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Some contractible open manifolds and coverings of manifolds in dimension three

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Abstract

McMillan has shown that every irreducible, contractible, open 3-manifold is the monotone union of handlebodies (only 0- and 1-handles) and that there are uncountably many such manifolds. Work by Myers and Wright shows that no irreducible, contractible, open 3-manifold different from $\mathbb{R}^3$ can nontrivially cover any 3-manifold when the handlebodies all have genus one or have bounded genus. We describe a family of irreducible, contractible, open 3-manifolds that we call *composite Whitehead manifolds*. These manifolds have the property that when written as the monotone union of handlebodies, the handlebodies must have unbounded genus. We show that there are uncountably many composite Whitehead manifolds that nontrivially cover open 3-manifolds but do not cover a compact 3-manifold. We also show that there exist uncountably many composite Whitehead manifolds which cannot nontrivially cover any 3-manifold. It is a famous unsolved problem if any irreducible, contractible, open 3-manifold different from $\mathbb{R}^3$ can cover a compact 3-manifold. It is unlikely that any composite Whitehead manifold covers a compact manifold, but our techniques are not strong enough to answer this question. © 1997 Elsevier Science B.V.

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1. Introduction

Unlike the case in higher dimensions [4], there are no known examples of contractible open 3-manifolds different from $\mathbb{R}^3$ which cover a compact 3-manifold. In fact, the

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classical contractible 3-manifolds, the Whitehead manifold and related manifolds that are *nice at infinity*, are known to not cover any manifolds other than themselves by a trivial covering projection [13,18]. The famous unsolved *Covering Conjecture* asserts that the universal covering space of a closed irreducible 3-manifold with infinite fundamental group is $\mathbb{R}^3$ [7,1,9].

In this paper we investigate certain interesting irreducible, contractible, open 3-manifolds that we call *composite Whitehead manifolds.* McMillan has shown [10] that all irreducible, contractible, open 3-manifolds are the monotone union of handlebodies (only 0- and 1-handles). The classical contractible non-Euclidean open 3-manifolds [19,11] are the monotone union of handlebodies of bounded genus. By [13] for genus 1 and [18] for bounded genus, the classical contractible open 3-manifolds cannot nontrivially cover any manifold. The composite Whitehead manifolds are not the monotone union of handlebodies of bounded genus. Unlike the classical contractible manifolds, many of these composite Whitehead manifolds are nontrivial covering spaces of open 3-manifolds. We show the existence of many composite Whitehead manifolds that do not cover a compact 3-manifold and yet nontrivially cover open 3-manifolds. In fact, it is unlikely that any of these manifolds cover a compact 3-manifold, but our methods are not strong enough to demonstrate this. Robert Myers [14] has given specific examples of open contractible 3-manifolds that nontrivially cover an open 3-manifold but do not cover a compact manifold. We also construct a collection of composite Whitehead manifolds which cannot cover any manifold except themselves by a trivial covering projection.

By *nice at infinity* we mean that the 3-manifold is *eventually end irreducible* as described in [1] or [2]. Equivalently, we mean that the 3-manifold is *eventually π_1-injective at infinity* as described in [18]. A definition of *eventually π_1-injective at infinity* is given in Section 2.

2. Definitions and notation

We use $\mathbb{R}^n$ to denote Euclidean n-space. We use $\boldsymbol{B}^n$ and $\boldsymbol{S}^{n-1}$ to denote the unit ball and the unit sphere in $\mathbb{R}^n$. All spaces and embeddings will be piecewise-linear. A solid torus is a space homeomorphic to $\boldsymbol{B}^2 \times \boldsymbol{S}^1$, an n-ball is a space homeomorphic to $\boldsymbol{B}^n$. If M is a manifold, we let $\operatorname{Bd} M$ and $\operatorname{Int} M$ denote the boundary and interior of M, respectively. An *open manifold* is without boundary and noncompact. A *closed manifold* is without boundary and compact. A 3-manifold is said to be *irreducible* if every 2-sphere bounds a 3-ball. A *proper embedding* is an embedding where the inverse of a compact set is compact. A *proper ray* and a *properly-embedded plane* are simply a ray and a plane, respectively, embedded in some larger space by a proper embedding.

If M is an open n-manifold, then there exists a manifold $\overline{M}$ so that $\operatorname{Int} \overline{M}$ is homeomorphic with M and $\operatorname{Bd} \overline{M}$ is homeomorphic with $\mathbb{R}^{n-1}$. This fact can be seen by taking a proper ray R to infinity in M and deleting a regular neighborhood N of R. The manifold M is homeomorphic to $M - N$ and the closure of $M - N$ is a manifold with boundary homeomorphic to $\mathbb{R}^{n-1}$.

Definition 2.1. Let $M_1, M_2, \ldots, M_k$ be open manifolds of dimension n $(n \geqslant 2)$. An *end sum* $M_1 \# M_2 \# \cdots \# M_k$ of $M_1, M_2, \ldots, M_k$ is obtained by taking disjoint manifolds with boundary $\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_k$ as described above and a disjoint n-manifold $\overline{U}$ so that $\operatorname{Int}\overline{U} = U$ is homeomorphic to $\mathbb{R}^n$ and $\operatorname{Bd} U$ has k components each homeomorphic to $\mathbb{R}^{n-1}$. An end sum $M_1 \# M_2 \# \cdots \# M_k$ is formed by identifying $\operatorname{Bd}(\overline{M}_1 \cup \overline{M}_2 \cup \cdots \cup \overline{M}_k)$ with $\operatorname{Bd}\overline{U}$ by a homeomorphism.

We think of the M_i and U as subsets of the end sum. The closure of M_i is $\overline{M}_i$ and the closure of U is $\overline{U}$. We call U the *connecting ball* for the end sum $M_1 \# M_2 \# \cdots \# M_k$.

The terminology *end sum* is due to Robert Myers although he uses it in a slightly different way. There is no claim about the uniqueness of an end sum. We may also form the countably infinite end sum of open manifolds in the exact same way.

Definition 2.2. A genus one Whitehead manifold [19] is a 3-manifold that is the monotone union of solid tori T_i so that $T_i \subset \operatorname{Int} T_{i+1}$, T_i is contractible in T_{i+1}, but T_i does not lie in a ball contained in T_{i+1}.

It is a well-known that T_i does not lie in a ball in the Whitehead manifold. McMillan has made an extensive study of genus one Whitehead manifolds [11]. He has shown the existence of uncountably many different such manifolds. The key ingredient in McMillan's proof is a lemma of Schubert [16]. We first state a definition and then Schubert's lemma.

Definition 2.3. If T_1 and T_2 are solid tori with $T_1 \subset T_2$, the *geometric index* of T_1 in T_2 is the minimal number of points of intersection of a centerline of T_1 with a meridianal disk of T_2.

See [11] or [16] for more details.

Lemma 2.4 (Schubert). *Let $T_1 \subset T_2 \subset T_3$ be solid tori so that the geometric index of T_1 in T_2 is p and the geometric index of T_2 in T_3 is q. Then the geometric index of T_1 in T_3 is pq.*

Definition 2.5. A topological space X is said to be *eventually π_1-injective at infinity* if there is a fixed compact set K of X so that for every compact set A of X containing K, there is a compact set B of X containing A, so that loops in $X - B$ which are inessential in $X - K$ are also inessential in $X - A$. We call the set K a *core* of X.

Informally, we think of this property as stating that loops close to infinity which are inessential missing the core are inessential close to infinity. This condition is really a very mild condition which is satisfied by all the classical contractible manifolds including the genus one Whitehead manifolds for which any solid torus in a defining sequence serves as a core. In [18] a simple lemma, called the Ratchet lemma, was established that proved to be crucial in showing that certain contractible open n-manifolds, $n \geqslant 3$, that nontrivially cover a manifold must be simply connected at infinity. Recall that a contractible n-manifold that is simply connected at infinity must be homeomorphic to $\mathbb{R}^n$ if $n \geqslant 4$ [17,6]. For $n = 3$, this is also the case if we assume the Poincaré conjecture or simply restrict our attention to irreducible manifolds. We state the Ratchet lemma for future reference. Later we will prove a special version of this lemma.

Lemma 2.6 (Ratchet lemma). *Let $h : W \to W$ be a homeomorphism of a space that is eventually π_1-injective at infinity. If K is a core of W, then there is a compact set L in W so that a loop γ in $W - \bigcup_{i=-\infty}^{\infty} h^i(L)$ is inessential in $W - K$ if and only if γ is inessential in $W - h^i(K)$ for each i.*

3. Some interesting contractible 3-manifolds

In this paper we study a class of contractible manifolds that we call *composite Whitehead manifolds*. A composite Whitehead manifold is an infinite end sum of genus one Whitehead manifolds.

Let $W = W_1 \# W_2 \# \cdots$ be a composite Whitehead manifold, U denote the connecting ball of W, and P_i be the plane $\overline{W}_i - W_i$. We know that the Whitehead manifold W_i is the union of nested solid tori T_{ik}. We assume also that the infinite end sum has been chosen so that $\overline{W}_i$ is the union of nested *eyebolts* $\overline{T}_{ik}$. The *eyebolt* $\overline{T}_{ik}$ is simply T_{ik} plus a solid tube (3-ball) that runs from $\operatorname{Bd}\overline{T}_{ik}$ to P_i and intersects each of these in a disk. We will always assume that composite Whitehead manifolds satisfy these nice conditions. It is clear that composite Whitehead manifolds are contractible and irreducible; however, unlike Whitehead manifolds, composite Whitehead manifolds are not eventually π_1-injective at infinity.

We devote this section to studying the effect of a homeomorphism on the very nice structure of composite Whitehead manifolds described above. In particular, if $h : W \to W'$ is a homeomorphism of composite Whitehead manifolds, we determine up to isotopy the image under h of any core of any W_i. Ultimately, we will show that for any core K of W_i, $h(K)$ can be isotoped into a W'_j where j depends only on i (not on K). Moreover, the induced map $i \to j$ is a permutation of the positive integers.

The first step is to establish criteria which determine whether a given compact subset of a composite Whitehead manifold lies inside a ball in that manifold. The property described in the next definition is the key to necessary and sufficient conditions ultimately given in Lemma 3.6.

Definition 3.1. Let V be an open subset of a manifold M and X be a compact subset of M. We say that X is *π_1-trivial at infinity in V* if for each compact set A of M there is a bigger compact set B of M so that loops in $V - B$ are inessential in $M - X$.

Lemma 3.2. *Let M be a manifold and V be an open subset of M. If X_1 and X_2 are compact subsets of M so that there is an ambient isotopy taking X_1 to X_2, then X_1 is π_1-trivial at infinity in V if and only if X_2 is π_1-trivial at infinity in V.*

Proof. By the covering isotopy lemma [3,5], we may assume that there is an isotopy of M taking X_1 to X_2 that fixes points outside a compact subset C of M. Assume that X_1 is π_1-trivial at infinity in V and let A be a compact subset of M. Choose B to be a compact subset containing $A \cup C$ so that loops in $V - B$ are inessential in $M - X_1$. Since the isotopy fixes $V - B$, it is clear that these loops are also inessential in $M - X_2$. The proof in the other direction is similar. □

Lemma 3.3. *Let $M = N_1 \# N_2$ be an end sum of irreducible 3-manifolds. If X is a compact connected subset of N_1 that lies in a ball B of M, then X lies in a ball in N_1.*

Proof. Let P be the properly-embedded plane $\overline{N}_1 - N_1$. Since B is piecewise-linear, we may suppose that $X \subset \operatorname{Int} B$ and $\operatorname{Bd} B \cap P$ consists of disjoint simple closed curves. The proof is by induction on the number of simple closed curves. Let J be an innermost simple closed curve in P. Then J bounds a disk D in P and two disks D_1 and D_2 in $\operatorname{Bd} B$. The 2-sphere $D \cup D_1$ bounds a 3-ball. If this 3-ball contains X, then by pushing D off P we get X in a 3-ball whose boundary intersects P in fewer simple closed curves than $\operatorname{Bd} B$. If the 3-ball bounded by $D \cup D_1$ does not contain X, then there is an isotopy of M, fixing X, taking $\operatorname{Bd} B$ to $D \cup D_2$. Hence, the ball bounded by $D \cup D_2$ contains X and by a slight adjustment its boundary intersects P in fewer simple closed curves than $\operatorname{Bd} B$. □

An isotopy naturally preserves the topology of the genus one Whitehead manifold construction. The next lemma, based on an idea of Kinoshita [8], categorizes the precise nature of the geometric linking which isotopy preserves as well.

Lemma 3.4. *Let T_1 and T_2 be solid tori in an irreducible manifold M so that $T_1 \subset \operatorname{Int} T_2$, T_1 is contractible in T_2, T_1 does not lie in a ball contained in T_2, and T_2 does not lie in a ball in M. If D is a disk in M so that $\operatorname{Bd} D$ is a nontrivial curve in $\operatorname{Bd} T_2$ and for which $\operatorname{Int} D \cap (\operatorname{Bd} T_1 \cup \operatorname{Bd} T_2)$ consists of disjoint simple closed curves, then D must meet $\operatorname{Bd} T_1$ in at least two nontrivial simple closed curves J_1 and J_2 so that neither lies in a disk bounded by the other in D.*

Proof. Without loss of generality we assume that all the curves of $\operatorname{Int} D \cap \operatorname{Bd} T_2$ are trivial in $\operatorname{Bd} T_2$. By the standard disk swapping techniques we may isotope D to get a new disk D' with the same boundary that does not meet $\operatorname{Bd} T_2$ in the interior of D' and so that D' meets $\operatorname{Bd} T_1$ in a subcollection of the simple closed curves of $D \cap \operatorname{Bd} T_1$ all of which are nontrivial in $\operatorname{Bd} T_1$ and fixed by the isotopy. The disk D' cannot lie on the outside of T_2 since T_2 does not lie in a ball in M. Hence D' is a meridional disk of T_2. Since T_1 does not lie in a ball inside T_2, $D' \cap \operatorname{Bd} T_1$ contains at least one nontrivial curve in $\operatorname{Bd} T_1$. Since T_1 is contractible in T_2 and does not lie in a ball of T_2, there must be at least two nontrivial simple closed curves J_1 and J_2 so that neither lies in the disk bounded by the other in D'. This can be seen by lifting D' and T_1 to the universal cover of T_2 and using linking theory. Thus J_1 and J_2 are the required simple closed curves in D. □

Lemma 3.5. *Let M be an irreducible 3-manifold, P be a properly-embedded plane in M, and $U = \bigcup T_i$ be an open subset of M that is homeomorphic to a genus one Whitehead manifold with T_i a defining sequence of solid tori. Then for each i, T_i can be isotoped off P by an isotopy of M.*

Proof. If T_i lies in a 3-ball of M, then we are done. We suppose that T_i does not lie in a 3-ball of M. By general position we assume that $\operatorname{Bd} T_j$ intersects P in a finite collection of simple closed curves for each j. By Lemma 3.4 for sufficiently large j all of the simple closed curves in $\operatorname{Bd} T_j \cap P$ are trivial in $\operatorname{Bd} T_j$. Assuming that this is the case, $\operatorname{Bd} T_j$, and hence T_i which lies in T_j, can be isotoped off P by induction on the number of simple closed curves in $\operatorname{Bd} T_j \cap P$. Let J be a simple closed curve of $\operatorname{Bd} T_j \cap P$ that is innermost in $\operatorname{Bd} T_j$. Let D_1 and D_2 be disks in $\operatorname{Bd} T_j$ and P, respectively bounded by J. The 2-sphere $D_1 \cup D_2$ bounds a 3-ball which can be used to isotope D_1 to D_2 by an isotopy of M that is fixed outside of small neighborhood of the 3-ball. Now by pushing D_2 to the side of P we have accomplished our goal of reducing the number of simple closed curves in $\operatorname{Bd} T_j \cap P$ by an isotopy. □

The above proof uses a standard technique in the study of 3-manifolds. In each stage of the induction, points outside a small neighborhood of the 3-ball need not be moved. Hence, we obtain the following addendum to the lemma which we state without proof.

Addendum to Lemma 3.5. *If K is a compact, connected subset of T_i which does not lie in a 3-ball in M and so that $K \cap P = \emptyset$, then K can be left fixed by the isotopy.*

Lemma 3.6. *Let K be a compact subset of a composite Whitehead manifold $W = W_1 \# W_2 \# \cdots$. If K is π_1-trivial at infinity in W_i for each i, then K lies in a 3-ball in W.*

Proof. Let U be the connecting ball for the decomposition $W_1 \# W_2 \# \cdots$. The compact set K meets at most finitely many of the W_i. Using the fact that K is π_1-trivial at infinity in W_i for each i and the fact that each W_i is the union of nested eyebolts, we see that K lies in the interior of a cube with handles R satisfying the following conditions:

(1) $R \cap \overline{W}_i$ is either empty or an eyebolt as described at the beginning of this section,

(2) $R \cap \overline{U}$ is a 3-ball,

(3) loops in $W_i - \operatorname{Int} R$ are inessential in $W - K$.

From these conditions it follows that any loop in $\operatorname{Bd} R$ is inessential in $W - K$. Standard techniques using the Loop theorem [15] show that K lies in the interior of a compact manifold whose boundary is a 2-sphere, but the irreducibility of W shows that this manifold is a ball. □

As promised, we now establish that any homeomorphism $h: W \to W'$ between composite Whitehead manifolds naturally induces a one-to-one correspondence $\Phi_h : \{W_i\} \to \{W_i'\}$ between the Whitehead manifold building blocks.

Lemma 3.7. *Let $W = W_1 \# W_2 \# \cdots$ and $W' = W'_1 \# W'_2 \# \cdots$ be composite Whitehead manifolds. For each homeomorphism $h : W \to W'$ there is a well defined function Φ_h between $\{W_i\}$ and $\{W'_i\}$. We define $\Phi_h(W_i)$ to be the unique W'_j so that $h(T)$ can be ambiently isotoped into W'_j for any torus T that is part of a defining sequence for the Whitehead manifold W_i.*

Proof. Since T does not lie in a ball in W_i, Lemma 3.3 shows that T does not lie in a ball in W. Therefore, $h(T)$ does not lie in a ball in W' and by Lemma 3.6 must not be π_1-trivial at infinity in some W'_j. By Lemma 3.5 $h(T)$ can be isotoped off the properly-embedded plane $\overline{W}'_j - W'_j = P'_j$. Since P'_j separates W', the isotoped $h(T)$ must lie in W'_j or the complement of $\overline{W}'_j$. But, by Lemma 3.2, the second case is impossible and we see that $h(T)$ can be isotoped into W'_j. Similarly by Lemma 3.2, $h(T)$ cannot be isotoped into W'_m for $m \neq j$, for this would imply that $h(T)$ is π_1-trivial at infinity in W'_j. Furthermore, it should be clear that the W'_j is independent of the choice of T. □

Lemma 3.8. *Let $W = W_1 \# W_2 \# \cdots$, $W' = W'_1 \# W'_2 \# \cdots$, and $W'' = W''_1 \# W''_2 \# \cdots$ be composite Whitehead manifolds. If $f : W \to W'$ and $g : W' \to W''$ are homeomorphisms then $\Phi_g \circ \Phi_f = \Phi_{g \circ f}$.*

Proof. Let T be a solid torus that is part of a defining sequence for the Whitehead manifold W_i. Then $f(T)$ can be isotoped into W'_j where $\Phi_f(W_i) = W'_j$ by an isotopy F_t of W' where $F_1(f(T))$ is a subset of W'_j. Let T' be a solid torus that is part of a defining sequence for W'_i that contains $F_1(f(T))$. Then $g(T')$ can be isotoped into W''_k where $\Phi_g(W'_j) = W''_k$ by an isotopy G_t of W'' where $G_1(g(T'))$ is a subset of W''_k. The isotopy $g \circ F_t \circ g^{-1}$ takes $g(f(T))$ into $g(T')$. The isotopy G_t takes $g(T')$ into W''_k. Hence $g(f(T))$ can be isotoped into W''_k and we see that $\Phi_g \circ \Phi_f = \Phi_{g \circ f}$. □

Theorem 3.9. *Let $W = W_1 \# W_2 \# \cdots$ and $W' = W'_1 \# W'_2 \# \cdots$ be composite Whitehead manifolds. For each homeomorphism $h : W \to W'$ there is a well defined one-to-one correspondence Φ_h between $\{W_i\}$ and $\{W'_i\}$ given by $\Phi_h(W_i)$ equals the unique W'_j so that $h(T)$ can be isotoped into W'_j for any solid torus T that is part of a defining sequence for the Whitehead manifold W_i by an isotopy of W'.*

Proof. By Lemma 3.7 Φ_h is well defined. To see that the function is one-to-one and onto we use Lemma 3.8 to note that both $\Phi_h \circ \Phi_{h-1}$ and $\Phi_{h-1} \circ \Phi_h$ are identity functions. □

Definition 3.10. If $W = W_1 \# W_2 \# \cdots$ is a composite Whitehead manifold and G is a group of homeomorphisms of W, we define a group action of G on $\{W_i\}$. For $g \in G$, we set $g \cdot W_i = \Phi_g(W_i)$.

4. Uncountably many composite Whitehead manifolds that are covering spaces

For each sequence $\alpha = (\alpha_i)$ of positive integers we construct a specific Whitehead manifold W_α so that $W_\alpha = \bigcup T_i$ so that T_i goes $2\alpha_i$ times around T_{i+1} geometrically and zero times algebraically. McMillan [11] has shown that if α and β are sequences of distinct odd primes such that an infinite number of primes occur in α which do not occur in β, then W_α and W_β are topologically different. We prove a related theorem for composite Whitehead manifolds that occur as nontrivial covering spaces.

An open solid torus is a space homeomorphic to $\operatorname{Int} \boldsymbol{B}^2 \times \boldsymbol{S}^1$. For each sequence $\alpha = (\alpha_i)$ of positive integers we construct a 3-manifold V_α to be an end sum between an open solid torus and W_α. The universal covering space of V_α is denoted by $\widetilde{V}_\alpha$ and is seen to be a composite Whitehead manifold $W_1 \# W_2 \# \cdots$ where each W_i is homeomorphic with W_α.

Theorem 4.1. *Let $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ be infinite sequences of distinct odd primes such that an infinite number of primes occur in α which do not occur in β. Then $\widetilde{V}_\alpha$ and $\widetilde{V}_\beta$ are not homeomorphic.*

Proof. Let $\widetilde{V}_\alpha = W_1 \# W_2 \# \cdots$ where each W_i is homeomorphic with W_α and $\widetilde{V}_\beta = W_1' \# W_2' \# \cdots$ where each W_i' is homeomorphic with W_β. Suppose $h : \widetilde{V}_\alpha \to \widetilde{V}_\beta$ is a homeomorphism. We let $W_1 = \bigcup T_i$ and $W_1' = \bigcup T_i'$ be defining sequences for the Whitehead manifolds W_1 and W_1' so that T_i goes around T_{i+1} $2\alpha_i$ times geometrically and zero times algebraically and T_i' goes around T_{i+1}' $2\beta_i$ times geometrically and zero times algebraically. We may assume by Theorem 3.9 that $h^{-1}(T_1')$ lies in some W_i which, by renumbering if necessary, we let be W_1. Since $h^{-1}(T_1')$ is compact it lies in some T_m which in turn lies in some T_{m+k} so that the geometric index of T_m in T_{m+k} has an odd prime factor p not equal to β_i for any i. Now $h(T_{m+k})$ can be isotoped into W_1' by an isotopy H_t of $\widetilde{V}_\beta$. By the addendum to Lemma 3.5, we may assume that points in T_1' are fixed by the isotopy. Hence, we have

$$T_1' \subset H_1\big(h(T_m)\big) \subset H_1\big(h(T_{m+k})\big) \subset T_n'$$

for some value of n. But the geometric index of $H_1(h(T_m))$ in $H_1(h(T_{m+k}))$ is the same as the geometric index of T_m in T_{m+k} which contains the odd prime factor p not equal to β_i for any i. But by Schubert's lemma, the geometric index of T_1' in T_n' contains the factor p, a contradiction. So we see that there is no homeomorphism between $\widetilde{V}_\alpha$ and $\widetilde{V}_\beta$. $\square$

There are uncountably many sequences so that any two of these sequences satisfy the conditions in Theorem 4.1. So we obtain the following theorem.

Theorem 4.2. *There exist uncountably many composite Whitehead manifolds of the form $\widetilde{V}_\alpha$.*

McMillan and Thickstun [12] have pointed out that since there are only countably many closed 3-manifolds, there can only be countably many contractible open 3-manifolds that cover closed 3-manifolds. This yields the following theorem.

Theorem 4.3. *Uncountably many of the composite Whitehead manifolds of the form $\widetilde{V}_\alpha$ do not cover closed manifolds, but they do nontrivially cover open manifolds.*

We know that $\widetilde{V}_\alpha$ covers V_α, but $\widetilde{V}_\alpha$ also covers many other manifolds. For example, $\widetilde{V}_\alpha$ covers an end sum of $(M^2 \times \mathbb{R})$ with W_α where M^2 is any closed 2-manifold except the 2-sphere or projective plane.

5. Composite Whitehead manifolds as covering spaces

In this section we construct a large class of composite Whitehead manifolds that, although not eventually π_1-injective at infinity, share many properties with simple Whitehead manifolds. In particular, they cannot be nontrivial covering spaces. Our techniques directly generalize those from [18].

We must modify the Ratchet lemma from [18] as follows.

Lemma 5.1 (Special Ratchet lemma). *Let M be an open manifold and V be an open subset of M so that V is eventually π_1-injective at infinity. Furthermore, suppose $\overline{V}$, the closure of V, is a manifold with simply connected boundary. Let $h: M \to M$ be a homeomorphism with the property that for a core K of V, $h(K)$ and $h^{-1}(K)$ can be isotoped into V by an isotopy of M. Then there is a compact set L in M so that a loop γ in $M - \bigcup_{i=-\infty}^{\infty} h^i(L)$ is inessential in $M - K$ if and only if γ is inessential in $M - h^i(K)$ for each i.*

Proof. By hypothesis there are isotopies of M that take $h(K)$ and $h^{-1}(K)$ to sets K^+ and K^-, respectively, which lie in V. Let T^+ and T^-, be the respective tracks of $h(K)$ and $h^{-1}(K)$ under these isotopies. Consider the compact set $A = K^+ \cup K \cup K^-$ which lies in V. Since V is eventually π_1-injective at infinity, there is a compact set B in V $(A \subset B)$ so that loops in $V - B$ which are inessential in $V - K$ are also inessential in $V - A$. Note that since $\overline{V}$ has simply connected boundary, this also implies that loops in $M - B$ which are inessential in $M - K$ are also inessential in $M - A$. Let L be the compact set $T^+ \cup T^- \cup B$.

Now let γ be a loop in $M - \bigcup_{i=-\infty}^{\infty} h^i(L)$. If γ is inessential in $M - K$, then γ is inessential in $M - A$. Since $K^+ \subset A$, γ is inessential in $M - K^+$. Now $h(K)$ is isotopic to K^+ by an isotopy so that the track of $h(K)$ misses γ. Hence, the Covering Isotopy Lemma [3,5] implies that γ is inessential in $M - h(K)$. So far we have shown that if γ is inessential in $M - K$, then γ is inessential in $M - h(K)$.

If γ is inessential in $M - h(K)$, then γ is inessential in $M - h(A)$. Since $K^- \subset A$, γ is inessential in $M - h(K^-)$. Now K is isotopic to $h(K^-)$ by an isotopy so that the track of K misses γ. As before, this implies that γ is inessential in $M - K$.

We have thus shown that γ is inessential in $M - K$ if and only if γ is inessential in $M - h(K)$. The rest of the proof now follows by induction. □

We can use the Orbit lemma from [18] unchanged.

Lemma 5.2 (Orbit lemma). *Suppose W is an open contractible n-manifold, $n \geqslant 3$. Let h be a nontrivial homeomorphism of W onto itself so that the group G of homeomorphisms* *generated by h acts without fixed points and properly discontinuously on W. If C is a compact subset of W, then loops of W can be homotoped off $\bigcup_{i=-\infty}^{\infty} h^i(C)$, the orbit of C.*

Furthermore, given a compact set A there is a bigger compact set B so that loops in $W - B$ can be homotoped off the orbit of C by a homotopy that lies in $W - A$.

Recall that a group G of homeomorphisms of a manifold W acts *properly discontinuously* on W if for every compact subset K of W the set

$$\{g \in G \mid K \cap g(K) \neq \emptyset\}$$

is finite. Recall also that a group acts *without fixed points* on a set if the only element having any fixed points is the identity element.

Theorem 5.3. *If G acts properly discontinuously without fixed points on a composite Whitehead manifold $W = W_1 \# W_2 \# \cdots$, then G acts without fixed points on $\{W_i\}$; i.e., if $g \in G$, and $g \neq 1$, then $g \cdot W_i \neq W_i$ for each i.*

Proof. Suppose that $g \in G$ and $g \cdot W_k = W_k$. If $g \neq 1$, then g has infinite order [13,18]. Let T be a solid torus in W_k which is also a core. Thus, $g(T)$ and $g^{-1}(T)$ can both be isotoped into W_k by an isotopy of W, and all the hypotheses of the Special Ratchet lemma are satisfied for the manifold W with open set W_k and core T.

Let L be the compact set promised by the Special Ratchet lemma. We now show that T is π_1-trivial at infinity in W_k. By the Orbit lemma there is a compact set A containing T so any loop γ in $W - A$ can be homotoped to a loop γ' in the complement of $\bigcup_{i=-\infty}^{\infty} g^i(L)$ by a homotopy that misses T. Now γ' is inessential in a compact subset D of W. Since the group G acts properly discontinuously and since g has infinite order, $g^i(T)$ must miss D for some value of i. So γ' is inessential in the complement of $g^i(T)$. Now the Special Ratchet lemma implies that γ' is inessential in $W - T$ which in turn implies that γ is inessential in $W - T$. Thus we see that T is π_1-trivial at infinity in W_k.

Since T lies in W_k, T is clearly π_1-trivial at infinity in W_i for $i \neq k$. So T is π_1-trivial at infinity in W_i for each i. Theorem 3.6 implies that T lies in a ball of W. Lemma 3.3 shows that T lies in a ball of W_i which is a contradiction, and our theorem is proved. □

By carefully partitioning the primes, we obtain a main example of this section, a composite Whitehead manifold which only trivially covers itself.

Theorem 5.4. *Let $W = W_{\alpha(1)} \# W_{\alpha(2)} \# \cdots$ be a composite Whitehead manifold so that $\alpha(i) = (\alpha_{ij})$ is a sequence of distinct odd primes and $W_{\alpha(i)}$ is the Whitehead manifold described in Section 4. Furthermore, suppose that for some k and any $i \neq k$ an infinite number of primes occur in $\alpha(k)$ that do not occur in $\alpha(i)$. Then W cannot nontrivially cover any 3-manifold.*

Proof. Suppose that h is a covering translation of W. By a proof similar to the proof of Theorem 4.1, we see that $\Phi_h(W_{\alpha(k)}) = W_{\alpha(k)}$. Theorem 5.3 implies that h equals the identity. □

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