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# Solution of a hypersingular integral equation in two disjoint intervals

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ABSTRACT

A hypersingular integral equation in two disjoint intervals is solved by using the solution of Cauchy type singular integral equation in disjoint intervals. Also a direct function theoretic method is used to determine the solution of the same hypersingular integral equation in two disjoint intervals. Solutions by both the methods are in good agreement with each other.



## 1. Introduction

Hypersingular integral equation is considered as an important tool in Applied Mathematics as it finds application in solving a large class of mixed boundary value problems arising in mathematical physics. Particularly the crack problems in fracture mechanics or water wave scattering problems involving barriers, diffraction of electromagnetic waves, aerodynamics problems (cf. [1–3]) can be reduced to hypersingular integral equations in single or disjoint multiple intervals.

A simple hypersingular integral equation is given by

$$Hf = \int_{-1}^{1} \frac{f(t)}{(x-t)^2}\mathrm{d}t = \psi(x), \quad -1 < x < 1, \tag{1.1}$$

where $f \in C^{1,\alpha}(-1,1)$ and $\psi \in C^{0,\alpha}(-1,1)(0 < \alpha < 1)$; $C^{n,\alpha}(-1,1)$ denote the class of functions having Holder continuous derivative of order $n$ with exponent as $\alpha$.

The hypersingular integral $Hf$ appearing in (1.1) is understood to be equal to Hadamard finite part (cf. [4]) of this divergent integral as given by the relation

$$Hf = \lim_{\epsilon\to 0+}\left[\int_{-1}^{x-\epsilon} \frac{f(t)}{(x-t)^2}\mathrm{d}t + \int_{x+\epsilon}^{1} \frac{f(t)}{(x-t)^2}\mathrm{d}t - \frac{f(x+\epsilon)+f(x-\epsilon)}{\epsilon}\right]. \tag{1.2}$$

Eq. (1.1) has been solved by Martin [4], Chakrabarti and Mandal [5] by utilizing the known solution of Cauchy type singular integral equation of first kind.

Recently, Chakrabarti [6] has developed a direct function theoretic method to determine the solution of (1.1).

In the present paper we have considered for solution the following hypersingular integral equation in two disjoint intervals $G \equiv (-1,-k) \cup (k,1)$,

$$\frac{1}{\pi}\int_{-1}^{-k} \frac{f(t)}{(x-t)^2}\mathrm{d}t + \frac{1}{\pi}\int_{k}^{1} \frac{f(t)}{(x-t)^2}\mathrm{d}t = \psi(x), \quad x \in G. \tag{1.3}$$

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Here $f(x) \in C^{1,\alpha}(G)$ and $\psi(x) \in C^{0,\alpha}(G)$, $0 < \alpha < 1$ with

$$f(\pm 1) = f(\pm k) = 0 \tag{1.4}$$

and the integral is defined in the sense of Hadamard finite part as described in Eq. (1.2).

We have used two different methods to solve Eq. (1.3). In the first method we have utilized the solution of the following aerofoil equation which is Cauchy type singular integral equation in two disjoint intervals (cf [7])

$$\frac{1}{\pi}\int_{-1}^{-k}\frac{\phi(t)}{x-t}\mathrm{d}t + \frac{1}{\pi}\int_{k}^{1}\frac{\phi(t)}{x-t}\mathrm{d}t = h(x), \quad x \in G.$$

In the second method we have applied direct function theoretic method as described by Chakrabarti [5] to obtain the closed form solution of integral equation (1.3). Solution obtained by both the methods are in good agreement with each other.

In the following sections we give detailed analysis for solution of integral equation (1.3).

## 2. The detailed analysis

In this section we proceed to solve the hypersingular integral equation (1.3) under condition (1.4) by two different methods.

**Method-I**

Noting condition (1.4), the hypersingular integral equation (1.3) can be written equivalently as

$$\frac{1}{\pi}\int_{-1}^{-k}\frac{f'(t)}{x-t}\mathrm{d}t + \frac{1}{\pi}\int_{k}^{1}\frac{f'(t)}{x-t}\mathrm{d}t = -\psi(x), \quad x \in G. \tag{2.1}$$

This is the well-known aerofoil equation with Cauchy type singularity which was solved by Tricomi [7] and the solution is given by

$$f'(x) = \begin{cases} \dfrac{1}{\pi R(x)}\left[C_1 + C_2 x + \Psi(x)\right], & x \in (-1, -k), \\ -\dfrac{1}{\pi R(x)}\left[C_1 + C_2 x + \Psi(x)\right], & x \in (k, 1), \end{cases} \tag{2.2}$$

where,

$$\Psi(x) = \int_{-1}^{-k}\frac{\psi(t)R(t)}{x-t}\mathrm{d}t - \int_{k}^{1}\frac{\psi(t)R(t)}{x-t}\mathrm{d}t,$$

$C_1, C_2$ are two arbitrary constants and

$$R(x) = \{(1-x^2)(x^2-k^2)\}^{\frac{1}{2}}. \tag{2.3}$$

Integrating (2.2) with respect to $x$ gives

$$f(x) = \begin{cases} \dfrac{1}{\pi}\left[\displaystyle\int_{-1}^{x}\frac{1}{R(u)}\left(C_1 + C_2 u + \Psi(u)\right)\mathrm{d}u\right] + F_1, & x \in (-1, -k), \\ \dfrac{1}{\pi}\left[\displaystyle\int_{x}^{1}\frac{1}{R(u)}\left(C_1 + C_2 u + \Psi(u)\right)\mathrm{d}u\right] + F_2, & x \in (k, 1), \end{cases} \tag{2.4}$$

where $F_1, F_2$ are another two arbitrary constants.

Now, we make an observation that if $f(x)$ has to satisfy the end conditions $f(\pm 1) = 0$ as given in relation (1.4), we must have

$$F_2 = 0, \qquad F_1 = 0. \tag{2.5}$$

Also, the conditions $f(\pm k) = 0$ yields

$$\int_{k}^{1}\frac{1}{R(u)}\left(C_1 + C_2 u + \Psi(u)\right)\mathrm{d}u = 0, \tag{2.6}$$

$$\int_{-1}^{-k}\frac{1}{R(u)}\left(C_1 + C_2 u + \Psi(u)\right)\mathrm{d}u = 0. \tag{2.7}$$

Solving Eqs. (2.6) and (2.7) we get the constants $C_1, C_2$ which are given by

$$C_1 = \frac{P(k)}{F(q)}, \qquad C_2 = 0, \tag{2.8}$$

where,

$$P(k) = \int_{k}^{1}\frac{\mathrm{d}u}{R(u)}\int_{k}^{1}\frac{tR(t)}{u^2-t^2}\left(\psi(-t) + \psi(t)\right)\mathrm{d}t,$$

and

$$F(q)=\int_0^{\sin\lambda}\frac{\mathrm{d}x}{((1-x^2)(1-q^2x^2))^{\frac{1}{2}}},\quad q=(1-k^2)^{\frac{1}{2}} \tag{2.9}$$

is a complete elliptic integral of first kind. Hence the solution of the hypersingular integral equation (1.3) is given by

$$f(x)=\begin{cases}\dfrac{1}{\pi}\left[\displaystyle\int_{-1}^{x}\frac{1}{R(u)}\left(C_1+\Psi(u)\right)\mathrm{d}u\right], & x\in(-1,-k),\\ -\dfrac{1}{\pi}\left[\displaystyle\int_{x}^{1}\frac{1}{R(u)}\left(C_1+\Psi(u)\right)\mathrm{d}u\right], & x\in(k,1),\end{cases} \tag{2.10}$$

where, $C_1$ is given in Eqs. (2.8) and (2.9). It may be noted that if $\psi(t)$ is an odd function then $C_1=0$.

**Method-II**

We first consider the sectionally analytic function

$$F(z)=\frac{1}{\pi}\int_G\frac{f(t)}{(z-t)^2}\mathrm{d}t, \tag{2.11}$$

cut along the real axis on $(-k,-1)\cup(k,1)$ and

$$F(z)\sim O\left(\frac{1}{z^2}\right)\quad\text{as } |z|\to\infty.$$

If we utilize the following standard limiting values (cf [6,8,9]):

$$\lim_{y\to\pm 0}\frac{1}{(x+\mathrm{i}y)}=\mp\pi\mathrm{i}\delta(x)+\frac{1}{x},\quad(-\infty<x<\infty),$$

then on differentiation with respect to $x$ it gives

$$\lim_{y\to\pm 0}\frac{1}{(x+\mathrm{i}y)^2}=\pm\pi\mathrm{i}\delta'(x)+\frac{1}{x^2},\quad(-\infty<x<\infty), \tag{2.12}$$

where $\delta(x)$ is Dirac's delta function and $\delta'(x)$ denotes its derivative with respect to $x$. Hence we find the following limiting values of $F(z)$ as

$$F^{\pm}(x)=\frac{1}{\pi}\left[\int_G\frac{f(t)}{(x-t)^2}\mathrm{d}t\pm\pi\mathrm{i}f'(x)\right],\quad x\in G. \tag{2.13}$$

From relation (2.13) we get the following Plemelj type alternative formulae:

$$F^{+}(x)+F^{-}(x)=2\psi(x),\quad x\in G \tag{2.14}$$

$$F^{+}(x)-F^{-}(x)=r'(x),\quad x\in G \tag{2.15}$$

where,

$$r'(x)=2\mathrm{i}f'(x),\quad x\in G. \tag{2.16}$$

From Eqs. (2.15) and (2.11) we get

$$F(z)=\frac{1}{2\pi\mathrm{i}}\int_G\frac{r(t)}{(z-t)^2}\mathrm{d}t. \tag{2.17}$$

Utilizing the above idea, we can solve the Riemann–Hilbert (R–H) problem (2.14) in the following manner. Let

$$F(z)=F_0(z)F_1(z), \tag{2.18}$$

$F_0(z)$ being the solution of the homogeneous problem corresponding to relation (2.14), ie,

$$F_0^{+}(x)+F_0^{-}(x)=0.$$

We choose

$$F_0(z)=((z^2-1)(z^2-k^2))^{-\frac{1}{2}}, \tag{2.19}$$

with

$$F_0^{\pm}(x)=\begin{cases}\mp\dfrac{\mathrm{i}}{R(x)}, & x\in(k,1),\\ \pm\dfrac{\mathrm{i}}{R(x)}, & x\in(-1,-k),\end{cases} \tag{2.20}$$

and $R(x)$ is given by Eq. (2.3). Substituting $F(z)$ from Eq. (2.18) into Eq. (2.14) we get

$$F_1^+(x) - F_1^-(x) = \frac{2\psi(x)}{F_0^+(x)}, \quad x \in G. \tag{2.21}$$

Comparing this relation with relation (2.15) and noting the relations (2.16) and (2.17), we write the solution of Eq. (2.21) as

$$F_1(z) = \frac{F(z)}{F_0(z)} = \frac{1}{2\pi \mathrm{i}} \int_G \frac{g(t)}{(z-t)^2} \mathrm{d}t + E(z), \tag{2.22}$$

where,

$$g'(t) = \frac{2\psi(t)}{F_0^+(t)}, \tag{2.23}$$

and $E(z)$ is an entire function. Noting the behaviour of $F(z)$ as $|z| \to \infty$ and considering Eq. (2.19), we can choose $E(z)$ as

$$E(z) = Bz + C, \tag{2.24}$$

where $B$ and $C$ are constants.

Using relations (2.22), (2.19) and (2.20) in relation (2.15) we get

$$f'(x) = \begin{cases} \dfrac{-1}{\pi R(x)} \left[ \displaystyle\int_k^1 \frac{g_0(t)}{(x-t)^2} \mathrm{d}t - \int_{-1}^{-k} \frac{g_0(t)}{(x-t)^2} \mathrm{d}t + \pi(Bx + C) \right], & x \in (k, 1), \\ \dfrac{1}{\pi R(x)} \left[ \displaystyle\int_k^1 \frac{g_0(t)}{(x-t)^2} \mathrm{d}t - \int_{-1}^{-k} \frac{g_0(t)}{(x-t)^2} \mathrm{d}t + \pi(Bx + C) \right], & x \in (-1, -k), \end{cases} \tag{2.25}$$

with

$$g_0'(x) = \psi(x) R(x), \tag{2.26}$$

$R(x)$ being given by Eq. (2.3).

Simplifying the result (2.25) using relation (2.26) we obtain

$$f'(x) = \begin{cases} \dfrac{-1}{\pi R(x)} \left[ \dfrac{g_0(1)}{x-1} - \dfrac{g_0(k)}{x-k} - \dfrac{g_0(-k)}{x+k} + \dfrac{g_0(-1)}{x+1} + \Psi(x) + \pi(Bx + C) \right], & x \in (k, 1), \\ \dfrac{1}{\pi R(x)} \left[ \dfrac{g_0(1)}{x-1} - \dfrac{g_0(k)}{x-k} - \dfrac{g_0(-k)}{x+k} + \dfrac{g_0(-1)}{x+1} + \Psi(x) + \pi(Bx + C) \right], & x \in (-1, -k). \end{cases} \tag{2.27}$$

For consistency of the solution from Eq. (2.27) we conclude that

$$g_0(1) = g_0(k) = g_0(-1) = g_0(-k) = 0.$$

So Eq. (2.25) can be rewritten as

$$f'(x) = \begin{cases} \dfrac{1}{\pi R(x)} \left[ C_3 + C_4 x + \Psi(x) \right], & x \in (-1, -k), \\ -\dfrac{1}{\pi R(x)} \left[ C_3 + C_4 x + \Psi(x) \right], & x \in (k, 1), \end{cases} \tag{2.28}$$

where

$$C_3 = \pi C, \qquad C_4 = \pi B. \tag{2.29}$$

Finally integrating Eq. (2.28) we get,

$$f(x) = \begin{cases} \dfrac{1}{\pi} \left[ \displaystyle\int_{-1}^x \frac{1}{R(u)} \left( C_3 + C_4 u + \Psi(u) \right) \mathrm{d}u \right] + F_3, & x \in (-1, -k), \\ \dfrac{1}{\pi} \left[ \displaystyle\int_x^1 \frac{1}{R(u)} \left( C_3 + C_4 u + \Psi(u) \right) \mathrm{d}u \right] + F_4, & x \in (k, 1). \end{cases} \tag{2.30}$$

Result (2.30) coincides with result (2.4). Now using end conditions $f(\pm 1) = 0 = f(\pm k)$ as before we get

$$C_4 = F_4 = F_3 = 0, \qquad C_3 = \frac{P(k)}{F(q)}, \tag{2.31}$$

where $P(k)$, $F(q)$ are given by relation (2.9). Thus it is found that solution $f(x)$ given by Eqs. (2.30) and (2.31) coincides with Eqs. (2.10) and (2.9).

## 3. Conclusion

Two different methods of solutions of hypersingular integral equation (1.3) in two disjoint intervals are presented here. The first method of solution is based on utilization of the solution of Cauchy type integral equation in two disjoint intervals. In the second approach, a recently devised function theoretic method is applied to reduce the hypersingular integral equation (1.3) to a Riemann–Hilbert problem of complex variable theory. The solution of the Riemann–Hilbert problem is then used to solve (1.3) in a very simple manner. The solution of Eq. (1.3) obtained by both the methods coincide with each other. The present method of solution can be applied to solve the hypersingular integral equation in finite number of multiple disjoint intervals. The solution of (1.3) can be used to study water wave scattering problem involving vertical barrier with two or more gaps and also in the problem of fracture mechanics involving multiple cracks.

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