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Green's matrices

V. Olshevsky [b], G. Strang [a], P. Zhlobich [b,*]

[a] *Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA 02139, USA*
[b] *Department of Mathematics, University of Connecticut, Storrs CT 06269, USA*

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A B S T R A C T

Our goal is to identify and understand matrices A that share essential properties of the unitary Hessenberg matrices M that are fundamental for Szegö's orthogonal polynomials. Those properties include: **(i)** Recurrence relations connect characteristic polynomials $\{r_k(x)\}$ of principal minors of A. **(ii)** A is determined by generators (parameters generalizing reflection coefficients of unitary Hessenberg theory). **(iii)** Polynomials $\{r_k(x)\}$ correspond not only to A but also to a certain "CMV-like" five-diagonal matrix. **(iv)** The five-diagonal matrix factors into a product BC of block diagonal matrices with 2×2 blocks. **(v)** Submatrices above and below the main diagonal of A have rank 1. **(vi)** A is a multiplication operator in the appropriate basis of Laurent polynomials. **(vii)** Eigenvectors of A can be expressed in terms of those polynomials.

Condition **(v)** connects our analysis to the study of quasi-separable matrices. But the factorization requirement **(iv)** narrows it to the subclass of "Green's matrices" that share Properties **(i)–(vii)**.

The key tool is "twist transformations" that provide 2^n matrices all sharing characteristic polynomials of principal minors with A. One such twist transformation connects unitary Hessenberg to CMV. Another twist transformation explains findings of Fiedler who noticed that companion matrices give examples outside the unitary Hessenberg framework. We mention briefly the further example of a Daubechies wavelet matrix. Infinite matrices are included.

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* Corresponding author.
E-mail addresses: olshevsky@uconn.edu (V. Olshevsky), gs@math.mit.edu (G. Strang), zhlobich@math.uconn.edu (P. Zhlobich).

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1. Introduction

Various polynomial systems $\{r_k(x)\}_{k=0}^{n}$ are associated with n by n Hessenberg matrices H via

$$r_0(x) = \lambda_0, \quad r_k(x) = \lambda_0\lambda_1 \ldots \lambda_k \det(xI - H_{k\times k}), \quad k = 1, \ldots, n. \tag{1.1}$$

The relation (1.1) establishes a bijection [5] if $\lambda_k = 1/h_{k+1,k}$ and λ_0, λ_n are two given parameters:

$$\{r_k(x)\}_{k=0}^{n} \longleftrightarrow \{H, \lambda_0, \lambda_n\} \tag{1.2}$$

1.1. From Hessenberg to five-diagonal matrices. Two examples

It is widely known that Szegö polynomials $\{\phi_k^{\#}(x)\}_{k=0}^{n}$ orthogonal on the unit circle are connected via (1.1) to a certain (almost[1]) unitary Hessenberg matrix

$$M = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & \cdots & -\rho_0^*\mu_1\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & \cdots & -\rho_1^*\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ 0 & \mu_2 & -\rho_2^*\rho_3 & \cdots & -\rho_2^*\mu_3\cdots\mu_{n-1}\rho_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -\rho_{n-1}^*\rho_n \end{bmatrix}, \tag{1.3}$$

where ρ_k are *reflection coefficients*[2] and μ_k are *complementary parameters*. The details on this relation can be found in [23,25,4,34,2,10,29,27,3,28]. Matrix M has rather dense structure in comparison with the tridiagonal Jacobi matrix [1,11,22] for orthogonal polynomials on the real line. However, the bijection (1.2) implies that for a given system of Szegö polynomials there are no matrices other than M. The situation is much different if we do not restrict the matrix to the class of strictly upper Hessenberg matrices.

It was found first by Kimura [24] and independently by Cantero et al. [13–15] that Szegö polynomials are also related via (1.1) (with $\lambda_k = 1/\mu_k$) to the following five-diagonal "CMV matrix":

$$\mathcal{K} = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{1.4}$$

The initials CMV honor the paper [13] that triggered deep interest in the orthogonal polynomials community. This matrix is reputed to be better than unitary Hessenberg in studying properties of polynomials orthogonal on the unit circle (mostly because of its banded structure).

Shortly after the discovery of the CMV matrix it was noticed that this is not the only example of its kind. Consider the companion matrix

$$C = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.5}$$

Characteristic polynomials $p_k(x)$ of its leading submatrices are so-called Horner polynomials. It was shown by Fiedler [35] that the five-diagonal matrix

[1] Throughout the paper, matrices referred to as unitary Hessenberg are almost unitary, differing from unitary in the length of the last column. Specifically, $M = UD$ for a unitary matrix U and diagonal matrix $D = \text{diag}\{1, \ldots, 1, \rho_n\}$.

[2] Reflection coefficients are also known in various contexts as Schur parameters [30] and Verblunsky coefficients [31].

$$F = \begin{bmatrix} -a_1 & -a_2 & 1 & & & & \\ 1 & 0 & 0 & 0 & & & \\ 0 & -a_3 & 0 & -a_4 & 1 & & \\ & 1 & 0 & 0 & 0 & 0 & \\ & & 0 & -a_5 & 0 & -a_6 & 1 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (1.6)$$

is also related to the same set of Horner polynomials.

1.2. Quasi-separable approach. Twist transformation

In a recent paper we used the theory of quasi-separable matrices to derive a number of new results on five-diagonal matrices. In particular, we gave a unified proof of the fact that CMV and Fiedler matrices share systems of characteristic polynomials with unitary Hessenberg and companion matrices correspondingly. Let us outline the idea of the proof.

Following [17,19] we define the class of $(1,1)$-qs matrices:

Definition 1.1 (*Generator definition of* $(1,1)$*-qs matrices*). A matrix A is called $(1,1)$-qs if it can be represented in the form

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & \cdots & \cdots & g_1b_2\dots b_{n-1}h_n \\ p_2q_1 & d_2 & g_2h_3 & \cdots & \cdots & g_2b_3\dots b_{n-1}h_n \\ p_3a_2q_1 & p_3q_2 & d_3 & \cdots & \cdots & g_3b_4\dots b_{n-1}h_n \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & d_{n-1} & g_{n-1}h_n \\ p_na_{n-1}\dots a_2q_1 & p_na_{n-1}\dots a_3q_2 & p_na_{n-1}\dots a_4q_3 & \cdots & p_nq_{n-1} & d_n \end{bmatrix}$$

The parameters $\{q_k, a_k, p_k, d_k, g_k, b_k, h_k\}$ are called *generators* of A.

It turns out that all the matrices described in the previous subsection (unitary Hessenberg, CMV, companion and Fiedler) are $(1,1)$-qs matrices. We prove this by specifying generators of these matrices in Table 1.

One of many useful properties of $(1,1)$-qs matrices is the existence of two-term recurrence relations for polynomials related to them via (1.1).

Theorem 1.2 [20]. *Let* $\{r_k(x)\}_{k=0}^n$ *be a system of polynomials related to a* $(1,1)$*-qs matrix* A *via* (1.1). *Then they satisfy two-term recurrence relations*

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_0 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \lambda_k \begin{bmatrix} a_kb_kx - c_k & -q_kg_k \\ p_kh_k & x - d_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}, \quad (1.7)$$

where $c_k = d_ka_kb_k - q_kp_kb_k - g_kh_ka_k$.

Table 1
Generators of unitary Hessenberg, CMV, companion and Fiedler matrices.

Matrix	k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
(1.3)	Any	$-\rho_{k-1}^*\rho_k$	0	μ_k	μ_k	$-\rho_{k-1}^*\mu_k$	1	ρ_k
(1.4)	Odd	$-\rho_{k-1}^*\rho_k$	0	μ_k	μ_k	$-\rho_{k-1}^*\mu_k$	1	ρ_k
	Even	$-\rho_{k-1}^*\rho_k$	μ_k	0	$-\rho_{k-1}^*\mu_k$	μ_k	ρ_k	1
(1.5)	1	$-a_1$	–	–	1	1	–	–
	> 1	0	0	1	1	0	1	$-a_k$
(1.6)	1	$-a_1$	–	–	1	1	–	–
	> 1, odd	0	1	0	0	1	$-a_k$	1
	Even	0	0	1	1	0	1	$-a_k$

What one can get immediately from this theorem is that the interchange of lower and upper generators:

$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k \tag{1.8}$$

for some k does not change the recurrence relations (1.7) and, hence, does not change the polynomials $\{r_k(x)\}_{k=0}^n$. We propose to call operation (1.8) a *twist transformation*.

Comparing generators given in Table 1, each CMV matrix is obtained from unitary Hessenberg via twist transformations for even indices. Similarly, each Fiedler matrix is obtained from companion via twist transformations for odd indices $k > 1$. This explains why unitary Hessenberg and CMV as well as companion and Fiedler matrices share the same systems of characteristic polynomials.

1.3. Main results

Let us consider two important aspects as follows.

A. Factorizations. Both CMV matrix K and Fiedler matrix F admit factorizations into block diagonal matrices with 2 by 2 blocks. Note the shift in block positions between even and odd k.

$$K = [\Gamma_0\Gamma_2\ldots]\cdot[\Gamma_1\Gamma_3\ldots], \quad F = [A_1A_3\ldots]\cdot[A_2A_4\ldots], \tag{1.9}$$

where

$$\Gamma_0 = \left[\begin{array}{c|c} \rho_0^* & \\ \hline & I_{n-1}\end{array}\right], \quad \Gamma_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & -\rho_k & \mu_k & \\ & \mu_k & \rho_k^* & \\ \hline & & & I_{n-k-1}\end{array}\right], \quad \Gamma_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & -\rho_n\end{array}\right]$$

and

$$A_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & -a_k & 1 & \\ & 1 & 0 & \\ \hline & & & I_{n-k-1}\end{array}\right], \quad A_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & -a_n\end{array}\right].$$

We refer to [24,35] for details. Factorization (1.9) implies a number of results for CMV matrices and greatly simplifies proofs, see, for instance, [12,26,32,33]. The recent paper [9] considered a class of so-called *twisted* $(H,1)$-*qs* matrices generalizing CMV and Fiedler. Unfortunately, twisted $(H,1)$-qs matrices, in general, may not have a factorization similar to (1.9) which tells us that this class is just too wide.

B. Laurent polynomials. CMV matrices are often associated with Laurent polynomials on the unit circle. Actually the CMV matrix is just the representation of the multiplication operator in this "Laurent" basis [13,33].

In the present paper we identify a subclass of twisted $(H,1)$-qs matrices (called *twisted Green's matrices*) that is crucial in addressing these two problems **A** and **B**. In Section 3 we provide several descriptions of this class (entrywise characterization, generator characterization, polynomial characterization).

Furthermore, in Section 4 we observe that the class is exactly the one admitting factorization, of which (1.9) is the special case.

Finally in Section 5, we specify the twist transformation of [9] to Green's case (introducing an additional new Green's twist transformation), and apply the new theory to study twisted $(H,1)$-qs Green's matrices. Specifically, we use it to identify the related Laurent polynomials (general enough to include those of [13] as a special case) and show that a twisted $(H,1)$-qs Green's matrix serves as an operator of multiplication in the basis of Laurent polynomials.

In the last Section 6 we apply the results of [18] to derive efficient algorithms for inversion of Green's matrices.

2. Preliminaries. Twist transformation and twisted $(H, 1)$-qs matrices

2.1. Twist transformation

A system of polynomials can be related to many distinct $(1, 1)$-qs matrices (Definition 1.1). For instance, a nonsymmetric $(1, 1)$-qs matrix and its transpose share the same system of polynomials. In this subsection we show how for a given $(1, 1)$-qs matrix one can obtain other $(1, 1)$-qs matrices related to the same system of polynomials as the original one.

Definition 2.1 (*Twist transformation*). We say that a $(1, 1)$-qs matrix $\widetilde{A}$ having generators $\{\widetilde{p}_k, \widetilde{q}_k, \widetilde{a}_k, \widetilde{g}_k, \widetilde{h}_k, \widetilde{b}_k, \widetilde{d}_k\}$ is obtained via twist transformation from another $(1, 1)$-qs matrix A with generators $\{p_k, q_k, a_k, g_k, h_k, b_k, d_k\}$ if there is k between 1 and n such that

$$\begin{cases} \widetilde{q}_1 = g_1, \quad \widetilde{g}_1 = q_1, \quad \widetilde{d}_1 = d_1 & \text{if } k = 1, \\ \widetilde{p}_n = h_n, \quad \widetilde{h}_n = p_n, \quad \widetilde{d}_n = d_n & \text{if } k = n, \\ \widetilde{p}_k = h_k, \quad \widetilde{q}_k = g_k, \quad \widetilde{a}_k = b_k, & \\ \widetilde{h}_k = p_k, \quad \widetilde{g}_k = q_k, \quad \widetilde{b}_k = a_k, \quad \widetilde{d}_k = d_k & \text{otherwise} \end{cases} \tag{2.1}$$

and all other generators of $\widetilde{A}$ and A are equal.

In other words, $\widetilde{A}$ is obtained from A via the interchange of *lower* and *upper* generators:

$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k$$

for some k. This is why we propose to call (2.1) *twist transformation*.

The significant feature of the twist transformation is that it transforms one $(1, 1)$-qs matrix into another preserving the coefficients of the recurrence relations (1.7) and, thus, characteristic polynomials of all their submatrices. The next theorem exploits this fact.

Theorem 2.2. *Let $\{r_k(x)\}_{k=0}^n$ be a system of polynomials related to a $(1, 1)$-qs matrix A. Then it is invariant under any combination of twist transformations (2.1) for different indices k.*

Proof. It is enough to prove the proposition for only one twist transformation for index k. Let $\widetilde{A}$ be the matrix obtained from A via (2.1) and $\{\widetilde{r}_k(x)\}_{k=0}^n$ be the system of polynomials related to $\widetilde{A}$. Considering the recurrence relations (1.7) for polynomials related to $(1, 1)$-qs matrices and noticing that

$$\widetilde{a}_k\widetilde{b}_k = a_k b_k, \quad \widetilde{p}_k\widetilde{h}_k = p_k h_k, \quad \widetilde{d}_k = d_k,$$
$$\widetilde{d}_k\widetilde{a}_k\widetilde{b}_k - \widetilde{q}_k\widetilde{p}_k\widetilde{b}_k - \widetilde{g}_k\widetilde{h}_k\widetilde{a}_k = d_k a_k b_k - q_k p_k b_k - g_k h_k a_k.$$

we conclude that both systems of polynomials $\{r_k(x)\}_{k=0}^n$ and $\{\widetilde{r}_k(x)\}_{k=0}^n$ satisfy the same recurrence relations and, hence, coincide. □

Corollary 2.3. *One can see from Table 1 that CMV (1.4) and Fiedler (1.6) matrices are obtained via twist transformations from unitary Hessenberg (1.3) and companion (1.5) matrices. Hence, unitary Hessenberg and CMV as well as companion and Fiedler matrices share the same systems of characteristic polynomials.*

Corollary 2.4. *For an arbitrary $(1, 1)$-qs matrix A of size n specified by its generators, there exist 2^n (possibly not distinct) matrices obtained from A via twist transformations for different indices k and related to the same system of polynomials.*

2.2. Twisted $(H, 1)$-qs matrices

Following [6–8] we define the class of matrices which are both strictly[3] upper Hessenberg and $(1, 1)$-qs:

[3] i.e. having nonzero elements along the first subdiagonal.

Definition 2.5 (*Generator definition of* $(H,1)$*-qs matrices*). A matrix A is called $(H,1)$-qs (i.e., *Hessenberg Order-One-Quasi-separable*) if it can be represented in the form

$$A=\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & \cdots & \cdots & g_1b_2\dots b_{n-1}h_n \\ q_1 & d_2 & g_2h_3 & \cdots & \cdots & g_2b_3\dots b_{n-1}h_n \\ 0 & q_2 & d_3 & \cdots & \cdots & g_3b_4\dots b_{n-1}h_n \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & q_{n-2} & d_{n-1} & g_{n-1}h_n \\ 0 & \cdots & \cdots & 0 & q_{n-1} & d_n \end{bmatrix}, \tag{2.2}$$

where the parameters $\{q_k \neq 0, d_k, g_k, b_k, h_k\}$ are called *generators* of A.

Remark 2.6. Comparing Definitions 1.1 and 2.5 one can easily see that a $(1,1)$-qs matrix is $(H,1)$-qs if and only if it has a choice of generators such that $a_k = 0, p_k = 1, q_k \neq 0$ for all k.

There exists an alternative definition of $(H,1)$-qs matrices in terms of ranks of their submatrices which reveals the idea behind the Definition 2.5:

Definition 2.7 (*Rank definition for* $(H,1)$*-qs matrices*). A matrix A is called $(H,1)$*-qs* if $\max_{1\leqslant i\leqslant n-1}$ rank $A(1:i, i+1:n) = 1$.

It is easy to check that both unitary Hessenberg and companion matrices are $(H,1)$-qs. As we have seen CMV and Fiedler matrices can be obtained from them via twist transformations. In order to generalize these results we define next the entire class of matrices which can be obtained from $(H,1)$-qs matrices via twist transformations.

Definition 2.8 (*Twisted* $(H,1)$*-qs matrices*). A $(1,1)$-qs matrix A is called twisted $(H,1)$-qs if it can be obtained from an $(H,1)$-qs matrix via twist transformations.

Performing the twist transformation of the matrix (2.2) explicitly, one can give the following alternative definition in terms of generators:

Definition 2.9 (*Generator definition of twisted* $(H,1)$*-qs matrices*). A $(1,1)$-qs matrix A is twisted $(H,1)$-qs if and only if it has a choice of generators $\{p_k, q_k, a_k, g_k, h_k, b_k, d_k\}$ such that

$$\begin{cases} q_1 \neq 0 \quad \text{or} \quad g_1 \neq 0, \\ a_k = 0,\ q_k \neq 0,\ p_k = 1 \quad \text{or} \quad b_k = 0,\ g_k \neq 0,\ h_k = 1, \quad k = 2\dots n-1, \\ p_n = 1 \quad \text{or} \quad h_n = 1. \end{cases}$$

For an arbitrary $(H,1)$-qs matrix A with given generators according to the Corollary 2.4 there are 2^n (possibly not distinct) twisted—$(H,1)$-qs matrices related to the same polynomial system as A. But it is always feasible to distinguish them using the *pattern* defined next as the set of "twisted indices".

Definition 2.10 (*Pattern of twisted* $(H,1)$*-qs matrices*). For an arbitrary twisted $(H,1)$-qs matrix A, the sequence of binary digits $(i_1, i_2, \dots, i_n)$ is its *pattern* if A can be transformed to some $(H,1)$-qs matrix H applying the twist transformations for all k such that $i_k = 1$. Or, equivalently $(i_1, i_2, \dots, i_n)$ is the pattern of A if there exist generators of A satisfying

$$\begin{cases} q_1 \neq 0 & \text{if } i_1 = 0, \\ g_1 \neq 0 & \text{if } i_1 = 1, \\ a_k = 0,\ q_k \neq 0,\ p_k = 1 & \text{if } i_k = 0, \\ b_k = 0,\ g_k \neq 0,\ h_k = 1 & \text{if } i_k = 1, \\ p_n = 1 & \text{if } i_n = 0, \\ h_n = 1 & \text{if } i_n = 1. \end{cases} \tag{2.3}$$

Under these conditions we write $A = H(i_1, i_2, \ldots, i_n)$.

Example 2.11. Any $(H, 1)$-qs matrix H of size n is $H(0, 0, \ldots, 0)$ and its transpose is $H(1, 1, \ldots, 1)$.

Example 2.12. Comparing generators of unitary Hessenberg and CMV matrices in Table 1, a CMV matrix has pattern $(0, 1, 0, 1, \ldots)$ and a Fiedler matrix has pattern $(1, 0, 1, 0, 1, \ldots)$.

Remark 2.13. Let H be an $(H, 1)$-qs matrix specified by its generators $\{q_k, d_k, g_k, b_k, h_k\}$. Then $H(0, 1, 0, 1, 0, \ldots)$ and $H(1, 0, 1, 0, 1, \ldots)$ are five-diagonal. In particular,

$$H(0,1,0,1,0,\ldots) = \begin{bmatrix} d_1 & g_1 & 0 & & & & & \\ q_1h_2 & d_2 & q_2h_3 & q_2b_3 & & & & \\ q_1b_2 & g_2 & d_3 & g_3 & 0 & & & \\ & 0 & q_3h_4 & d_4 & q_4h_5 & q_4b_5 & & \\ & & q_3b_4 & g_4 & d_5 & g_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

and $H(1, 0, 1, 0, 1, \ldots)$ is its transpose. Thus for every $(H, 1)$-qs matrix there exist five-diagonal twisted $(H, 1)$-qs matrices having the same system of characteristic polynomials.

2.3. The lack of factorization of general twisted $(H, 1)$-qs matrices

It is well-known that unitary Hessenberg matrix (1.3) can be written as the product $M = \Gamma_0\Gamma_1 \Gamma_2 \ldots \Gamma_n$ of Givens rotations (so-called Schur representation):

$$\Gamma_0 = \left[\begin{array}{c|c} \rho_0^* & \\ \hline & I_{n-1} \end{array}\right], \quad \Gamma_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & -\rho_k & \mu_k & \\ & \mu_k & \rho_k^* & \\ \hline & & & I_{n-k-1} \end{array}\right], \quad \Gamma_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & -\rho_n \end{array}\right] \tag{2.4}$$

The companion matrix (1.5) admits similar factorization $C = A_1A_2 \ldots A_n$:

$$A_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & -a_k & 1 & \\ & 1 & 0 & \\ \hline & & & I_{n-k-1} \end{array}\right], \quad A_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & -a_n \end{array}\right]. \tag{2.5}$$

However, general twisted $(H, 1)$-qs matrices do not admit a factorization similar to (2.4) and (2.5). It is proved by the following easy example:

Example 2.14 (*Non-factorizable twisted $(H, 1)$-qs matrix*). Consider the 3×3 twisted $(H, 1)$-qs matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Assume that it has a factorization

$$A = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} a & bf & bg \\ c & df & dg \\ 0 & h & e \end{bmatrix}. \tag{2.6}$$

Then coefficients $\{b, d, f, g\}$ must obey the inconsistent system of equations

$$\begin{cases} bg = df = 0, \\ bf = dg = 1. \end{cases}$$

It is also possible to find a non-Hessenberg non-factorizable twisted $(H, 1)$-qs matrix. Since CMV and Fiedler matrices are factorizable (1.9), we conclude that there must exist a proper subclass of twisted $(H, 1)$-qs matrices admitting a factorization similar to (1.9), (2.4), (2.5). The next two sections are devoted to this problem.

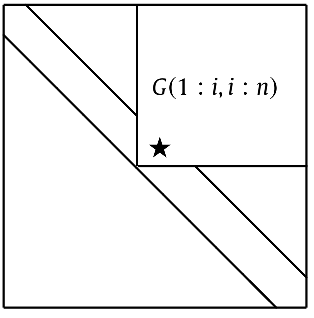
3. Twisted $(H, 1)$-qs Green's matrices and polynomials

We start by defining *Green's $(H, 1)$-qs matrices* which are a proper subclass of $(H, 1)$-qs matrices.

Definition 3.1 (*Rank definition of Green's matrices*). A strictly upper Hessenberg matrix G is called *Green's $(H, 1)$-qs* (or simply *Green's matrix*) if

$$\max_{1 \leqslant i \leqslant n} \operatorname{rank} G(1 : i, i : n) = 1.$$

The difference between $(H, 1)$-qs matrices and Green's matrices is as follows. Submatrices $A(1 : i, i + 1 : n)$ in Definition 2.7 do not include the diagonal while submatrices $G(1 : i, i : n)$ do.



Since every Green's matrix G is $(H, 1)$-qs, it has a generator description as in Definition 2.5. It is more convenient, however, to define generators of Green's matrices in a different way because their rank-one submatrices capture the diagonal. These new generators are given next.

Definition 3.2 (*Generator definition of Green's matrices*). A strictly upper Hessenberg matrix G is *Green's $(H, 1)$-qs* if it can be represented in the form

$$G = \begin{bmatrix} \widehat{\tau}_0\tau_1 & \widehat{\tau}_0\sigma_1\tau_2 & \widehat{\tau}_0\sigma_1\sigma_2\tau_3 & \cdots & \cdots & \widehat{\tau}_0\sigma_1\ldots\sigma_{n-1}\tau_n \\ \widehat{\sigma}_1 & \widehat{\tau}_1\tau_2 & \widehat{\tau}_1\sigma_2\tau_3 & \cdots & \cdots & \widehat{\tau}_1\sigma_2\ldots\sigma_{n-1}\tau_n \\ 0 & \widehat{\sigma}_2 & \widehat{\tau}_2\tau_3 & \cdots & \cdots & \widehat{\tau}_2\sigma_3\ldots\sigma_{n-1}\tau_n \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \widehat{\sigma}_{n-2} & \widehat{\tau}_{n-2}\tau_{n-1} & \widehat{\tau}_{n-2}\sigma_{n-1}\tau_n \\ 0 & \cdots & \cdots & 0 & \widehat{\sigma}_{n-1} & \widehat{\tau}_{n-1}\tau_n \end{bmatrix}, \tag{3.1}$$

where $\{\sigma_k, \tau_k, \widehat{\sigma}_k \neq 0, \widehat{\tau}_k\}$ are called *generators* of G.

Remark 3.3. Table 2 gives the conversion formulas from Green's generators to quasi-separable generators.

Example 3.4. Unitary Hessenberg (1.3) and companion (1.5) matrices in fact belong to the class of Green's matrices. We prove this statement by specifying explicitly in Table 3 their generators as in Definition 3.2.

Table 2
$(H, 1)$-qs generators via Green's generators.

q_k	d_k	g_k	b_k	h_k
$\widehat{\sigma}_k$	$\widehat{\tau}_{k-1}\tau_k$	$\widehat{\tau}_{k-1}\sigma_k$	σ_k	τ_k

Table 3
Green's generators of unitary Hessenberg and companion matrices.

Matrix	k	σ_k	τ_k	$\widehat{\sigma}_k$	$\widehat{\tau}_k$
(1.3)	Any	μ_k	ρ_k	μ_k	$-\rho_k^*$
(1.5)	0	–	–	–	1
	> 0	1	$-a_k$	1	0

Green's matrices are Hessenberg, therefore there is bijection (1.2) between them and polynomial systems. Theorem 3.5 characterizes the polynomial systems related to Green's matrices via (1.1) in terms of recurrence relations satisfied by them.

Theorem 3.5 (Recurrence relations for Green's polynomials). *Let G be an $n \times n$ Green's matrix (3.1) having generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$ and $\{\lambda_0, \lambda_n\}$ – nonzero parameters. Then a system of polynomials $\{r_k(x)\}_{k=0}^n$ is related to G via (1.1) with $\lambda_k = 1/\widehat{\sigma}_k$ if and only if polynomials $r_k(x)$ satisfy two-term recurrence relations*

$$\begin{bmatrix} f_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \delta_0 \end{bmatrix}, \quad \begin{bmatrix} f_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix} \begin{bmatrix} f_{k-1}(x) \\ x \cdot r_{k-1}(x) \end{bmatrix}, \tag{3.2}$$

with $\delta_k = \lambda_k$.

Proof (*Necessity*). Let $\{r_k(x)\}_{k=0}^n$ satisfy recurrence relations (3.3). Then for every k

$$\begin{cases} r_k(x) = \delta_k x \cdot r_{k-1}(x) + \gamma_k f_{k-1}(x), \\ f_k(x) = \beta_k x \cdot r_{k-1}(x) + \alpha_k f_{k-1}(x). \end{cases} \tag{3.3}$$

Using the first equation in (3.3) we can get the expression for $x \cdot r_{k-1}(x)$ and substitute it into the second equation:

$$f_k(x) = \frac{\beta_k}{\delta_k} r_k(x) + \frac{\Delta_k}{\delta_k} f_{k-1}(x), \tag{3.4}$$

where $\Delta_k = \alpha_k\delta_k - \beta_k\gamma_k$.

Eq. (3.4) for different indices k can be used to eliminate recursively f_k-terms in the first equation in (3.3). The final result is

$$r_k(x) = \left(\delta_k x + \frac{\gamma_k\beta_{k-1}}{\delta_{k-1}}\right) r_{k-1}(x) + \frac{\gamma_k\Delta_{k-1}\beta_{k-2}}{\delta_{k-1}\delta_{k-2}} r_{k-2}(x) + \cdots + \frac{\gamma_k\Delta_{k-1}\ldots\Delta_1\beta_0}{\delta_{k-1}\ldots\delta_0} r_0(x). \tag{3.5}$$

These are the unique n-term recurrence relations for the system of polynomials $r_k(x)$ and, hence, there is a unique strictly upper Hessenberg matrix

$$G = \begin{bmatrix} -\frac{\beta_0\gamma_1}{\delta_0\delta_1} & -\frac{\beta_0\Delta_1\gamma_2}{\delta_0\delta_1\delta_2} & -\frac{\beta_0\Delta_1\Delta_2\gamma_3}{\delta_0\delta_1\delta_2\delta_3} & \cdots & -\frac{\beta_0\Delta_1\cdots\Delta_{n-1}\gamma_n}{\delta_0\ldots\delta_n} \\ \frac{1}{\delta_1} & -\frac{\beta_1\gamma_2}{\delta_1\delta_2} & -\frac{\beta_1\Delta_2\gamma_3}{\delta_1\delta_2\delta_3} & \cdots & -\frac{\beta_1\Delta_2\cdots\Delta_{n-1}\gamma_n}{\delta_1\ldots\delta_n} \\ 0 & \frac{1}{\delta_2} & -\frac{\beta_2\gamma_3}{\delta_2\delta_3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\delta_{n-1}} & -\frac{\beta_{n-1}\gamma_n}{\delta_{n-1}\delta_n} \end{bmatrix} \tag{3.6}$$

Table 4
Conversion formulas: Green's two-term r.r. coefficients $\Longleftrightarrow$ Green's generators.

Green's generators				Green's r.r. coefficients			
σ_k	τ_k	$\widehat{\sigma}_k$	$\widehat{\tau}_k$	α_k	β_k	γ_k	δ_k
$\frac{\alpha_k\delta_k-\beta_k\gamma_k}{\delta_k}$	$-\frac{\gamma_k}{\delta_k}$	$\frac{1}{\delta_k}$	$\frac{\beta_k}{\delta_k}$	$\frac{\widehat{\sigma}_k\sigma_k-\widehat{\tau}_k\tau_k}{\widehat{\sigma}_k}$	$\frac{\widehat{\tau}_k}{\widehat{\sigma}_k}$	$-\frac{\tau_k}{\widehat{\sigma}_k}$	$\frac{1}{\widehat{\sigma}_k}$

related to system of polynomials $\{r_k(x)\}_{k=0}^n$ via (1.1) with $\lambda_k = \delta_k$. By comparing (3.6) and (3.1) it is easy to see that this matrix is Green's.

(*Sufficiency*). Let A have generator representation $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$ as in the Definition 3.2. Since it is also $(H, 1)$-qs, its quasi-separable generators (2.5) can be chosen as in Table 2. It was proved in [7] that polynomials related to $(H, 1)$-qs matrices satisfy EGO-type recurrence relations

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \frac{1}{q_0}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{q_k}\begin{bmatrix} q_k p_k b_k & -q_k g_k \\ p_k h_k & x - d_k \end{bmatrix}\begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}. \tag{3.7}$$

Substituting Green's generators from Table 2 into (3.7) we reach the two-term recurrence relations

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \frac{1}{\widehat{\sigma}_0}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\widehat{\sigma}_k}\begin{bmatrix} \widehat{\sigma}_k\sigma_k & -\widehat{\sigma}_k\widehat{\tau}_{k-1}\sigma_k \\ \tau_k & x - \widehat{\tau}_{k-1}\tau_k \end{bmatrix}\begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}. \tag{3.8}$$

We define

$$X_k = \begin{bmatrix} -1 & \widehat{\tau}_k \\ 0 & 1 \end{bmatrix} \quad \text{with } X_k^{-1} = \begin{bmatrix} -1 & \widehat{\tau}_k \\ 0 & 1 \end{bmatrix}.$$

Using X_k and X_k^{-1} we can transform recurrence relations (3.8) into

$$X_k\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \left(X_k\frac{1}{\widehat{\sigma}_k}\begin{bmatrix} \widehat{\sigma}_k\sigma_k & -\widehat{\sigma}_k\widehat{\tau}_{k-1}\sigma_k \\ \tau_k & x - \widehat{\tau}_{k-1}\tau_k \end{bmatrix}X_{k-1}^{-1}\right)X_{k-1}\begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}. \tag{3.9}$$

After matrix multiplications, (3.9) is equivalent to

$$\begin{bmatrix} f_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\widehat{\sigma}_k}\begin{bmatrix} \widehat{\sigma}_k\sigma_k - \widehat{\tau}_k\tau_k & \widehat{\tau}_k \\ -\tau_k & 1 \end{bmatrix}\begin{bmatrix} f_{k-1}(x) \\ x \cdot r_{k-1}(x) \end{bmatrix}, \tag{3.10}$$

where $f_k(x) = \widehat{\tau}_k r_k(x) - F_k(x)$. Hence, the system of polynomials $\{r_k(x)\}_{k=0}^n$ satisfies recurrence relations (3.2). □

Remark 3.6. There are also conversion formulas (Table 4) between Green's generators and recurrence relations (r.r.) coefficients in (3.2).

Example 3.7 (*Recurrence relations for Szegö polynomials*). The well-known two-term recurrence relations for polynomials $\{\phi_k^\#(x)\}_{k=0}^n$ orthogonal on the unit circle [21]

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^\#(x) \end{bmatrix} = \frac{1}{\mu_0}\begin{bmatrix} -\rho_0^* \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^\#(x) \end{bmatrix} = \frac{1}{\mu_k}\begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix}\begin{bmatrix} \phi_{k-1}(x) \\ x \cdot \phi_{k-1}^\#(x) \end{bmatrix} \tag{3.11}$$

are a special case of Green's recurrence relations (3.3).

Example 3.8 (*Recurrence relations for Horner polynomials*). Horner polynomials $\{p_k(x)\}_{k=0}^n$ associated with the companion matrix (1.5) satisfy

$$p_k(x) = x \cdot p_{k-1}(x) + a_k. \tag{3.12}$$

Since every companion matrix is Green's (see Example 3.4) there must exist two-term recurrence relations (3.3) for Horner polynomials. Indeed, one can easily derive them from (3.12):

$$\begin{bmatrix} f_0(x) \\ p_0(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} f_k(x) \\ p_k(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_k & 1 \end{bmatrix} \begin{bmatrix} f_{k-1}(x) \\ x \cdot p_{k-1}(x) \end{bmatrix}, \tag{3.13}$$

where $f_k(x) = 1$ for all k.

Since unitary Hessenberg and companion matrices are in Green's class, CMV and Fiedler matrices belong to the class of matrices obtained from Green's via twist transformations. We suggest to call such matrices *twisted Green's*.

Definition 3.9 (*Twisted Green's matrices*). A $(1, 1)$-qs matrix G is called twisted Green's $(H, 1)$-qs if it can be transformed into some Green's matrix via twist transformations.

If a matrix G is Green's defined by its generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$ as in Definition 3.2, then twisted matrix of pattern $(0, 1, 0, 1, \ldots)$ obtained from G via twist transformations is five-diagonal:

$$G(0,1,0,1,\ldots) = \begin{bmatrix} \widehat{\tau}_0\tau_1 & \widehat{\tau}_0\sigma_1 & 0 & & & & \\ \widehat{\sigma}_1\tau_2 & \widehat{\tau}_1\tau_2 & \widehat{\sigma}_2\tau_3 & \widehat{\sigma}_2\sigma_3 & & & \\ \widehat{\sigma}_1\sigma_2 & \widehat{\tau}_1\sigma_2 & \widehat{\tau}_2\tau_3 & \widehat{\tau}_2\sigma_3 & 0 & & \\ & 0 & \widehat{\sigma}_3\tau_4 & \widehat{\tau}_3\tau_4 & \widehat{\sigma}_4\tau_5 & \widehat{\sigma}_4\sigma_5 & \\ & & \widehat{\sigma}_3\sigma_4 & \widehat{\tau}_3\sigma_4 & \widehat{\tau}_4\tau_5 & \widehat{\tau}_4\sigma_5 & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{3.14}$$

This structure yields a simple lemma:

Lemma 3.10. *A five-diagonal matrix A is twisted Green's of pattern $(0, 1, 0, 1, 0, \ldots)$ if and only if it is block bidiagonal*

$$\begin{bmatrix} \star & \star & 0 & & & & \\ \star & \star & \star & \star & & & \\ \star & \star & \star & \star & 0 & & \\ & 0 & \star & \star & \star & \star & \\ & & \star & \star & \star & \star & 0 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{3.15}$$

with rank-one 2×2 blocks.

Proof. Necessity is obvious because the 2×2 blocks in (3.14) are of rank one. To prove sufficiency notice that if the 2×2 blocks in (3.15) are of rank one, then there exist generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$ such that A coincides with (3.14) and is, in fact, twisted Green's. □

Theorem 3.5 and Lemma 3.10 yield the following theorem.

Theorem 3.11. *A system of polynomials $R = \{r_k(x)\}_{k=0}^{n}$ satisfies Green's two-term recurrence relations (3.3) if and only if it is related to a matrix A of zero pattern (3.15) with rank one 2×2 submatrices via (1.1) with $\lambda_k = \delta_k$.*

4. Factorization of Green's matrices

In this section we show that twisted Green's matrices are exactly the ones admitting a factorization similar to (1.9), (2.4), (2.5) valid for unitary Hessenberg, companion, CMV and Fiedler matrices. We start with proving that Green's matrices admit such a factorization.

Theorem 4.1. *Let G be an arbitrary Green's matrix specified by its generators as in Definition 3.2. Then the following decomposition holds:*

$$G = \Theta_0\Theta_1 \cdots \Theta_{n-1}\Theta_n, \tag{4.1}$$

where

$$\Theta_0 = \left[\begin{array}{c|c} \widehat{\tau}_0 & \\ \hline & I_{n-1} \end{array}\right], \quad \Theta_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & \tau_k & \sigma_k & \\ & \widehat{\sigma}_k & \widehat{\tau}_k & \\ \hline & & & I_{n-k-1} \end{array}\right], \quad \Theta_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & \tau_n \end{array}\right]. \tag{4.2}$$

Proof. It is easy to see by performing matrix multiplications that the product on the right in (4.1) is equal to the Green matrix G defined in (3.1). □

Example 4.2. Taking Green's generators (Table 3) of a unitary Hessenberg matrix M and substituting them into (4.2) we get the Schur representation

$$M = \Gamma_0\Gamma_1\Gamma_2 \ldots \Gamma_n,$$
$$\Gamma_0 = \left[\begin{array}{c|c} \rho_0^* & \\ \hline & I_{n-1} \end{array}\right], \quad \Gamma_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & -\rho_k & \mu_k & \\ & \mu_k & \rho_k^* & \\ \hline & & & I_{n-k-1} \end{array}\right], \quad \Gamma_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & -\rho_n \end{array}\right] \tag{4.3}$$

as the consequence of Theorem 4.1.

Similarly, substituting generators (Table 3) of a companion matrix C into (4.2) we get the factorization

$$C = A_1A_2 \ldots A_n,$$
$$A_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & -a_k & 1 & \\ & 1 & 0 & \\ \hline & & & I_{n-k-1} \end{array}\right], \quad A_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & -a_n \end{array}\right]. \tag{4.4}$$

Kimura [24] and Fiedler [35] proved that CMV and Fiedler matrices admit factorizations into products of the same matrices Γ_k (4.3) and A_k (4.4) but with interchanged order of terms:

$$K = [\Gamma_0\Gamma_2 \ldots] \cdot [\Gamma_1\Gamma_3 \ldots] \tag{4.5}$$
$$F = [A_1A_3 \ldots] \cdot [A_2A_4 \ldots] \tag{4.6}$$

Both matrices K and F are twisted Green's obtained via twist transformations from Hessenberg matrices M (1.3) and C (1.5) correspondingly. Hence, there should be a relation between the order of terms in factorizations and twist transformations. The next theorem shows that this is indeed the case.

Theorem 4.3. *Let G be a twisted Green's matrix of pattern $(i_1, i_2, \ldots, i_n)$ with generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$. Then it can be constructed by the following procedure*:

$$G_0 = \Theta_0, \quad G_k = \begin{cases} G_{k-1}\Theta_k & \text{if } i_k = 0, \\ \Theta_k^T G_{k-1} & \text{if } i_k = 1, \end{cases} \quad k = 1, \ldots n, \quad \text{and} \quad G = G_n, \tag{4.7}$$

where Θ_k are matrices from (4.2).

Proof. We know from Theorem 4.1 that the assertion holds in the case $i_k = 0$ for all k. Hence, we only need to prove that

(i) the matrix G from (4.7) is (1, 1)-qs;
(ii) the operation $G_{k-1}\Theta_k \longrightarrow \Theta_k^T G_{k-1}$ is equivalent to a twist transformation for every k.

First, note that

$$G_1 = \begin{bmatrix} \widehat{\tau}_0\tau_1 & \widehat{\tau}_0\sigma_1 \\ \widehat{\sigma}_1 & \widehat{\tau}_1 \end{bmatrix} \text{ if } i_1 = 0, \quad G_1 = \begin{bmatrix} \widehat{\tau}_0\tau_1 & \widehat{\sigma}_1 \\ \widehat{\tau}_0\sigma_1 & \widehat{\tau}_1 \end{bmatrix} \text{ if } i_1 = 1.$$

Both matrices are $(1, 1)$-qs with generators:

$$\{g_1 = \widehat{\tau}_0\sigma_1,\ q_1 = \widehat{\sigma}_1,\ p_2 = h_2 = 1\} \text{ and } \{q_1 = \widehat{\tau}_0\sigma_1,\ g_1 = \widehat{\sigma}_1,\ p_2 = h_2 = 1\}.$$

Thus one is obtained from the other by a twist transformation.

Suppose the same is true for all indices up to $k - 1$. Consider the last two rows of the matrix $G_{k-1}\Theta_k$:

$$\begin{bmatrix} g_1b_2 \dots b_{k-1} & 0 \\ \cdots & \cdots \\ g_{k-1} & 0 \\ d_k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_k & \sigma_k \\ \widehat{\sigma}_k & \widehat{\tau}_k \end{bmatrix} = \begin{bmatrix} g_1b_2 \dots b_{k-1}\tau_k & g_1b_2 \dots b_{k-1}\sigma_k \\ \cdots & \cdots \\ g_{k-1}\tau_k & g_{k-1}\sigma_k \\ d_k\tau_k & d_k\sigma_k \\ \widehat{\sigma}_k & \widehat{\tau}_k \end{bmatrix}.$$

One can easily see that $G_{k-1}\Theta_k$ is $(1, 1)$-qs with generators

$$\{q_k = \widehat{\sigma}_k,\ a_k = 0,\ p_k = 1,\ g_k = d_k\sigma_k,\ b_k = \sigma_k,\ h_k = \tau_k\}.$$

Similarly, by observing the last two rows of the matrix $\Theta_k^T G_{k-1}$ one can check that it is also $(1, 1)$-qs with generators

$$\{q_k = d_k\sigma_k,\ a_k = \sigma_k,\ p_k = \tau_k,\ g_k = \widehat{\sigma}_k,\ b_k = 0,\ h_k = 1\}.$$

Hence, $\Theta_k^T G_{k-1}$ is obtained from $G_{k-1}\Theta_k$ via twist transformation and the assertion of the theorem holds by induction. $\square$

Corollary 4.4. *For every Green's matrix G of size n having decomposition* (4.1) *there are* 2^n (*possibly not distinct*) *twisted Green's matrices obtained via the procedure* (4.7) *and related to the same system of polynomials as G*.

The next two examples apply Theorem 4.3 to CMV and Fiedler matrices.

Example 4.5 (*Factorization of a CMV matrix*). CMV matrix (1.4) is twisted Green's matrix of pattern $(0, 1, 0, 1, 0 \dots)$ obtained from the Green's (unitary Hessenberg) matrix (1.3) via twist transformations. It admits factorization (4.7) with Θ_k coinciding with Γ_k from (4.3). Note that all Γ_k are symmetric and $\Gamma_i\Gamma_j = \Gamma_j\Gamma_i$ if $|i - j| > 1$. Hence the factorization (4.7) coincides with the known factorization (4.5):

$$\begin{aligned} K &= \Gamma_0\Gamma_2 \dots \Gamma_1\Gamma_3 \dots = BC \\ &= \begin{bmatrix} \rho_0^* & & & & & \\ & -\rho_2 & \mu_2 & & & \\ & \mu_2 & \rho_2^* & & & \\ & & & -\rho_4 & \mu_4 & \\ & & & \mu_4 & \rho_4^* & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} -\rho_1 & \mu_1 & & & & \\ \mu_1 & \rho_1^* & & & & \\ & & -\rho_3 & \mu_3 & & \\ & & \mu_3 & \rho_3^* & & \\ & & & & -\rho_5 & \\ & & & & & \ddots \end{bmatrix}, \end{aligned} \tag{4.8}$$

where B and C are products of even and odd Γ_i's. Formula (4.8) is exactly the well-known tridiagonal factorization of CMV matrices.

Example 4.6 (*Factorization of a Fiedler matrix*). Fiedler matrix (1.6) is twisted Green's matrix of pattern $(0, 0, 1, 0, 1, \dots)$ obtained from the Green's (companion) matrix (1.5) via twist transformations. It admits the factorization (4.7) with Θ_k coinciding with A_k from (4.4) (and $A_0 = I_n$). By the same reasoning as in the previous example this factorization (4.7) coincides with (4.6) derived by Fiedler [35]:

$$F = A_1A_3\ldots A_2A_4\ldots = BC$$
$$= \begin{bmatrix} -a_1 & 1 & & & & \\ 1 & 0 & & & & \\ & & -a_3 & 1 & & \\ & & 1 & 0 & & \\ & & & & -a_5 & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & -a_2 & 1 & & & \\ & 1 & 0 & & & \\ & & & -a_4 & 1 & \\ & & & 1 & 0 & \\ & & & & & \ddots \end{bmatrix}. \tag{4.9}$$

Example 4.7 (*Factorization of the Daubechies wavelet matrix*). The seminal paper [16] of Ingrid Daubechies constructed the first orthogonal wavelets beyond the simple average-difference pair due to Haar in 1910. The decompostion of a signal into low and high frequencies is executed by a pair of filters, each with four coefficients:

$$\begin{array}{lllll} \text{Lowpass filter coefficients:} & 1+\sqrt{3}, & 3+\sqrt{3}, & 3-\sqrt{3}, & 1-\sqrt{3} \\ \text{Highpass filter coefficients:} & 1-\sqrt{3}, & -3+\sqrt{3}, & 3+\sqrt{3}, & 1-\sqrt{3} \end{array}$$

These are typical rows (with a normalization factor 1/8 for unit row sums) of the "wavelet matrix" W that multiplies a signal. Normally these rows are shifted by two columns and repeated, to produce a shift-invariant (block Toeplitz) matrix. Shift-invariance allows Fourier methods to apply – we note below that the Green's matrix factorization allows a simple construction of "time-varying" wavelets, which has been a difficult obstacle in previous constructions.

The relations between the eight Daubechies coefficients produce exactly a bidiagonal matrix in CMV form, with 2×2 blocks W_1 and W_2 of rank one.

$$W = \begin{bmatrix} \cdots & & & \\ W_1 & W_2 & & \\ & W_1 & W_2 & \\ & & & \cdots \end{bmatrix} \quad \begin{array}{l} W_1 = \begin{bmatrix} 1+\sqrt{3} & 3+\sqrt{3} \\ 1-\sqrt{3} & -3+\sqrt{3} \end{bmatrix} \\ W_2 = \begin{bmatrix} 3-\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -1-\sqrt{3} \end{bmatrix} \end{array}$$

Now we introduce the factorization (which may be new to wavelet theory). The factors are 2×2 block diagonal. We show columns of B and rows of C:

$$B = \begin{bmatrix} [b_1\ b_2] & & \\ & [b_1\ b_2] & \\ & & [b_1\ b_2] \end{bmatrix}, \quad C = \begin{bmatrix} \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix} & \\ & \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix} \end{bmatrix}.$$

The shift between B blocks and C blocks makes BC block bidiagonal, with blocks $W_1 = b_1c_2^T$ and $W_2 = b_2c_1^T$ of rank one. To match the numbers in W, we take

$$[b_1 \ \ b_2] = \begin{bmatrix} 1+\sqrt{3} & -1+\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} \end{bmatrix} \quad \begin{bmatrix} c_1^T \\ c_2^T \end{bmatrix} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

May we add three comments on possible extensions of this factorization of one particular filter bank, which is associated with the first of the Daubechies wavelets.

1. It is natural to ask about factorizations (with suitable block sizes) of other important filter banks. Conceivably, the wavelet transform can be executed using the factors directly at each level. The inverse wavelet transform is evident from C^{-1} and B^{-1} separately, as in the lifting scheme.

2. The shift-invariant matrix W is normally modified, for example by “symmetric reflection”, in its boundary rows and columns. Early wavelet papers required complicated constructions to preserve good properties, in this step from infinite-length to finite-length signals. B and C offer a new approach to the boundary rows, still to be developed.
3. The factorization immediately suggests that W can become time-varying (instead of block Toeplitz) by making B and C vary block by block.

It remains to use the generators, and the quasi-separable property and the twist transformations, of wavelet matrices.

Though matrices Γ_k and A_k in the above examples are symmetric, it turns out that matrices Θ_k in Theorem 4.3 can be moved from right to left without transposition and this operation does not change characteristic polynomials. This additional *symmetry* of twisted Green’s matrices is proved in Theorem 4.8.

Theorem 4.8. *Let G be a Green’s matrix of size n described by generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$ and let $(j_1, j_2, \ldots, j_n)$ be an arbitrary sequence of binary digits. Then all 2^n matrices $G(j_1, j_2, \ldots, j_n)$ constructed from Θ_k in (4.2) by*

$$G_0 = \Theta_0, \ G_k = \begin{cases} G_{k-1}\Theta_k & \text{if } i_k = 0, \\ \Theta_k G_{k-1} & \text{if } i_k = 1, \end{cases} \quad k = 1, \ldots, n, \ G(j_1, j_2, \ldots, j_n) = G_n, \tag{4.10}$$

share the same system of characteristic polynomials.

Proof. From Theorem 3.5 we know that characteristic polynomials $\{r_k(x)\}_{k=0}^{n}$ of principal submatrices of G satisfy two-term recurrence relations:

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \widehat{\sigma}_k\sigma_k - \widehat{\tau}_k\tau_k & \widehat{\tau}_k \\ \tau_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ x \cdot r_{k-1}(x) \end{bmatrix}. \tag{4.11}$$

Θ_k^T is obtained from Θ_k via the interchange of σ_k and $\widehat{\sigma}_k$, and the recurrence relations (4.11) are symmetric with respect to this operation. Hence, changing Θ_k^T to Θ_k in the assertion of Theorem 4.3 does not change the polynomials $r_k(x)$. □

Though matrices $G(j_1, j_2, \ldots, j_n)$ in (4.10) share the same system of characteristic polynomials, they cannot be obtained from the original matrix G via twist transformations. Therefore, the definition of pattern for twisted $(H, 1)$-qs matrices is not applicable to them. In order to distinguish among the matrices (4.10), we define an alternative pattern.

Definition 4.9 (*Alternative pattern of twisted Green’s matrices*). A sequence of binary digits $(j_1, j_2, \ldots, j_n)$ is the pattern of a twisted Green’s matrix $G(j_1, j_2, \ldots, j_n)$ if it is obtained from some Green’s matrix G having decomposition (4.1) via procedure (4.10).

Matrices defined by (4.10) are found to be extremely important in connection with Laurent polynomials. It will be shown in the next section that they serve as multiplication operators in bases of Laurent polynomials.

4.1. Pentadiagonal Green’s matrices and some generalizations of the results due Fiedler [35]

In this subsection we will study properties of pentadiagonal (block diagonal) twisted Green‘s matrices. To be more concrete we will consider matrices with pattern $(1, 0, 1, 0 \ldots)$. Applying twist transformations for corresponding indices to the general Green’s matrix (3.1) it is easy to see that a pentadiagonal twisted Green’s matrix of pattern $(1, 0, 1, 0 \ldots)$ has the following form:

$$G = \begin{bmatrix} \widehat{\tau}_0\tau_1 & \widehat{\sigma}_1\tau_2 & \widehat{\sigma}_1\sigma_2 & & & & & \\ \widehat{\tau}_0\sigma_1 & \widehat{\tau}_1\tau_2 & \widehat{\tau}_1\sigma_2 & 0 & & & & \\ 0 & \widehat{\sigma}_2\tau_3 & \widehat{\tau}_2\tau_3 & \widehat{\sigma}_3\tau_4 & \widehat{\sigma}_3\sigma_4 & & & \\ & \widehat{\sigma}_2\sigma_3 & \widehat{\tau}_2\sigma_3 & \widehat{\tau}_3\tau_4 & \widehat{\tau}_3\sigma_4 & 0 & & \\ & & 0 & \widehat{\sigma}_4\tau_5 & \widehat{\tau}_4\tau_5 & \widehat{\sigma}_5\tau_6 & \widehat{\sigma}_5\sigma_6 & \\ & & & \widehat{\sigma}_4\sigma_5 & \widehat{\tau}_4\sigma_5 & \widehat{\tau}_5\tau_6 & \widehat{\tau}_5\sigma_6 & 0 \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.12}$$

Remark 4.10. The matrix G in (4.12) can be transformed by the odd-even permutation similarity to the block form

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix},$$

where

$$X_{11} = \begin{bmatrix} \widehat{\tau}_0\tau_1 & \widehat{\sigma}_1\sigma_2 & & & \\ & \widehat{\tau}_2\tau_3 & \widehat{\sigma}_3\sigma_4 & & \\ & & \widehat{\tau}_4\tau_5 & \widehat{\sigma}_5\sigma_6 & \\ & & & \ddots & \ddots \end{bmatrix}, \quad X_{12} = \begin{bmatrix} \widehat{\sigma}_1\tau_2 & & & \\ \widehat{\sigma}_2\tau_3 & \widehat{\sigma}_3\tau_4 & & \\ & \widehat{\sigma}_4\tau_5 & \widehat{\sigma}_5\tau_6 & \\ & & \ddots & \ddots \end{bmatrix},$$

$$X_{21} = \begin{bmatrix} \widehat{\tau}_0\sigma_1 & \widehat{\tau}_1\sigma_2 & & & \\ & \widehat{\tau}_2\sigma_3 & \widehat{\tau}_3\sigma_4 & & \\ & & \widehat{\tau}_4\sigma_5 & \widehat{\tau}_5\sigma_6 & \\ & & & \ddots & \ddots \end{bmatrix}, \quad X_{22} = \begin{bmatrix} \widehat{\tau}_1\tau_2 & & & \\ \widehat{\sigma}_2\sigma_3 & \widehat{\tau}_3\tau_4 & & \\ & \widehat{\sigma}_4\sigma_5 & \widehat{\tau}_5\tau_6 & \\ & & \ddots & \ddots \end{bmatrix}.$$

According to Theorem 4.3 this matrix can be decomposed into the product of matrices Θ_k from (4.2) in the following way:

$$G = \underbrace{\Theta_1^T\Theta_3^T\Theta_5^T\ldots}_{\text{odd indices}}\underbrace{\Theta_0\Theta_2\Theta_4\ldots}_{\text{even indices}} = G_o \cdot G_e, \tag{4.13}$$

where the product of matrices Θ_k with odd and even indices is

$$G_o = \begin{bmatrix} \tau_1 & \widehat{\sigma}_1 & & & \\ \sigma_1 & \widehat{\tau}_1 & & & \\ & & \tau_3 & \widehat{\sigma}_3 & \\ & & \sigma_3 & \widehat{\tau}_3 & \\ & & & & \ddots \end{bmatrix}, \quad G_e = \begin{bmatrix} \widehat{\tau}_0 & & & & & \\ & \tau_2 & \sigma_2 & & & \\ & \widehat{\sigma}_2 & \widehat{\tau}_2 & & & \\ & & & \tau_4 & \sigma_4 & \\ & & & \widehat{\sigma}_4 & \widehat{\tau}_4 & \\ & & & & & \ddots \end{bmatrix}.$$

This generalizes the tridiagonal decompositions (4.8) and (4.9) of CMV and Fiedler matrices. Let us note that if $\{r_k\}_{k=0}^n$ is the polynomial system related to matrix G, then matrices G_o and G_e depends only on odd and even coefficients of the recurrence relations (4.11) correspondingly.

As in [5], the new pentadiagonal matrix (4.12) can be used to estimation the eigenvalues in (3.1). Since

$$\det(xI - G_oG_e) = \det(xG_o^{-1} - G_e) = \det(xG_e^{-1} - G_o),$$

for non-singular G_o or G_e, we obtain

Theorem 4.11. *Let $P(x)$ be a characteristic polynomial of the Green's matrix (3.1). Then the roots of $P(x) = 0$ coincide with the roots of (4.14) in which the $n \times n$ matrix is tridiagonal:*

$$\begin{vmatrix} x\frac{\widehat{\tau}_1}{\delta_1}-\widehat{\tau}_0 & -x\frac{\widehat{\sigma}_1}{\delta_1} & & & & \\ -x\frac{\sigma_1}{\delta_1} & x\frac{\tau_1}{\delta_1}-\tau_2 & -\sigma_2 & & & \\ & -\widehat{\sigma}_2 & x\frac{\widehat{\tau}_3}{\delta_3}-\widehat{\tau}_2 & -x\frac{\widehat{\sigma}_3}{\delta_3} & & \\ & & -x\frac{\sigma_3}{\delta_3} & x\frac{\tau_3}{\delta_3}-\tau_4 & -\sigma_4 & \\ & & & \ddots & \ddots & \ddots \end{vmatrix} = 0 \tag{4.14}$$

and $\delta_k = \tau_k\widehat{\tau}_k - \sigma_k\widehat{\sigma}_k$. *The case* $\det(xG_e^{-1} - G_o) = 0$ *is rather similar.*

5. Laurent polynomials and multiplication operators

Let M be an infinite-dimensional unitary Hessenberg matrix and $\{\phi_k^\#(x)\}_{k\geqslant 0}$ be the infinite sequence of polynomials orthogonal on the unit circle related to M via (1.1). It is widely known that M represents multiplication by x in the basis $\{\phi_k^\#(x)\}$:

$$\left[\phi_0^\#(x)\ \phi_1^\#(x)\ \phi_2^\#(x)\cdots\right] M = x\left[\phi_0^\#(x)\ \phi_1^\#(x)\ \phi_2^\#(x)\cdots\right]. \tag{5.1}$$

If M_n is of size n and λ is a root of polynomial $\phi_n^\#(x)$, then we have a left eigenvector of M_n:

$$\left[\phi_0^\#(\lambda)\ \phi_1^\#(\lambda)\cdots\phi_{n-1}^\#(\lambda)\right] M_n = \lambda\left[\phi_0^\#(\lambda)\ \phi_1^\#(\lambda)\cdots\phi_{n-1}^\#(\lambda)\right] \tag{5.2}$$

For Szegö polynomials $\{\phi_k^\#(x)\}_{k\geqslant 0}$ define right Laurent polynomials as follows:

$$\chi_k(x) = \begin{cases} x^{-l}\phi_k(x), & k = 2l, \\ x^{-l}\phi_k^\#(x), & k = 2l+1, \end{cases} \tag{5.3}$$

where $\phi_k(x)$ are auxiliary polynomials from (3.11). It was shown in [13] that the right Laurent polynomials $\{\chi_k(x)\}_{k\geqslant 0}$ are orthogonal in the same inner product as $\{\phi_k^\#(x)\}_{k\geqslant 0}$. Therefore, they can be obtained using the Gram–Schmidt procedure starting with the ordered set $\{1, x, x^{-1}, x^2, x^{-2}, \ldots\}$.

It is known due to [13] that an infinite CMV matrix K plays the same role for Laurent polynomials $\{\chi_k(x)\}_{k\geqslant 0}$ as unitary Hessenberg does for Szegö polynomials:

$$[\chi_0(x)\ \chi_1(x)\ \chi_2(x)\cdots]\, K = x\,[\chi_0(x)\ \chi_1(x)\ \chi_2(x)\cdots]\,. \tag{5.4}$$

Similarly to (5.2) if λ is an eigenvalue of an $n\times n$ CMV matrix K_n, then

$$[\chi_0(\lambda)\ \chi_1(\lambda)\cdots\chi_{n-1}(\lambda)]\, K_n = \lambda\,[\chi_0(\lambda)\ \chi_1(\lambda)\cdots\chi_{n-1}(\lambda)]\,. \tag{5.5}$$

The proofs of (5.1), (5.2) and (5.4), (5.5) are based on the orthogonality of polynomials $\{\phi_k^\#(x)\}$ and $\{\chi_k(x)\}$. We are to show that all the above results can be generalized to twisted Green's matrices and associated Laurent polynomials (and our proof does not require orthogonallity).

As shown in Theorem 4.1 every infinite Green's matrix G defined by generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$ has the following factorization:

$$G = \Theta_0\Theta_1\Theta_2\ldots, \quad \text{where } \Theta_0 = \left[\begin{array}{c|c} \widehat{\tau}_0 & \\ \hline & I \end{array}\right], \quad \Theta_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & \tau_k & \sigma_k & \\ & \widehat{\sigma}_k & \widehat{\tau}_k & \\ \hline & & & I \end{array}\right]. \tag{5.6}$$

The polynomials $\{r_k(x)\}_{k\geqslant 0}$ related to G via (1.1) with $\lambda_k = \frac{1}{\widehat{\sigma}_k}$ satisfy Green's two-term recurrence relations (Theorem 3.5):

$$\begin{bmatrix} f_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\widehat{\sigma}_k}\begin{bmatrix} \widehat{\sigma}_k\sigma_k - \widehat{\tau}_k\tau_k & \widehat{\tau}_k \\ -\tau_k & 1 \end{bmatrix}\begin{bmatrix} f_{k-1}(x) \\ x\cdot r_{k-1}(x) \end{bmatrix}. \tag{5.7}$$

Let $\mathcal{J} = (j_1, j_2, j_3, \ldots)$ be an infinite sequence of binary digits. We define twisted Green's matrices $G_{\mathcal{J}}$ by the recursion

$$G_0 = \Theta_0, \quad G_k = \begin{cases} G_{k-1}\Theta_k & \text{if } j_k = 0, \\ \Theta_k G_{k-1} & \text{if } j_k = 1, \end{cases} \quad G_{\mathcal{J}} = G_\infty, \tag{5.8}$$

Matrices $G_{\mathcal{J}}$ are related to the same polynomials $\{r_k(x)\}_{k=0}^n$ via (1.1) with $\lambda_k = \frac{1}{\hat{\sigma}_k}$ as G (Theorem 4.8). For every $\mathcal{J}$ we also define a sequence of Laurent polynomials $\{\psi_k(x)\}_{k \geqslant 0}$:

$$\psi_k(x) = \begin{cases} x^{-\sum_{m=1}^{k+1} j_m} r_k(x) & \text{if } j_{k+1} = 0, \\ x^{-\sum_{m=1}^{k+1} j_m} f_k(x) & \text{if } j_{k+1} = 1, \end{cases} \tag{5.9}$$

where $f_k(x)$ are the auxiliary polynomials from (5.7). The next theorem shows that matrices $G_{\mathcal{J}}$ represent multiplication operators in the frame of Laurent polynomials (5.9).

Theorem 5.1. *Let $G_{\mathcal{J}}$ be a twisted Green's matrix of alternative pattern $\mathcal{J} = (j_1, j_2, j_3 \ldots)$ defined by (5.8) and $\{\psi_k(x)\}_{k \geqslant 0}$ be Laurent polynomials (5.9). Then*

$$[\psi_0(x) \;\; \psi_1(x) \;\; \psi_2(x) \cdots] \, G_{\mathcal{J}} = x \, [\psi_0(x) \;\; \psi_1(x) \;\; \psi_2(x) \cdots] \, . \tag{5.10}$$

Proof. Solving (5.7) with respect to $x \cdot r_{k-1}(x)$ and f_k we get

$$[x \cdot r_{k-1} \quad f_k] = [f_{k-1} \quad r_k] \begin{bmatrix} \tau_k & \sigma_k \\ \hat{\sigma}_k & \hat{\tau}_k \end{bmatrix}. \tag{5.11}$$

Denote $\Psi_k = [\psi_0(x) \;\; \psi_1(x) \ldots \psi_{k-1}(x)]$. We will show by induction that

$$\left[\Psi_k \, ; \; x^{-\sum_{m=1}^{k} j_m} r_k\right] G_k(1:k+1, 1:k+1) = \left[x\Psi_k \, ; \; x^{-\sum_{m=1}^{k} j_m} f_k\right], \tag{5.12}$$

where G_k are matrices from (5.8).

This holds for $k = 0$ because $G_0(1:1, 1:1) = \Theta_0(1:1, 1:1) = [\,\tau_0\,]$ and $[\,r_0\,][\,\tau_0\,] = [f_0\,]$.

Assume (5.12) holds for some k, and consider two cases:

$\mathbf{j_{k+1}} = \mathbf{0}$. Padding rows in (5.12) with $x^{-\sum_{m=1}^{k+1} j_m} r_{k+1}$ and noticing that $G_k(1:k+2, 1:k+2) = \operatorname{diag}(G_k(1:k+1, 1:k+1), 1)$ we get

$$\left[\Psi_k \, ; \; x^{-\sum_{m=1}^{k+1} j_m} \, [r_k \, ; \; r_{k+1}]\right] G_k(1:k+2, 1:k+2) = \left[x\Psi_k \, ; \; x^{-\sum_{m=1}^{k+1} j_m} \, [f_k \, ; \; r_{k+1}]\right].$$

Multiply by Θ_{k+1} from the right and use (5.11):

$$\left[\Psi_{k+1} \, ; \; x^{-\sum_{m=1}^{k+1} j_m} r_{k+1}\right] G_{k+1}(1:k+2, 1:k+2) = \left[x\Psi_{k+1} \, ; \; x^{-\sum_{m=1}^{k+1} j_m} f_{k+1}\right],$$

where $G_{k+1} = G_k \Theta_{k+1}$ and $\psi_k(x) = x^{-\sum_{m=1}^{k+1} j_m} r_k$.

$\mathbf{j}_{k+1} = 1$. Using (5.11) one can easily see that

$$\left[\Psi_k \, ; \; x^{-\sum_{m=1}^{k+1} j_m} \, [f_k \, ; \; r_{k+1}]\right] \Theta_{k+1}(1:k+2, 1:k+2)$$

$$= \left[\Psi_k \, ; \; x^{-\sum_{m=1}^{k} j_m} r_k \, ; \; x^{-\sum_{m=1}^{k+1} j_m} f_{k+1}\right].$$

Multiply by $G_k(1:k+2, 1:k+2)$ from the right and use (5.12):

$$\left[\Psi_{k+1}\ ;\ x^{-\sum_{m=1}^{k+1} j_m} r_{k+1}\right] G_{k+1}(1:k+2, 1:k+2) = \left[x\Psi_{k+1}\ ;\ x^{-\sum_{m=1}^{k+1} j_m} f_{k+1}\right],$$

where $G_{k+1} = \Theta_{k+1} G_k$ and $\psi_k(x) = x^{-\sum_{m=1}^{k+1} j_m} f_k$.

Finally, letting $k \to \infty$ in (5.12) we obtain (5.10). □

We next apply Theorem 5.1 to CMV and Fiedler matrices.

Example 5.2 (*CMV matrix and Laurent polynomials*). Each infinite-dimensional unitary Hessenberg matrix M is Green's having the factorization

$$M = \Gamma_0 \Gamma_1 \Gamma_2 \ldots, \tag{5.13}$$

where matrices Γ_k are defined in (2.4). Hence, M has the alternative pattern $(0, 0, 0, \ldots)$ and in accordance with Theorem 5.1 represents the multiplication operator (5.1) in the basis of Szegö polynomials $\{\phi_k^{\#}(x)\}_{k \geqslant 0}$ satisfying (3.11).

Each infinite CMV matrix K has the factorization

$$K = [\Gamma_0 \Gamma_2 \ldots] \cdot [\Gamma_1 \Gamma_3 \ldots]$$

and hence it is twisted Green's of alternative pattern $\mathcal{J} = (0, 1, 0, 1, \ldots)$. Hence, from (5.9) we get

$$\sum_{m=1}^{k+1} j_m = \begin{cases} l & k = 2l \\ l & k = 2l-1 \end{cases} \quad \text{and} \quad \psi_k(x) = \begin{cases} x^{-l}\phi_k^{\#}(x) & k = 2l, \\ x^{-l}\phi_k(x) & k = 2l-1. \end{cases} \tag{5.14}$$

Theorem 5.1 says that matrix K represents the multiplication operator in the basis of polynomials $\{\psi_k(x)\}_{k \geqslant 0}$ which, in fact, coincides with (5.4) because $\psi_k(x) \equiv \chi_k(x)$ from (5.3).

Example 5.3 (*Fiedler matrix and Laurent polynomials*). Each infinite companion matrix C admits the factorization

$$C = A_1 A_2 A_3 A_4, \ldots \tag{5.15}$$

with matrices A_k defined in (2.5). Hence, C is twisted Green's of alternative pattern $(0, 0, 0, 0, \ldots)$. According to Theorem 5.1 C represents the multiplication operator in the basis of Horner polynomials (3.12):

$$[p_0(x)\ p_1(x)\ p_2(x) \ldots]\, C = x\, [p_0(x)\ p_1(x)\ p_2(x) \ldots].$$

This result is well-known in contrast to the similar result for Fiedler matrix F presented next.

Fiedler matrix (1.6) admits the factorization

$$F = [A_1 A_3 \ldots] \cdot [A_2 A_4 \ldots]$$

and, hence, it is twisted Green's of alternative pattern $\mathcal{J} = (1, 0, 1, 0, \ldots)$. Laurent polynomials associated with it are as follows

$$\psi_k(x) = \begin{bmatrix} x^{-l-1} & \text{if } k = 2l, \\ x^{-l-1} p_k(x) & \text{if } k = 2l+1. \end{bmatrix} \tag{5.16}$$

This is a direct consequence of (3.13) and (5.9), and F is the multiplication operator in this basis:

$$\left[x^{-1}\ ;\ x^{-1}p_1(x)\ ;\ x^{-2}\ ;\ x^{-2}p_3(x) \ldots\right] F = x \left[x^{-1}\ ;\ x^{-1}p_1(x)\ ;\ x^{-2}\ ;\ x^{-2}p_3(x) \ldots\right].$$

The major remark that has to be made is that Laurent polynomials $\{\psi_k(x)\}$ in (5.10) do not necessarily form a basis. In fact, they can be linearly dependent, as we illustrate.

Example 5.4. Consider the infinite Green's matrix

$$G = \Theta_0\Theta_1\Theta_2\ldots,$$

where

$$\Theta_0 = I, \quad \Theta_1 = \left[\begin{array}{cc|c} 0 & 0 & \\ 1 & 1 & \\ \hline & & I \end{array}\right], \quad \Theta_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & 0 & 1 & \\ & 1 & 0 & \\ \hline & & & I \end{array}\right], \quad k \geqslant 2. \tag{5.17}$$

According to (5.7) the polynomials related to G via (1.1) satisfy

$$\begin{bmatrix} f_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} f_1(x) \\ r_1(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_0(x) \\ x \cdot r_0(x) \end{bmatrix} \equiv \begin{bmatrix} x \\ x \end{bmatrix},$$
$$\begin{bmatrix} f_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_{k-1}(x) \\ x \cdot r_{k-1}(x) \end{bmatrix} \equiv \begin{bmatrix} x \\ x^k \end{bmatrix}, \quad k \geqslant 2.$$

Now take a twisted Green's matrix $G_{\mathcal{J}}$ of alternative pattern $\mathcal{J} = \{0, 1, 0, 0, 0, \ldots\}$. Then the Laurent polynomials $\{\psi_k(x)\}_{k\geqslant 0}$ in (5.9) are:

$$[\psi_0(x)\,;\; \psi_1(x)\,;\; \psi_2(x)\,;\; \psi_3(x)\,;\; \psi_4(x)\,;\; \psi_5(x)\,;\; \ldots] = \left[1\,;\; 1\,;\; x\,;\; x^2\,;\; x^3\,;\; \ldots\right].$$

The identity (5.10) still holds while $\{\psi_k(x)\}_{k\geqslant 0}$ is not a basis, because the first two polynomials are linearly dependent.

In order to guarantee that Laurent polynomials in (5.10) form a basis we need to impose a restriction on the related Green's matrix. This limitation is expressed as a necessary and sufficient condition in the next theorem.

Theorem 5.5. *Let G be a Green's matrix defined by the factorization (5.6) and let k_0 be first index such that Θ_{k_0} is singular. Then for any alternative pattern $\mathcal{J} = (j_1, j_2, j_3 \ldots)$ with $j_k = 0$ for $k > k_0$ Laurent polynomials $\{\psi_k(x)\}_{k\geqslant 0}$ defined in (5.9) are linearly independent.*

Proof. For all indices $k < k_0$, the matrices Θ_k are invertible and

$$|\Theta_0| = \widehat{\tau}_0 \neq 0, \quad |\Theta_k| = \widehat{\tau}_k\tau_k - \widehat{\sigma}_k\sigma_k \neq 0.$$

Hence, it follows directly from the recurrence relations (5.7) that the free coefficient of an auxiliary polynomial $f_k(x)$ for every $k < k_0$ is not zero:

$$\frac{\widehat{\tau}_0(\widehat{\sigma}_1\sigma_1 - \widehat{\tau}_1\tau_1)\cdots(\widehat{\sigma}_k\sigma_k - \widehat{\tau}_k\tau_k)}{\widehat{\sigma}_0\widehat{\sigma}_1\cdots\widehat{\sigma}_k} = (-1)^k \frac{|\Theta_0||\Theta_1|\cdots|\Theta_k|}{\widehat{\sigma}_0\widehat{\sigma}_1\cdots\widehat{\sigma}_k}. \tag{5.18}$$

After shifting polynomial $f_k(x)$ to the left in (5.9) this free coefficient becomes leading on the left. Therefore, every new polynomial $\psi_k(x)$ in (5.9) for $j_{k+1} = 1$ and $k + 1 \leqslant k_0$ is independent from the previous ones, because its leftmost term is not in their span.

Now observe that polynomials $r_k(x)$ in (5.9) are always of increasing degrees. Hence, $\psi_k(x)$ in (5.9) for $j_{k+1} = 0$ is not in the span of the previous Laurent polynomials. Since $j_k = 0$ for all $k > k_0$, all polynomials $\psi_k(x)$ in (5.9) are linearly independent.

Conversely, suppose that k is the first index greater than k_0 such that $j_k = 0$, then the leading coefficient on the left of the Laurent polynomial $\psi_k(k)$ is zero because it is equal to (5.18), where $|\Theta_{k_0}| = 0$. Note that all the Laurent polynomials $\psi_m(x)$ for $m < k$ are linearly independent (the same justification as was given above). Hence, $\psi_k(x)$ is in the span of $\{\psi_m(x)\}_{m=0}^{k-1}$. This completes the proof. □

Remark 5.6. Let us note that both infinite unitary Hessenberg and companion matrices satisfy the condition of Theorem 5.5 because all the matrices in factorizations (5.13) and (5.15) are invertible. Therefore, polynomials (5.14) and (5.16) for CMV and Fiedler matrices are bases in the space of Laurent polynomials.

In the finite-dimensional case Theorem 5.1 gives the way to describe eigenvectors of twisted Green's matrices under the assumption that is made in Theorem 5.5. This motivates us to introduce a new slightly narrower class of matrices for which we will find the eigenvectors.

5.1. Eigenvectors for the non-degenerate case

Definition 5.7. Let $G_{\mathcal{J}}$ be an $n \times n$ twisted Green's matrix of alternative pattern $\mathcal{J} = (j_1, j_2, \dots, j_n)$ defined via the factorization

$$G_0 = \Theta_0, \quad G_k = \begin{cases} G_{k-1}\Theta_k & \text{if } j_k = 0, \\ \Theta_k G_{k-1} & \text{if } j_k = 1, \end{cases} \quad G_{\mathcal{J}} = G_n, \tag{5.19}$$

where

$$\Theta_0 = \left[\begin{array}{c|c} \widehat{\tau}_0 & \\ \hline & I_{n-1} \end{array}\right], \quad \Theta_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & \tau_k & \sigma_k & \\ & \widehat{\sigma}_k & \widehat{\tau}_k & \\ \hline & & & I_{n-k-1} \end{array}\right], \quad \Theta_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & \tau_n \end{array}\right]. \tag{5.20}$$

If k_0 is the first index for which Θ_0 is singular and $j_k = 0$ for all $k > k_0$, then $G_{\mathcal{J}}$ is called *semi-non-degenerate*, otherwise it is called *semi-degenerate*[4].

Remark 5.8. Being semi-degenerate is not the same as being singular. Consider a twisted Green's matrix G defined via the factorization (5.19). If there is a singular term Θ_k in this factorization, matrix G is singular. But if all the singular terms are multiplied from the right in (5.19), matrix G is semi-non-degenerate.

We now state the theorem which describes the structure of eigenvectors of semi-non-degenerate twisted Green's matrices.

Theorem 5.9. *Let $G_{\mathcal{J}}$ be a semi-non-degenerate twisted Green's matrix of alternative pattern $\mathcal{J} = (j_1, j_2, \dots, j_n)$. Then for every eigenvalue λ of $G_{\mathcal{J}}$ of multiplicity m, the eigenvector is $\Psi^{(0)}(\lambda)$ and the generalized eigenvectors are $\Psi^{(1)}(\lambda)$ to $\Psi^{(m-1)}(\lambda)$:*

$$\begin{aligned} &\Psi^{(0)}(\lambda) \cdot G_{\mathcal{J}} = \lambda \cdot \Psi^{(0)}(\lambda), \\ &\Psi^{(1)}(\lambda) \cdot G_{\mathcal{J}} = \lambda \cdot \Psi^{(1)}(\lambda) + \Psi^{(0)}(\lambda), \\ &\quad \cdots \qquad \cdots \qquad \cdots \\ &\Psi^{(m-1)}(\lambda) \cdot G_{\mathcal{J}} = \lambda \cdot \Psi^{(m-1)}(\lambda) + \Psi^{(m-2)}(\lambda). \end{aligned} \tag{5.21}$$

$\Psi^{(0)}(x) = x^{\sum_{k=1}^{n} j_k} \cdot [\psi_0(x) \;\; \psi_1(x) \cdots \psi_{n-1}(x)]$ and $\Psi^{(k)}(\lambda)$ denotes the kth derivative of $\Psi^{(0)}(x)$ evaluated at λ, where $\psi_i(x)$ are the Laurent polynomials defined in (5.9).

Proof. To prove (5.21) let us note that (5.11) implies

$$x \cdot r_{n-1} = \tau_n f_{n-1} + \widehat{\sigma}_n r_n,$$

---

[4] We reserve the term *degenerate* to another class of matrices defined further in the text.

which can be used to eliminate the last elements in the rows of (5.12) to get

$$[\Psi_n]\, G_{\mathcal{J}} = x\,[\Psi_n] - x^{-\sum_{k=1}^{n} j_k}\widehat{\sigma}_n[0\ldots 0\ r_n(x)]\cdot \begin{cases} I_n & j_n = 0,\\ G_{n-1}(1:n,1:n) & j_n = 1.\end{cases}$$

Multiplying this identity by $x^{\sum_{k=1}^{n} j_k}$ we have

$$\Psi^{(0)}(x)\cdot G_{\mathcal{J}} = x\cdot \Psi^{(0)}(x) + [0\ldots 0\ r_n(x)]\cdot A, \tag{5.22}$$

where A is a constant matrix and $\Psi^{(0)}(x)$ consists of polynomials which form a basis in the span of $\{1, x, x^2, \ldots, x^{n-1}\}$ (apply Theorem 5.5).

Since λ has multiplicity m, it is a root of $r_n(x)$ in (5.22) with

$$r_n(\lambda) = r_n'(\lambda) = r_n''(\lambda) = \ldots = r_n^{(m-1)}(\lambda) = 0.$$

Differentiating (5.22) with respect to x $m-1$ times and substituting λ for x we get the desired result (5.21), where vectors $\Psi^{(0)}(\lambda), \ldots, \Psi^{(m-1)}(\lambda)$ are linearly independent due to linear independence of polynomials which form $\Psi^{(0)}(x)$. □

Corollary 5.10 (Eigenvectors of CMV matrices). *The eigenvector of K which corresponds to λ is given by*

$$[\chi_0(\lambda)\ \chi_1(\lambda)\ \chi_2(\lambda)\cdots\chi_{n-1}(\lambda)],$$

where

$$\chi_k(x) = x^{[\frac{n}{2}]}\begin{cases} x^{-l}\phi_k^{\#}(x), & k = 2l,\\ x^{-l}\phi_k(x), & k = 2l-1.\end{cases}$$

Corollary 5.11 (Eigenvectors of Fiedler matrices). *The eigenvector of F which corresponds to λ is given by*

$$[\psi_0(\lambda)\ \psi_1(\lambda)\ \psi_2(\lambda)\cdots\psi_{n-1}(\lambda)],$$

where

$$\psi_k(x) = x^{[\frac{n+1}{2}]}\begin{cases} x^{-l-1} & \text{if } k = 2l,\\ x^{-l-1}p_k(x) & \text{if } k = 2l+1.\end{cases}$$

Let G be a Green's matrix, $G = \Theta_0\Theta_1\ldots\Theta_n$ with Θ_k invertible for $k < n$. It follows from Definition 5.7 of that all possible twisted Green's matrices obtained from G are semi–non–degenerate and, therefore, satisfy the conditions of Theorem 5.9. This motivates us to name this class of matrices.

Definition 5.12. Let $G_{\mathcal{J}}$ be a twisted Green's matrix of alternative pattern obtained from some Hessenberg Green's matrix $G = \Theta_0\Theta_1\ldots\Theta_n$. If Θ_k are invertible for $k < n$ then $G_{\mathcal{J}}$ is called *non-degenerate*, otherwise it is *degenerate*.

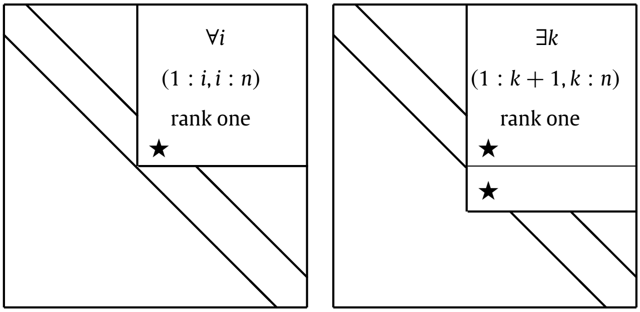
Remark 5.13. CMV and Fiedler matrices are always non-degenerate.

A degenerate twisted Green's matrix is always singular but the converse is not true. For instance, a companion matrix is always *non-degenerate* (Remark 5.6) although it can be singular. Actually, a non-degenerate Green's matrix of size n is singular if and only if it has a choice of generators such that $\tau_n = 0$.

Degenerate Hessenberg Green's matrices have a very transparent description via the condition on ranks of their submatrices:

$$\exists k \in [1, n-1] \text{ such that } \operatorname{rank} G(1:k+1, k:n) = 1. \tag{5.23}$$

The difference between rank definitions of general Green's and degenerate Green's matrices is illustrated in the picture below.



Green's degenerate Green's

5.2. Five-diagonal twisted Green's matrices and Laurent polynomials

Consider general five-diagonal twisted Green's matrix $G_{\mathcal{J}}$ of the alternative pattern $\mathcal{J} = (0, 1, 0, 1, \ldots)$:

$$G_{\mathcal{J}} = \begin{bmatrix} \widehat{\tau}_0\tau_1 & \widehat{\tau}_0\sigma_1 & 0 & & & & & \\ \widehat{\sigma}_1\tau_2 & \widehat{\tau}_1\tau_2 & \sigma_2\tau_3 & \sigma_2\sigma_3 & & & & \\ \widehat{\sigma}_1\widehat{\sigma}_2 & \widehat{\tau}_1\widehat{\sigma}_2 & \widehat{\tau}_2\tau_3 & \widehat{\tau}_2\sigma_3 & 0 & & & \\ & 0 & \widehat{\sigma}_3\tau_4 & \widehat{\tau}_3\tau_4 & \sigma_4\tau_5 & \sigma_4\sigma_5 & & \\ & & \widehat{\sigma}_3\widehat{\sigma}_4 & \widehat{\tau}_3\widehat{\sigma}_4 & \widehat{\tau}_4\tau_5 & \widehat{\tau}_4\sigma_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.24}$$

This matrix is the multiplication operator (Theorem 5.1):

$$[\psi_0(x)\ \ \psi_1(x)\ \ \psi_2(x)\ldots]\,F = x\,[\psi_0(x)\ \ \psi_1(x)\ \ \psi_2(x)\ldots]$$

in the basis of Laurent polynomials

$$\psi_k(x) = \begin{cases} x^{-l}f_k(x) & k = 2l, \\ x^{-l}r_k(x) & k = 2l+1, \end{cases}$$

where $f_k(x)$ and $r_k(x)$ are defined by the recurrence relations (3.3). Directly from the structure (5.24) of $G_{\mathcal{J}}$ we get the recurrence relations for Laurent polynomials $\{\psi_k(x)\}_{k \geqslant 0}$:

$$\begin{gathered} x\begin{bmatrix} \psi_{2n}(x) \\ \psi_{2n+1}(x) \end{bmatrix} = A_{2n}\begin{bmatrix} \psi_{2n-1}(x) \\ \psi_{2n}(x) \end{bmatrix} + B_{2n+1}\begin{bmatrix} \psi_{2n+1}(x) \\ \psi_{2n+2}(x) \end{bmatrix}, \\ A_k = \begin{bmatrix} \sigma_k\widehat{\tau}_{k+1} & \widehat{\tau}_k\tau_{k+1} \\ \sigma_k\sigma_{k+1} & \widehat{\tau}_k\sigma_{k+1} \end{bmatrix}, \quad B_k = \begin{bmatrix} \widehat{\sigma}_k\tau_{k+1} & \widehat{\sigma}_k\widehat{\sigma}_{k+1} \\ \widehat{\tau}_k\tau_{k+1} & \widehat{\tau}_k\widehat{\sigma}_{k+1} \end{bmatrix}. \end{gathered} \tag{5.25}$$

These relations generalize the ones for Laurent polynomials $\chi_k(x)$ in (5.3) derived, in [13].

6. Inversion of Green's matrices

Theorem 6.1. *Let G be a twisted Green's matrix of alternative pattern $\mathcal{J} = (j_1, j_2, j_3, \ldots)$ given by its generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$. Then its inverse G^{-1} is also twisted Green's with the reversed alternative pattern $\widetilde{\mathcal{J}} = (\widetilde{j}_1, \widetilde{j}_2, \widetilde{j}_3, \ldots)$, $\widetilde{j}_k = 1 - j_k$, and generators $\{-\frac{\sigma_k}{\Delta_k}, \frac{\widehat{\tau}_k}{\Delta_k}, -\frac{\widehat{\sigma}_k}{\Delta_k}, \frac{\tau_k}{\Delta_k}\}$ with $\Delta_k = \tau_k\widehat{\tau}_k - \sigma_k\widehat{\sigma}_k$.*

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