar product (3.40) is given in (3.38) for any \( b_{03} \neq 0 \). They provide four different bifurcating continua in \( \text{Fix}_X(\Sigma_j) \times \mathbb{R}, j = 1, 2, 3 \), respectively. Since the different symmetries separate the continua (\( \text{Fix}_X(\Sigma_j) \cap \text{Fix}_X(\Sigma_j) = \{0\} \) for \( j \neq k \)) we have altogether at least twelve different bifurcating continua.

A closer look at example (3.38) reveals the following: In complex notation the function \( g(\bar{v}, 0) \) defined in (2.20) is given by \( \text{Im}[h_3(z)\bar{z}] = \text{Im}[b_3\bar{z}^4] \), cf. (2.21), (3.35). Let \( \bar{v}_0 = u_0^1v_1 + u_0^2\bar{v}_2 \) be a zero on \( S^1 \); i.e. \( g(\bar{v}_0, 0) = 0 \). Then \( z_0 = e^{i\theta_0} = w_1^0 + w_2^0 \) solves \( \text{Im}[b_3\bar{z}^4] = 0 \). Since \( D_{\theta_0} \text{Im}[b_3e^{-4i\theta}] = -4\text{Re}[b_3e^{-4i\theta}] \), any zero of \( \text{Im}[b_3\bar{z}^4] = 0 \) on \( S^1 \) is non-degenerate, provided that \( b_3 \neq 0 \). The same holds true for the zero \( g(\bar{v}_0, 0) = 0 \), and the Implicit Function Theorem provides a curve \( g(\bar{v}(s), s) = 0 \) with \( (\bar{v}(s), s) \in S^1 \times [−\delta_4, \delta_4], \bar{v}(0) = \bar{v}_0 \). This sharpens the statement of Theorem 2.1. Thus, we obtain four different bifurcating curves in \( \text{Fix}_X(\Sigma_3) \times \mathbb{R} \) through \((0, \lambda_0) = (0, \frac{5}{4})\) for example (3.38). The same arguments also apply to the function \( g(\bar{v}, 0) \) (2.20) in \( \text{Fix}_X(\Sigma_j) \) for \( j = 1, 2 \), cf. (3.26), (3.30). We emphasize, however, that the verification of the non-degeneracy of zeros of \( g(\cdot, 0) \) on \( S^1 \) is simply due to the special case \( h_3(z) = b_3\bar{z}^3 \). For functions like \( h_5(z) = z^5 + b_5|z|^2\bar{z}^3 \) or \( h_{11}(z) = (\bar{z}^8 - 3|z|^4\bar{z}^4 + 3|z|^8)\bar{z}^3 \) this task is much harder. As a matter of fact, there are zeros of \( \text{Im}[h_{11}(z)\bar{z}] = 0 \) on \( S^1 \) that are degenerate. Nonetheless Theorem 2.1 provides bifurcating continua.

More remarks are in order: The vector field \( \textbf{f}(\cdot, \lambda) \) given in (3.38) without remainder (i.e., \( \textbf{R} = 0 \)) has a potential with respect to the Euclidean scalar product in \( \mathbb{R}^2 \), namely
\begin{equation}
\phi(\textbf{u}, \lambda) = \frac{1}{2}\lambda(u_1^2 + u_2^2) + \omega_{03}(\frac{1}{4}(u_1^4 + u_2^4) - \frac{3}{2}u_1^2u_2^2) - \beta_{03}(u_1u_2^3 - u_1^3u_2)
\end{equation}
for \( (\textbf{u}, \lambda) \in \mathbb{R}^2 \times \mathbb{R} \).

This, in turn, implies that \( \textbf{F}(\cdot, \lambda) \) defined in (3.16) also has a potential with respect to the scalar product \((\cdot, \cdot)_0\) in \( L^2(\Omega) \), given by
\begin{equation}
-\frac{1}{2}\|
abla \textbf{u}\|_0^2 + \int_\Omega \phi(\textbf{u}, \lambda)\, dx \text{ for } (\textbf{u}, \lambda) \in X \times \mathbb{R}.
\end{equation}

By the linear dependence on the parameter \( \lambda \), both bifurcation theorems for potential operators presented in [7], 1.21 and II.7, are applicable. For details of the applications see [7], III.2.3 and III.2.4. Here we mention only that the crossing number of the family \( D_{\lambda} \textbf{F}(\textbf{u}, 0, \lambda) = \Delta + \lambda \textbf{I} \) at \( \lambda = \lambda_0 \) through \( 0 \) is simply \(-\dim N(D_{\lambda} \textbf{F}(\textbf{u}, 0, \lambda_0)) = -4 \neq 0 \). Since \( \textbf{F}(\cdot, \lambda) \) is odd, Corollary I.21.3 in [7] provides \( 4 \) pairs \((\pm \textbf{u}, \lambda) \in X \times \mathbb{R} \) of non-trivial solutions clustering at \((0, \lambda_0)\); cf. also III.2.3 in [7]. Accordingly it provides 2
pairs \((±u, λ) ∈ \text{Fix}_X(Σ_j) × \mathbb{R}\) clustering at \((0, λ_0) = (0, \frac{5}{4})\) for \(j = 1, 2, 3\), respectively. The results of Theorem 2.1 and Corollary 2.3, however, are sharper since they guarantee four different bifurcating continua (= curves) in \(\text{Fix}_X(Σ_j) × \mathbb{R}\) through \((0, λ_0)\). Note that Theorem 2.1 requires only equivariance (3.12) on the remainder \(\mathbb{R}\), i.e., no potential is required.

When we modify example (3.38) to
\[
f(z, λ) = λz + b_{05}z^2 z^3 \quad \text{or} \quad f(u, λ) = λu + \begin{pmatrix} x_{05} & -β_{05} \\ β_{05} & x_{05} \end{pmatrix}(u_1^2 + u_2^2) \begin{pmatrix} u_1^3 - 3u_1u_2^2 \\ u_2^3 - 3u_1^2u_2 \end{pmatrix}
\]
then the vector field \(f(·, λ)\) has no potential, but Theorem 2.1 and Corollary 2.3 are still applicable, providing 12 different solution continua (= curves) through \((0, λ_0) = (0, \frac{5}{4})\).

Finally, we point out that in all of the cases associated with problem (3.1) on the square (for the second eigenvalue \(λ_0 = \frac{5}{4}\)), we do not necessarily have reflection symmetry, viz., in general,
\[
f(E_1u, λ) ≠ E_1f(u, λ), \quad \text{where} \quad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
For examples (3.38) and (3.46), this reflection symmetry is precluded by taking \(β_{03} ≠ 0\) or \(β_{05} ≠ 0\), respectively. Thus, \(F(·, λ)\) is not generally equivariant with respect to \(D_4 × D_4\), cf. (3.19). In such a case, the usual equivariant branching lemma would be appropriate, i.e., the analysis of each of the bifurcating branches is essentially one-dimensional.

4. \(C_n\)-equivariant bifurcation problems

The previous example (3.1) over the square with symmetry (3.12) suggests consideration of (3.1) on more general domains having the symmetry of a regular \(n\)-gon, \(n ≥ 3\). Indeed, if the nonlinearity \(f(·, λ)\) enjoys symmetry as in (3.12) with respect to a rotation through \(2π/n\), then at a double eigenvalue of the Laplacian, we would obtain a four-dimensional bifurcation problem equivariant under \(C_n × D_n\), cf. (3.20). In principle we could then analyze the two-dimensional reduced problems as in Section 3. However, the analytical expressions for the eigenfunctions are not generally available, rendering explicit calculations difficult at best. (For an equilateral-triangular domain, the eigenfunctions are known, e.g. [5]. However we do not pursue that avenue here.) Accordingly, we take a more abstract point of view in this section.

For \(n\) odd we expect the maximal isotropy subgroups \(\mathbb{Z}_2 × \mathbb{Z}_2, C_n \subset C_n × D_n \) whereas for \(n\) even we also get \(C_n\) and two non-equivalent copies of \(\mathbb{Z}_2 × \mathbb{Z}_2\) (as in (3.21)). In the absence of explicit eigenfunctions, calculations like (3.26) and (3.30) are not possible.
Nonetheless, as seen in Section 3, each of the two-dimensional reduced problems is \( C_n \)-equivariant. Accordingly, we focus on \( C_n \)-equivariant bifurcation problems, which, aside from our motivation here, are interesting in their own right [6].

We consider the following abstract situation: For \( n' = kn \), a \( C_n' \)-equivariant bifurcation problem \( F(x, \lambda) = 0 \) is reduced to a \( C_n' \)-equivariant problem \( G(v, \lambda) = 0 \) via an equivariant Lyapunov–Schmidt reduction, cf. (2.5). Assume that the representation of \( C_n' \) on the kernel \( N = N(D_x F(0, \lambda_0)) \) is irreducible. Consequently, \( \dim N = 1 \) or \( 2 \), and assuming the two-dimensional case, we make an appropriate choice of basis so that \( C_n' \) acts on \( N \) as a rotation \( R_{2\pi/n} \) on \( \mathbb{R}^2 \) about an angle \( 2\pi/n \), where \( n \) divides \( n' \). Note that \( n \geq 3 \).

Although a bifurcation function is only locally defined we assume for convenience:

\[
\Phi: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \quad \Phi \in C^k(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2),
\]

\[
\Phi(R_{2\pi/n}v, \lambda) = R_{2\pi/n}(v, \lambda) \quad \text{for} \quad (v, \lambda) \in \mathbb{R}^2 \times \mathbb{R},
\]

or in complex notation (cf. (3.34)),

\[
\Phi: \mathbb{C} \times \mathbb{R} \to \mathbb{C}, \quad \text{where}
\]

\[
\Phi(e^{2\pi i/n}z, \lambda) = e^{2\pi i/n}(\Phi(z, \lambda) \quad \text{for} \quad (z, \lambda) \in \mathbb{C} \times \mathbb{R}.
\]

As in (2.7) we assume

\[
\Phi(v, \lambda) = (\lambda - \lambda_0)\Phi_{11}v + \Phi_{0k}(v) + R(v, \lambda),
\]

or, by equivariance of \( \Phi_{11} \),

\[
\Phi(z, \lambda) = (\lambda - \lambda_0)a_{11}z + h_k(z) + R(z, \lambda)
\]

for some \( a_{11} \in \mathbb{C} \).

Assumption (2.14) means that \( a_{11} \neq 0 \). The \( C_n \)-equivariance of the homogeneous polynomial \( h_k \) is expressed as follows, cf. (3.34):

\[
h_k(z) = \sum_{\ell=0}^{k} \alpha_{\ell}z^\ell z^{k-\ell}, \quad \alpha_{\ell} \in \mathbb{C},
\]

where \( \alpha_{\ell} = 0 \) if \( 2\ell - k \equiv 1 \) (mod \( n \)).

Since the abstract rotation defined in (2.13) is a geometric rotation in \( \mathbb{R}^2 \) endowed with the canonical basis, we obtain as in (3.35) for \( \|v\| = |z| = 1 \) or for \( z = e^{i\theta} \):

\[
(\Phi_{0k}(v), R_{\pi/2}\Phi_{11}v) = \Phi_{0k}(v) \cdot R_{\pi/2}\Phi_{11}v = \text{Im}[h_k(z)\bar{a}_{11}z]
\]

\[
= \sum_{\ell=0}^{k} \text{Im}[\bar{a}_{11}\alpha_{\ell}e^{(2\ell-k-1)i\theta}].
\]
The assumption
\[
\text{Im}[\bar{a}_{11}\mathcal{z}_{\ell}] = 0 \quad \text{for } \ell = \frac{k+1}{2} \quad \text{and for odd } k, \quad \text{but}
\]
\[
\text{Im}[h_k(z)\bar{a}_{11}\mathcal{z}] \neq 0
\]
(4.5)
implies
\[
\int_0^{2\pi/n} \text{Im}[h_k(e^{i\theta})\bar{a}_{11}e^{-i\theta}] d\theta = 0,
\]
but the integrand does not vanish identically.

(4.6)
The \(2\pi/n\)-periodicity of the integrand then implies the existence of at least \(n\) positive maxima and \(n\) negative minima on \([0, 2\pi]\). These provide via (4.4) at least \(2n\) pairs of vectors \(\mathbf{v}_1, \mathbf{v}_2\) on \(S^1 \subset \mathbb{R}^2\) for which (2.15), or (2.15) with opposite signs, hold true. The application of Theorem 2.1 and the arguments of the proof of Corollary 2.3 then give the following: All antipodal pairs of zeros on \(S^1\) give, by the symmetries (2.27) or (2.28), only one continuum of solutions, respectively. We summarize:

**Theorem 4.1.** Under the assumption (4.5) there exist \(n\) local continua \(C \subset \mathbb{R}^2 \times \mathbb{R}\) of non-trivial solutions of \(\Phi(v, \lambda) = 0\) through \((0, \lambda_0)\) and \(C\backslash\{(0, \lambda_0)\}\) consist of at least two components, respectively.

Note that condition (4.5)1 is redundant if \(k\) is even, and that for \(k\) odd it refers only to the single term \(\mathcal{z}_\ell|z|^{k-1}z\).

As mentioned before (after (2.12)), the existence of a zero \(\mathbf{v}_0\) of \(\tilde{\lambda}_0\Phi_{11}\tilde{\mathbf{v}}_0 + \Phi_{0k}(\tilde{\mathbf{v}}_0) = 0\) on \(S^1 \subset \mathbb{R}^2\) for some \(\tilde{\lambda}_0 \in \mathbb{R}\) is necessary for bifurcation from \((0, \lambda_0)\) if \(\Phi_{11}\) is an isomorphism. This necessary condition is satisfied if \(\Phi_{0k}\) has a potential and if \(\Phi_{11}\) is symmetric and positive (or negative) definite. Accordingly, \(\langle \Phi_{0k}(\tilde{\mathbf{v}}_0), R_{\pi/2}\Phi_{11}\tilde{\mathbf{v}}_0 \rangle = g(\tilde{\mathbf{v}}_0, 0) = 0\), cf. (2.19), and if \(\tilde{\mathbf{v}}_0\) is a non-degenerate zero on \(S^1\), then the Implicit Function Theorem provides a curve of solutions of (2.20), and via (2.26) a curve of solutions of \(\Phi(v, \lambda) = 0\) through \((0, \lambda_0)\).

In order to follow these lines we assume that

\[
\Phi_{11}, \Phi_{0k} : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{each have a potential,}
\]
or equivalently in complex notation,
\[
a_{11} \in \mathbb{R} \quad \text{and} \quad \frac{\partial}{\partial z} h_k(z, \mathcal{z}) \in \mathbb{R} \quad \text{for all } z \in \mathbb{C}.
\]
(4.7)
A short computation shows that the only terms that satisfy (4.7)3 are
\[
h_k(z) = \sum_{\ell=0}^{k} a_{\ell\mathcal{z}} \mathcal{z}^{k-\ell}
\]
where
\[
a_{\ell\mathcal{z}} = 0 \quad \text{if } 2\ell - k \neq 1 \mod n
\]
\[
\text{Im } a_{\ell\mathcal{z}} = 0 \quad \text{for } \ell = \frac{k+1}{2} \quad \text{and for odd } k,
\]
\[
\ell a_{\ell\mathcal{z}} = (k - \ell + 1) \mathcal{z}_{k-\ell+1} \quad \text{for } \ell = 1, \ldots, k.
\]
(4.8)
The lowest order terms for which assumption (4.8) is satisfied are

for odd $k \leq n - 1$
\[ h_k(z) = z_\ell |z|^{k-1}z \text{ for } \ell = \frac{k+1}{2}, \text{ with } z_\ell \in \mathbb{R}, \]
for $k = n - 1$
\[ h_k(z) = z_0 z^{n-1} \text{ with } z_0 \in \mathbb{C}. \] (4.9)

For equivariant mappings $\Phi_{11}, \Phi_{0k}$ that have a potential, the lowest order $k$ for which
\[ \text{Im}[h_k(z) \bar{a}_{11} z] \neq 0 \]
is
\[ k = n - 1, \]
and
\[ h_k(z) = z_{n/2} |z|^{n-2}z + z_0 \bar{z}^{n-1} \text{ if } n \text{ is even}, \]
\[ h_k(z) = z_0 \bar{z}^{n-1} \text{ if } n \text{ is odd}, \]
with some $z_{n/2} \in \mathbb{R}$ and $z_0 \in \mathbb{C} \setminus \{0\}$. (4.10)

In all cases (4.10), the $n$ zeros of
\[ \text{Im}[h_k(e^{i\theta}) a_{11} e^{-i\theta}] = a_{11} \text{ Im}[b_k e^{-i\theta}] \]
in $[0, 2\pi)$ are all non-degenerate: If
\[ \text{Im}[b_k e^{-i\theta}] = 0 \text{ then } -n \text{ Re}[b_k e^{-i\theta}] = D_0 \text{ Im}[b_k e^{-i\theta}] \neq 0 \]
provided $b_k \neq 0$. The following theorem is due to [6]:

**Theorem 4.2.** The leading part of the bifurcation function, i.e., $(\lambda - \lambda_0) a_{11} z + h_k(z)$ in complex notation, has a potential if and only if $a_{11} \in \mathbb{R}$ and $h_k$ consists only of terms (4.8). The Implicit Function Theorem is applicable yielding $n$ solution curves of $\Phi(z, \lambda) = 0$ through $(0, \lambda_0)$ if $a_{11} \in \mathbb{R} \setminus \{0\}$, $k = n - 1$, and $h_k$ is of the form (4.10).

The cases of Theorem 4.2 are also covered by Theorem 4.1 since (4.10) implies (4.5). Obviously Theorem 4.1 applies to many more cases than Theorem 4.2: We need no variational structure of the leading part and therefore the order $k$ in the bifurcation function and the order $n$ of the symmetry group are not locked.

5. Concluding remarks

One of the basic rules of “generic” equivariant bifurcation theory is that no special structure should be assumed (for the bifurcation equations) beyond those forced by symmetry [2,4]. Each of our examples in Section 3 violate this premise. Indeed in the first example, the nonlinear system (3.1) generally has no symmetry. Nonetheless, the linearized problem (cf. (3.2), (3.4)),

\[ \Delta u + \lambda u = 0 \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial \Omega, \]
is equivariant under $u \rightarrow Qu$ for all $Q \in O(2)$, the complete symmetry group of a circle in the plane. The fact that this two-dimensional representation of $O(2)$ is
irreducible “explains” why the kernel of the linearization is also two-dimensional. The same holds for the second example (3.1) over the square with assumption (3.12). While the nonlinear problem has the equivariant symmetry group \( C_4 \times D_4 \), cf. (3.18), (3.19), the linearized problem admits the equivariant symmetry \( O(2) \times D_4 \). In both examples, the non-generic culprit is assumption (3.2)_3 combined with the appearance of the Laplace operator. A similar situation holds for the abstract class of examples in Section 4. A generic \( C_n \)-equivariant system (4.1) does not generally satisfy the leading-order structure of (4.2)_1: By virtue of (4.1)_2, the \( 2 \times 2 \)-matrix function \( A(\lambda) \equiv D_4 \Phi(0, \lambda) \) commutes with \( R_{2\pi/n} \). Thus, \( A(\lambda) \) itself is proportional to an arbitrary rotation matrix, which generally has complex eigenvalues (in particular, no zero eigenvalue at \( \lambda = \lambda_0 \)). Of course, if \( \Phi \) is a gradient vector field, then \( A(\lambda) \) is necessarily self-adjoint, i.e., (4.2)_1 is generic if the problem is derivable from a potential.

We believe that our examples are worthwhile if not generic. It is typical in applications that the linearized problem possesses more structure—more symmetry and/or gradient structure—than the nonlinear problem. Moreover, as pointed out to us recently by Reiner Lauterbach, two-dimensional kernels with \( C_3 \)-symmetry occur generically within the context of bifurcation problems in the presence of icosahedral symmetry: In a five-dimensional irreducible representation of the icosahedral group, the dihedral group \( D_2 \) is a maximal isotropy subgroup with a two-dimensional fixed-point space. The normalizer of \( D_2 \) is the tetrahedral group and its action on the two-dimensional fixed-point space is that of \( \mathbb{Z}_3 \times \mathbb{C}_3 \). In any case, we have presented here a direct and efficient tool for the existence of bifurcating continua in problems with two-dimensional kernels. Moreover, we have illustrated our technique with several rich, non-trivial examples coming from elliptic partial differential equations.

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References