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Univalent Functions Starlike with Respect to a Boundary Point

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1. INTRODUCTION

Let $S^*(\alpha)$ denote the class of functions $S(z)$ analytic in $D\{z: |z| < 1\}$, normalized so that $S(0) = 0$, $S'(0) = 1$, and such that for some α, $0 \leqslant \alpha < 1$,

$$\mathscr{Re}\, \frac{zS'(z)}{S(z)} > \alpha \qquad (z \in D).$$

Then $S(z)$ maps D univalently on a domain Δ that is starlike with respect to the origin. This means that for all t, $0 \leqslant t \leqslant 1$, and all z, $z \in D$, $tS(z) \in \Delta$. In other words $tS(z)$ is subordinate to $S(z)$ in D.

The function $K(z)$ is said to be convex of order α in D if, and only if, $zK'(z) \in S^*(\alpha)$ and $K(0) = 0$. In this case for any α, $0 \leqslant \alpha < 1$, $K(D)$ is a convex domain. In particular $K(z) \in S^*(1/2)$ [14].

Although the class $S^*(\alpha)$ has been explored extensively by many authors over a long period of time not much seems to be known about the class of analytic functions $G(z)$ that map D onto domains Δ that are starlike with respect to a boundary point. This point may be taken to be $G(1)$ where $G(1) = \lim_{r \to 1} G(r)$, $0 < r < 1$, is assumed to exist and where $G(0) = 1$. In this situation the domain Δ has the property that $tG(z) + (1-t)\, G(1) \in \Delta$, $0 < t \leqslant 1$, $z \in D$. An example of an analytic function of the class under discussion is

$$G(z) = \prod_{k=1}^{n} \left(\frac{1-z}{1-\varepsilon_k z}\right)^{\delta_k}, \qquad \varepsilon_k \neq 1, \qquad |\varepsilon_k| = 1, \qquad \delta_k > 0,$$

$$\sum_{k=1}^{n} \delta_k = 1.$$

It will be shown later that $G(D)$ lies in a half-plane with a boundary

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consisting of a finite number of slits lying on rays from the origin which is the boundary point $G(1)$. Two of these slits meet at the origin.

Another interesting example comes from an observation of Egerváry [2] in connection with the Cesàro partial sums of the geometric series $z + z^2 + \cdots + z^n + \cdots$. Let

$$S_n^{(1)}(z) = \frac{1}{n}\,[nz + (n-1)\,z^2 + (n-2)\,z^3 + \cdots + z^n],$$

$$S_n^{(2)}(z) = \frac{1}{n(n+1)}\,[(n+1)\,nz + n(n-1)\,z^2 + (n-1)(n-2)\,z^3 + \cdots + 2z^n].$$

Egerváry [2] has shown that $S_n^{(1)}(z)$ is univalently starlike for $|z| \leqslant 1$ with respect to the point $S_n^{(1)}(1) = (n+1)/2$ and that, in effect, $S_n^{(2)}(z) \in S^*(1/2)$. Furthermore,

$$G(z) = 1 - \frac{S_n^{(1)}(z)}{S_n^{(1)}(1)} = \frac{S_n^{(2)}(z)}{z}\,(1 - z).$$

Then $G(0) = 1$, $G(1) = 0$ and $G(z)$ is starlike in D with respect to the origin which is the boundary point $G(1)$. In this example $G(D)$ lies in the half-plane $\mathscr{Re}\ G(z) > 0$ since for $z \in D$

$$\mathscr{Re}\ G(z) - 1 \geqslant -|G(z) - 1| = \frac{-2}{n+1}\,|S_n^{(1)}(z)| > \frac{-2}{n+1}\,S_n^{(1)}(1) = -1.$$

We introduce at this point three closely-related classes of analytic functions.

DEFINITION 1. Let $\mathscr{S}^*$ denote the class of functions $G(z)$ analytic in $D\{z : |z| < 1\}$, normalized so that $G(0) = 1$, $G(1) = \lim_{r \to 1} G(r) = 0$, and such that for some real $\alpha\,\mathscr{Re}\,[e^{i\alpha}G(z)] > 0$, $z \in D$. In addition let $G(z)$ map D univalently on a domain starlike with respect to $G(1)$. Let the constant function 1 also belong to the class $\mathscr{S}^*$.

Let $\mathscr{S}_*$ denote the subset of $\mathscr{S}^*$ for which $G(z)$ is assumed to be analytic on $|z| = 1$ as well as for $|z| < 1$.

DEFINITION 2. Let $\mathscr{S}$ denote the class of functions

$$G(z) = 1 + d_1 z + d_2 z^2 + \cdots + d_n z^n + \cdots$$

analytic and non-vanishing in D, normalized so that $G(0) = 1$ and such that

$$\mathscr{Re}\ \left\{\frac{2zG'(z)}{G(z)} + \frac{1+z}{1-z}\right\} > 0 \qquad (z \in D).$$

The main purpose of this paper is to study the properties of the class $\mathscr{G}$. A primary objective will be to show that

$$\mathscr{G}_* \subset \mathscr{G} \subset \mathscr{G}^*.$$

It is an open question whether the class $\mathscr{G}$ coincides with the class $\mathscr{G}^*$. This is suggested by the fact that the inner and outer classes $\mathscr{G}_*$ and $\mathscr{G}^*$ have a great deal in common.

The class $\mathscr{G}$ is closely related to the class $S^*(1/2)$. We shall show that a necessary and sufficient condition that $G(z) \in \mathscr{G}$ is that there exists an $S(z) \in S^*(1/2)$ such that

$$G(z) = (1-z)\frac{S(z)}{z}.$$

When $G(z) \in \mathscr{G}$ the coefficients d_n, $n = 1, 2,\dots$, may be arbitrarily small in absolute value. On the other hand the example

$$G(z) = (1-z)(1-2z\cos\phi + z^2)^{-1/2}, \qquad 0 < \phi < 2\pi, \qquad d_1 = \cos\phi - 1,$$

$$\lim_{\phi\to 0}\left[\frac{G(z)-1}{\cos\phi - 1}\right] = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$$

shows that the inequality $|d_n| \leqslant n\,|d_1|$, known to be true for close-to-convex functions [9], cannot be improved for the class $\mathscr{G}$. Yet, when in the representation

$$zG(z) = (1-z)\,S(z), \qquad S(z) \in S^*(1/2),$$

one restricts $S(z)$ to be convex in D, we have the somewhat surprising result that

$$|d_n| \leqslant \frac{(2n+1)}{3}|d_1|, \qquad n = 2, 3,\dots, \tag{1.1}$$

and the factor $(2n+1)/3$ cannot be replaced by a smaller one. Equivalently, if $K(z) = z + \sum_{n=2}^{\infty} c_n z^n$ is convex in D, we have the new inequalities

$$|c_{n+1} - c_n| \leqslant \frac{2n+1}{3}|1 - c_2|, \qquad n = 2, 3,\dots. \tag{1.2}$$

The function

$$K(z) = \sum_{n=1}^{\infty} \frac{1}{n}\frac{\sin n\phi}{\sin\phi} z^n$$

is convex in D since $zK'(z) = z(1 - 2z\cos\phi + z^2)^{-1} \in S^*(0)$, and $K(z)$ has

$$\lim_{\phi\to 0}\left|\frac{c_{n+1}-c_n}{1-c_2}\right| = \frac{2n+1}{3}.$$

It is of interest to note that the inequalities (1.2) are stronger than the inequalities

$$|1-c_n| \leqslant \frac{n^2-1}{3}|1-c_2|, \qquad n = 3, 4, \ldots, \tag{1.3}$$

that easily follow from (1.2) by addition of the inequalities and the triangular inequality $|A+B| \leqslant |A| + |B|$.

The inequalities (1.2) can be rewritten in terms of the coefficients $a_n = nc_n$ for $S(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(0)$:

$$\left|\frac{a_{n+1}}{n+1} - \frac{a_n}{n}\right| \leqslant \frac{2n+1}{6}|2-a_2|, \qquad n = 2, 3, \ldots. \tag{1.4}$$

We shall show that (1.4) also holds for $S(z)$ univalent in D if all the coefficients are real, or indeed if $S(z)$ is merely typically real in D. Moreover (1.4) is stronger than the known inequalities for $S \in S^*(0)$ due to Hummel [3]:

$$|n-a_n| \leqslant \frac{n(n^2-1)}{6}|2-a_2|, \qquad n = 3, 4, \ldots, \tag{1.5}$$

which (1.4) imply. In the typically real case (1.4) again implies (1.5). The inequalities (1.5) were established by Leeman [5] and later by Krzyż and Złotkiewicz [4].

When $S(z) \in S^*(0)$ is an odd function and $S(z) = z + a_3 z^3 + a_5 z^5 + \cdots + a_{2n+1} z^{2n+1} + \cdots$ we also have the new inequalities:

$$|a_{2n+1} - a_{2n-1}| \leqslant n\,|1-a_3|, \qquad n = 2, 3, \ldots. \tag{1.6}$$

For functions

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

that are analytic and have $\mathscr{Re}\, P(z) > 0$ in D we also find that

$$|p_{n+1} - p_n| \leqslant (2n+1)\,|2-p_1|, \qquad n = 1, 2, \ldots. \tag{1.7}$$

Again, the factor $(2n+1)$ cannot be replaced by a smaller one.

As an application that follows from our result that

$$\mathscr{Re}\,[G(z)\,e^{-i\arg G(1)}] > 0, \qquad z \in D, \qquad G(z) \in \mathscr{G},$$

we are able to prove that whenever the function

$$K(z) = z + \sum_{n=2}^{\infty} k_n z^n$$

is analytic and convex of order 1/2 in D then all the partial sums

$$K_n(z) = z + k_2 z^2 + \cdots + k_n z^n, \qquad n = 1, 2, \ldots, \tag{1.8}$$

are univalent and close-to-convex with respect to the convex $K(z)$. For any $n > 1$ and $\alpha < 1/2$ there exists $K(z)$ convex of order α and a $K_n(z)$ that is not univalent in D. This result was obtained by Ruscheweyh and Sheil-Small [13] who used another method.

A second application stems from inequalities we obtain for the differences of successive coefficients for the class $S^*(1/2)$ to be proven later in Corollary 2. If

$$K(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

maps D onto a convex domain then for $z \in D$ we obtain for $n = 2, 3, \ldots$

$$\left| K(z) - \sum_{k=1}^{n-1} c_k z^k - \frac{c_n z^n}{1-z} \right| \leqslant n\,|z|^{n-1} \left| K(z) - \frac{z}{1-z} \right|, \tag{1.9}$$

$$\sum_{k=n+1}^{n+m} |c_n - c_k|^2$$
$$\leqslant n^2 \cdot \sum_{k=2}^{m+1} |1 - c_k|^2, \qquad n = 2, 3, \ldots, \quad m = 1, 2, \ldots. \tag{1.10}$$

2. Preliminary Theorems

THEOREM 1. *Let $G(z)$ be analytic in D with $G(0) = 1$. Then $G \in \mathscr{G}$ if, and only if, there exists a function $S(z) \in S^*(1/2)$ such that*

$$G(z) = (1 - z)(S(z)/z). \tag{2.1}$$

Proof. Let $G(z)$ satisfy (2.1) where $S(z) \in S^*(1/2)$. Then $G(z)$ is analytic and non-vanishing in D, $G(0) = 1$ and

$$\mathscr{Re} \left\{ \frac{2zG'(z)}{G(z)} + \frac{1+z}{1-z} \right\} = \mathscr{Re} \left\{ \frac{2zS'(z)}{S(z)} - 1 \right\} > 0, \qquad z \in D. \tag{2.2}$$

By Definition 2 $G(z) \in \mathscr{S}$. Conversely, if $G(z) \in \mathscr{S}$ and $S(z) = zG(z)/(1-z)$ then $S(0) = 0$, $S'(0) = 1$, and by (2.2) $S(z) \in S^*(1/2)$.

THEOREM 2. *Let $S(z) \in S^*(1/2)$. Then the function*

$$S(z)/(1-z) \in S^*(0).$$

Proof. Let $f(z) = S(z)/(1-z)$, $S(z) \in S^*(1/2)$. Then $f(z)$ is analytic in D and $f(0) = f'(0) - 1 = 0$. For $z \in D$ we have

$$\mathscr{Re}\, \frac{zf'(z)}{f(z)} = \mathscr{Re}\, \left[\frac{zS'(z)}{S(z)} + \frac{z}{1-z}\right] \geqslant \frac{1}{1+|z|} - \frac{|z|}{1+|z|} > 0.$$

THEOREM 3. *Let $G(z) \in \mathscr{S}$. Then either $G(z)$ is univalent and close-to-convex in D or $G(z)$ is the constant* 1.

Proof. Let $G(z) \in \mathscr{S}$ and assume that $G(z)$ is not the constant $G(0) = 1$. Let $S(z) = zG(z)/(1-z)$. Then $S(z) \in S^*(1/2)$ by Theorem 1. With $0 < \rho < 1$ let $G_\rho(z) = (S(\rho z)/\rho z)(1-z)$. Then $G_\rho(z) \to G(z)$ as $\rho \to 1$. Since $S(\rho z)/\rho \in S^*(1/2)$ it follows from Theorem 1 that $G_\rho(z) \in \mathscr{S}$. We have

$$\frac{-\rho(1-z)\, zG_\rho'(z)}{S(\rho z)} = -\frac{(1-z)^2}{z} \cdot \frac{\rho z S'(\rho z)}{S(\rho z)} - (1 - z^{-1})$$

which is analytic for $|z| \leqslant 1$. For $z = e^{i\theta}$ we have

$$\mathscr{Re}\, \left\{\frac{zG_\rho'(z)}{-f_\rho(z)}\right\} = 2(1-\cos\theta)\left[\mathscr{Re}\, \left\{\frac{\rho z S'(\rho z)}{S(\rho z)}\right\} - \frac{1}{2}\right] \geqslant 0,$$

where by Theorem 2 $f_\rho(z) = S(\rho z)/\rho(1-z) \in S^*(0)$, Then for $z \in D$, $\mathscr{Re}\, \{zG'(z)/-f(z)\} \geqslant 0$ where $G(z) = (S(z)/z)(1-z)$, $f(z) = S(z)/(1-z)$. Since $G(z)$ is not a constant we have

$$\mathscr{Re}\, \{zG'(z)/-f(z)\} > 0 \qquad (z \in D).$$

This means that $G(z)$ is univalent and close-to-convex in D. $G(z) \equiv 1$ only when $S(z) = z/(1-z)$.

THEOREM 4. *Let $G \in \mathscr{S}$ and suppose that $G(z)$ is not a constant. Then the function* $\log G(z)$, $\log G(0) = 0$, *is univalent and close-to-convex in D with*

$$\mathscr{Re}\, \left\{(1-z)^2 \frac{G'(z)}{G(z)}\right\} < 0 \qquad (z \in D).$$

Proof. Since $G \in \mathscr{G}$ we may write

$$2\frac{zG'(z)}{G(z)} + \frac{1+z}{1-z} = \int_0^{2\pi} \frac{1+ze^{i\phi}}{1-ze^{i\phi}}\, d\alpha(\phi) \qquad (z \in D)$$

where $\alpha(\phi)$ is an increasing function with $\alpha(2\pi) - \alpha(0) = 1$. For $\varepsilon = e^{i\phi}$ we have

$$(1-z)^2 \frac{G'(z)}{G(z)} = -\int_0^{2\pi} (1-\varepsilon)\left(\frac{1-z}{1-\varepsilon z}\right) d\alpha(\phi).$$

The function

$$W = (1-\varepsilon)\left(\frac{1-z}{1-\varepsilon z}\right)$$

has $\mathscr{R}e\ W \geqslant 0$ for $z \in D$ and for all ε, $|\varepsilon| = 1$. Since $\alpha(\phi)$ is an increasing function of ϕ and $G(z)$ is not a constant we have

$$\mathscr{R}e \left\{(1-z)^2 \frac{G'(z)}{G(z)}\right\} < 0 \qquad (z \in D),$$

or

$$\mathscr{R}e \left\{\frac{-(\log G(z))'}{(z/(1-z))'}\right\} > 0.$$

Hence $\log G(z)$ is close-to-convex relative to the convex function $-z/(1-z)$.

3. Primary Theorems

THEOREM 5. *Let $G(z) \in \mathscr{G}$ and be not a constant. Then $G(1) = 0$ and $G(z)$ maps D onto a domain that is starlike with respect to $G(1)$.*

Proof. Since $G(z) \in \mathscr{G}$ we have $zG(z) = S(z)(1-z)$ where $S(z) = z + b_2 z^2 + \cdots \in S^*(1/2)$. We may write

$$\frac{S(z)}{z} = \exp\left\{\frac{1}{2}\int_0^z \frac{P(t)-1}{t}\, dt\right\}$$

where $P(z)$ is analytic in D, $P(0) = 1$, and $\mathscr{R}e\ P(z) > 0$ for $z \in D$. Then $P(z)$ is the uniform limit on compacta in D of functions $P_n(z)$ of positive real part in D of the form

$$P_n(z) = \sum_{k=1}^{n} \delta_k^{(n)} \left(\frac{1+\varepsilon_k^{(n)} z}{1-\varepsilon_k^{(n)} z}\right), \qquad n = 1, 2, \ldots$$

where $\delta_k^{(n)} > 0$, $\sum_{k=1}^{n} \delta_k^{(n)} = 1$, $|\varepsilon_k^{(n)}| = 1$.

Corresponding to each $P_n(z)$ we have the functions

$$S_n(z) = z \exp\left\{\frac{1}{2}\int_0^z [P_n(t)-1]\frac{dt}{t}\right\} = z + b_2^{(n)}z^2 + \cdots \in S^*(1/2),$$

$$G_n(z) = S_n(z)\frac{(1-z)}{z} = 1 + (b_2^{(n)}-1)z + \cdots .$$

By Theorem 1 $G_n(z) \in \mathcal{S}$. By Theorem 3 $G_n(z)$ is either a constant or is univalent in D. The sequence $G_n(z)$ converges uniformly to $G(z)$ on all compact subsets of D. Explicitly we have

$$G_n(z) = \prod_{k=1}^{n}\left(\frac{1-z}{1-\varepsilon_k^{(n)}z}\right)^{\delta_k^{(n)}}, \qquad \delta_k^{(n)} > 0, \qquad \sum_{k=1}^{n}\delta_k^{(n)} = 1, \qquad |\varepsilon_k^{(n)}| = 1. \tag{3.1}$$

Since $G_n(z) \to G(z)$ and $G(z)$ is assumed to be not a constant it follows that for $n > n_0$ $G_n(z)$ also cannot be independent of z. We conclude that $G_n(z)$ is univalent in D for $n > n_0$ since it is not a constant.

Since $G_n(z)$ is not independent of z we must have $\varepsilon_k^{(n)} \neq 1$ for at least one value of k. From the representation (3.1) we then have $G_n(1) = 0$. $G_n(z)$ maps D onto a domain starlike with respect to $G_n(1) = 0$. The image domain $G_n(D)$ lies in a half-plane with a boundary consisting of a finite number of slits lying on rays from the origin two of which meet at the origin. This is readily seen by observing that each of the terms in the product defining $G_n(z)$ maps D on a slit-plane containing $W = 1$ and whose boundary passes through the origin, from which we deduce that $G_n(z)$ also has this property. This fact is made clear from the observation that whenever

$$W = \prod_{k=1}^{n} w_k^{\lambda_k} \text{ has } \lambda_k > 0, \qquad \sum_{k=1}^{n}\lambda_k = 1, \qquad \mathcal{Re}\; e^{i\alpha_k}w_k > 0, \alpha_k$$

real, then for $\alpha = \sum_{k=1}^{n}\lambda_k\alpha_k$, $\mathcal{Re}\; e^{i\alpha}W > 0$. Indeed,

$$\arg(e^{i\alpha}W) = \sum_{k=1}^{n}(\lambda_k\alpha_k + \lambda_k \arg w_k) = \sum_{k=1}^{n}\lambda_k \arg(e^{i\alpha}w_k),$$

$$|\arg(e^{i\alpha}W)| < \frac{\pi}{2}\sum_{k=1}^{n}\lambda_k = \frac{\pi}{2} \quad \text{since } \mathcal{Re}\,(e^{i\alpha}w_k) > 0.$$

Thus $\mathcal{Re}\,(e^{i\alpha}W) > 0$.

Second, $G_n(z)$ is univalent for $|z| < 1$ and analytic for $|z| \leqslant 1$ except at the points $z = \bar{\varepsilon}_k^{(n)}$. For $|z| = 1$, $z \neq \bar{\varepsilon}_k^{(n)}$ we have

$$\frac{d}{d\theta}\arg G_n(e^{i\theta}) = \left[\mathcal{Re}\;\frac{zG_n'(z)}{G_n(z)}\right]_{z=e^{i\theta}} = 0,$$

so that the boundary of $G_n(D)$ consists of slits on rays from the origin. Two of these slits meet at the origin since $G_n(1) = 0$. We conclude that $G_n(D)$ is starlike with respect to $G_n(1) = 0$.

By Carathéodory's kernel theorem [7, pp. 28–31] $G_n(D)$ converges to the kernel $G(D)$. Moreover, [7, Problem 3, p. 31], every compact subset of $G(D)$ is contained in $G_n(D)$ for all large n and for every point $c \in \partial G(D)$ there exist points $c_n \in \partial G_n(D)$ with $c_n \to c$ $(n \to \infty)$. We show now that $G(D)$ is starlike with respect to the origin $W = 0$.

Let W_0 be an interior point of $G(D)$ for which, if possible, there exists a point tW_0, $0 < t < 1$, $tW_0 \in \partial G(D)$. This leads to a contradiction. Consider the disk $|W - W_0| \leqslant R$ which lies in $G(D)$ if R is sufficiently small. Let $\bar{\Delta}$ denote the closed domain comprised of all the rays from the origin to all points of the circle $|W - W_0| = R$. The interior Δ of $\bar{\Delta}$ lies in $G_n(D)$ for all $n > N_0$ by the kernel theorem and because $G_n(D)$ is starlike with respect to the origin. In particular the point tW_0 lies in $\Delta \subset G_n(D)$ and the $\operatorname{dist}(tW_0, \partial G_n(D)) > tR > 0$. Since tR is independent of n this contradicts the kernel theorem with $c = tW_0$. We conclude that $tW_0 \in G(D)$ for $0 < t \leqslant 1$ when $W_0 \in G(D)$. Hence $G(D)$ is starlike with respect to the origin. Since $G(z) \neq 0$, $z \in D$, the origin must be a boundary point of $G(D)$. Since $G(z) = (S(z)/z)(1 - z)$ and $|S(z)/z| > 1/2$ for $S(z) \in S^*(1/2)$ it follows that $G(z_n) \to 0$ implies that $z_n \to 1$ as $n \to \infty$. Conversely, by a recent result due to Cochrane and MacGregor [1] if $S \in S^*(1/2)$ and $S(z) \neq z/(1 - xz)(|x| = 1)$ then there is a $\delta > 0$ so that $S(z) = O((1 - |z|)^{\delta - 1})$ so that $G(z) \to 0$ as $z \to 1$. Since $G(z)$ is not a constant $S(z) \neq z/(1 - z)$ where $x = 1$.

THEOREM 6. *Let $G(z) \in \mathscr{S}$. Then*

$$\mathscr{Re}\,\{G(z)\, e^{-i \arg G(1)}\} > 0 \qquad (z \in D)$$

where $-\pi/2 < \arg G(1) = \lim_{\rho \to 1} \arg G(\rho) < \pi/2$.

Proof. Let $G(z) = (S(z)/z)(1 - z)$ where $S(z) \in S^*(1/2)$. For $0 < \rho < 1$ let $G_\rho(z) = (S(\rho z)/\rho z)(1 - z)$. Then $G_\rho(0) = 1$, $G_\rho(1) = 0$ and $G_\rho(z)$ is analytic for $|z| \leqslant 1$. Since $S(\rho z)/\rho \in S^*(1/2)$ $G_\rho(z) \in \mathscr{S}$ by Theorem 1. The image domain $G_\rho(D)$ has a boundary Γ_ρ that is an analytic Jordan contour. The domain $G_\rho(D)$ is starlike with respect to the origin by Theorem 5. A calculation shows that at the point $z = 1$

$$\mathscr{Re}\left[1 + \frac{zG_\rho''(z)}{G_\rho'(z)}\right] = \mathscr{Re}\left[\frac{2\rho S'(\rho)}{S(\rho)} - 1\right] > 0$$

which tells us that Γ_ρ is locally convex at $W = 0$. We observe that Γ_ρ is tangent at the origin to the line through the origin with inclination $[\arg(-izG_\rho'(z))]_{z=1} = \pi/2 + \arg S(\rho)$. Since $G_\rho(z)$ is analytic on $|z| = 1$, locally convex at $W = 0$ and since $G_\rho(z)$ is starlike with respect to the origin we conclude that $G_\rho(D)$ lies entirely on one side of this tangent line, the side containing the point $W=1$. Let $\rho\to 1$. If $G(z)$ is not a constant then $G(D)$ lies in the half-plane $\mathscr{Re}\,[We^{-i\arg S(1)}]>0$ where $\arg S(1)=\lim_{\rho\to 1}\arg S(\rho)$. This limit exists [8]. Since $S(z)/z=[F(z)/z]^{1/2}$ for some $F\in S^*(0)$, and $\mathscr{Re}\,[F(z)/z]^{1/2}>0$ for $z\in D$ [6, 14] we have $|\arg S(z)/z|<\pi/2$. Then $|\arg S(1)|\leqslant\pi/2$, and, indeed, $|\arg S(1)|<\pi/2$. This follows since $W=1$ is an inner point of $G(D)$ which would not be the case if $G(D)$ were to lie in the half-plane $\mathscr{Re}\,[We^{-i\pi/2}]>0$ when $G(z)$ is not a constant.

It is readily seen that

$$\arg G(1)=\lim_{\rho\to 1}G(\rho)=\lim_{\rho\to 1}\arg\left[\frac{\rho}{1-\rho}\,G(\rho)\right]$$
$$=\lim_{\rho\to 1}\arg S(\rho)=\arg S(1).$$

The theorem is trivially true if $G(z)$ is the constant 1.

We have seen in Theorem 5 that if $G(z)\in\mathscr{G}$ then $G(D)$ is starlike with respect to the origin (a boundary point). The following is a converse theorem.

THEOREM 7. *Let $G(z)\in\mathscr{G}_*$. Then $G(z)\in\mathscr{G}$.*

Proof. If $G(z)\in\mathscr{G}_*$ and is not a constant we have $G(z)$ analytic on $|z|=1$, $G(1)=0$, $G(0)=1$ and $G(D)$ lies in a half-plane $\mathscr{Re}\,[We^{i\alpha}]>0$. Hence $G'(1)\neq 0$ and $z=1$ is a simple zero of $G(z)$. Therefore $G'(z)/G(z)$ has a simple pole with residue 1 at $z=1$. Since $G(1)=0$ and $G(z)$ is univalent and starlike for $|z|<1$ we have $G(z)\neq 0$ for $|z|\leqslant 1$, $z\neq 1$. Then the function

$$P(z)=\frac{2zG'(z)}{G(z)}+\frac{1+z}{1-z}$$

is analytic for $|z|\leqslant 1$ with $P(0)=1$. Since $G(D)$ is starlike and $G(z)$ is analytic on $|z|=1$ we have

$$\mathscr{Re}\left\{\frac{zG'(z)}{G(z)}\right\}\geqslant 0 \qquad \text{while} \quad \mathscr{Re}\left(\frac{1+z}{1-z}\right)=0 \text{ on } |z|=1,\ z\neq 1.$$

We conclude that $\mathscr{Re}\,P(z)\geqslant 0$ for $|z|=1$ and $\mathscr{Re}\,P(z)>0$ for $|z|<1$. Then $G(z)\in\mathscr{G}$. If $G(z)$ is the constant 1 again $G(z)\in\mathscr{G}$.

It is an open question whether the condition that $G(z)$ be analytic on $|z|=1$ in the proof of Theorem 7 may be removed.

4. Coefficient Inequalities

In the proofs of Theorems 8 and 11 to follow we shall make use of the following lemma.

LEMMA 1. *Let* $p(z) = 1 + p_1 z + \cdots$ *be analytic in* D *and have* $\mathcal{Re}\ p(z) > 0$ *for* $|z| < 1$. *Let*

$$q(z) = \frac{1 - z^2}{z} - \frac{(1 - z)^2}{z} \cdot p(z).$$

Then $q(z)$ *is analytic in* D *and* $\mathcal{Re}\ q(z) \geqslant 0$ *for* $|z| < 1$ *with equality only if* $p(z) = (1 + z)/(1 - z)$.

Proof. Choose ρ in the interval $(0, 1)$ and let

$$zq_\rho(z) = (1 - z^2) - (1 - z)^2 \cdot p(\rho z).$$

Then $q_\rho(z) \to q(z)$ as $\rho \to 1$ and $q_\rho(z)$ is analytic for $|z| \leqslant 1$. For $z = e^{i\theta}$ we have

$$\mathcal{Re}\ q_\rho(z) = 2(1 - \cos\theta)\ \mathcal{Re}\ p(\rho z) \geqslant 0.$$

Since the minimum of a harmonic function occurs on the boundary we have $\mathcal{Re}\ q_\rho(z) \geqslant 0$ for $|z| \leqslant 1$. Letting $\rho \to 1$ it follows that $\mathcal{Re}\ q(z) \geqslant 0$ for $|z| < 1$. Equality occurs only when $q(z)$ is identically zero. This the case only when $p(z) = (1 + z)(1 - z)$.

THEOREM 8. *Let* $K(z) = z + \sum_{n=2}^{\infty} c_n z^n$ *map* D *onto a convex domain. Then for* $n = 2, 3,\ldots$ *the following inequalities hold*:

$$|c_{n+1} - c_n| \leqslant \frac{2n + 1}{3} |1 - c_2|. \tag{4.1}$$

The factor $(2n + 1)/3$ *cannot be replaced by a smaller constant independent of* $K(z)$.

Proof. Let $S(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^*(0)$ and $T(z) = (S(z)/z)(1 - z)^2$, $T(z) = 1 + \sum_{k=1}^{\infty} t_k z^k$. In Lemma 1 we take $p(z) = zS'(z)/S(z)$ and $q(z) = -zT'(z)/S(z)$. Then by Lemma 1 $\mathcal{Re}\ [zT'(z)/-S(z)] > 0$ for $|z| < 1$ unless $T'(z) \equiv 0$. If $T'(z) \equiv 0$ we have $T(z) \equiv 1$ and $S(z) = z/(1 - z)^2$. We have now shown that either $T(z)$ is a constant or $T(z)$ is close-to-convex in D [3]. It follows in either case [9] that $|t_n| \leqslant n\,|t_1|$, $n = 2, 3,\ldots$.

Since $S(z)/z = T(z)/(1-z)^2 = (1 + t_1 z + \cdots + t_n z^n + \cdots)(1 + 2z + 3z^2 + \cdots)$,

$$a_{n+1} = n + 1 + nt_1 + (n-1)\,t_2 + \cdots + 2t_{n-1} + t_n, \qquad t_1 = a_2 - 2,$$

$$\frac{a_{n+1}}{n+1} - \frac{a_n}{n} = \frac{1}{n(n+1)}\,[t_1 + 2t_2 + \cdots + (n-1)\,t_{n-1} + nt_n],$$

$$\left|\frac{a_{n+1}}{n+1} - \frac{a_n}{n}\right| \leqslant \frac{|t_1|}{n(n+1)}\,[1^2 + 2^2 + \cdots + n^2] = \frac{2n+1}{3}\left|1 - \frac{a_2}{2}\right|. \tag{4.2}$$

If $K(z) = z + \sum_{n=2}^{\infty} c_n z^n$ is convex in D, then $S(z) = zK'(z)$ is starlike in D and $nc_n = a_n$. Thus

$$|c_{n+1} - c_n| \leqslant \frac{2n+1}{3}\,|1 - c_2|, \qquad n = 2, 3, \ldots. \tag{4.3}$$

The example $K(z)$ in the introduction, following (1.2), shows that the factor $(2n+1)/3$ in Theorem 8 cannot be replaced by a smaller one.

Corollary 1. *Let $G(z) = (1-z)\,K(z)/z = 1 + \sum_{k=1}^{\infty} d_k z^k$ where $K(z)$ is convex in D. Then $G(z) \in \mathscr{G}$ and*

$$|d_n| \leqslant \frac{2n+1}{3}\,|d_1|, \qquad n = 2, 3, \ldots. \tag{4.4}$$

Corollary 2. *Let $S(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*(1/2)$. Then*

$$|b_{n+1} - b_n| \leqslant n\,|1 - b_2|, \qquad n = 2, 3, \ldots. \tag{4.5}$$

Proof. Let $G(z) = (S(z)/z)(1-z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ where $S \in S^*(1/2)$. By Theorem 1 $G \in \mathscr{G}$. By Theorem 3 $G(z)$ is either close-to-convex in D or is a constant. In either case $|d_n| \leqslant n\,|d_1|$, $n = 2, 3, \ldots$, [9]. Since $d_n = b_{n+1} - b_n$, $n = 1, 2, \ldots$, $b_1 = 1$, the inequalities (4.5) follow. The example $(1-z)(1 - 2z\cos\phi + z^2)^{-1/2}$ for small ϕ, as indicated in the introduction, shows that the factor n is a best possible one.

Corollary 3. *Let $S(z) = z + a_3 z^3 + \cdots + a_{2n+1} z^{2n+1} + \cdots$ be an odd function and $S \in S^*(0)$. Then*

$$|a_{2n+1} - a_{2n-1}| \leqslant n\,|1 - a_3|, \qquad n = 2, 3, \ldots. \tag{4.6}$$

Proof. Let $S(z) = z + a_3 z^3 + \cdots + a_{2n+1} z^{2n+1} + \cdots$ be an odd function and $S \in S^*(0)$. Let $C(z) = \sqrt{z}\, S(\sqrt{z}) = z + a_3 z^2 + \cdots$. Then

$$\mathscr{R}e\ \frac{zC'(z)}{C(z)} = \mathscr{R}e\ \left[\frac{1}{2} + \frac{\sqrt{z}}{2} \frac{S'(\sqrt{z})}{S(\sqrt{z})}\right] > \frac{1}{2},$$

and $C \in S^*(1/2)$. Corollary 3 now follows from Corollary 2. The function $S(z) = z[1 - 2z^2 \cos\phi + z^4]^{-1/2} \in S^*(0)$ and shows for small ϕ that the factor n cannot be replaced by a smaller one.

THEOREM 9. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic and typically real in D. Then for $n = 2, 3,\ldots$*

$$\left|\frac{a_{n+1}}{n+1} - \frac{a_n}{n}\right| \leqslant \frac{2n+1}{6}(2 - a_2), \tag{4.7}$$

$$n - a_n \leqslant \frac{n(n^2-1)}{6}(2 - a_2). \tag{4.8}$$

The factors $(2n+1)/6$ and $n(n^2-1)/6$ are the smallest constants independent of $f(z)$ that can appear here.

In the proof of Theorem 9 we shall have need for the following lemma.

LEMMA 2. *For all ϕ and $n = 1, 2,\ldots$*

$$\left|(n+1)\frac{\sin n\phi}{\sin\phi} - n\frac{\sin(n+1)\phi}{\sin\phi}\right| \leqslant \frac{n(n+1)(2n+1)}{6}\left(2 - \frac{\sin 2\phi}{\sin\phi}\right).$$

Proof. Let $\phi = 2\theta$;

$$\frac{\sin(2n+1)\theta}{\sin\theta} = 1 + 2\sum_{k=1}^{n} \cos 2k\theta,$$

$$\frac{d}{d\theta}\frac{\sin(2n+1)\theta}{\sin\theta} = -4\sin 2\theta \cdot \sum_{k=1}^{n} k\frac{\sin 2k\theta}{\sin 2\theta},$$

$$\left|\frac{d}{d\theta}\frac{\sin(2n+1)\theta}{\sin\theta}\right| \leqslant 4\,|\sin 2\theta| \sum_{k=1}^{n} k^2 = \frac{4}{3}n(n+1)(2n+1)\,|\sin\theta\cos\theta|,$$

$$\left|\frac{\sin(2n+1)\theta}{\sin\theta} - (2n+1)\frac{\cos(2n+1)\theta}{\cos\theta}\right| = \left|\tan\theta\frac{d}{d\theta}\left(\frac{\sin(2n+1)\theta}{\sin\theta}\right)\right|$$
$$\leqslant \frac{4}{3}n(n+1)(2n+1)\cdot\sin^2\theta,$$

$$\begin{aligned}2\left|(n+1)\frac{\sin n\phi}{\sin\phi}-n\frac{\sin(n+1)\phi}{\sin\phi}\right| &= \left|(2n+2)\frac{\sin 2n\theta}{\sin 2\theta}-2n\frac{\sin(2n+2)\theta}{\sin 2\theta}\right| \\ &= \left|\frac{\sin(2n+1)\theta}{\sin\theta}-(2n+1)\frac{\cos(2n+1)\theta}{\cos\theta}\right| \\ &\leqslant \frac{4}{3}n(n+1)(2n+1)\cdot\sin^2\theta,\end{aligned}$$

$$\left|(n+1)\frac{\sin n\phi}{\sin\phi}-n\frac{\sin(n+1)\phi}{\sin\phi}\right| \leqslant \frac{n(n+1)(2n+1)}{6}\left(2-\frac{\sin 2\phi}{\sin\phi}\right). \tag{4.9}$$

We now proceed with the proof of Theorem 9. Since $f(z)$ is typically real in D it has the representation [11]

$$f(z)=\int_0^{\pi}\frac{z}{1-2z\cos\phi+z^2}\,d\alpha(\phi) \tag{4.10}$$

where $\alpha(\phi)$ is an increasing function with $\alpha(\pi)-\alpha(0)=1$. Then

$$\frac{a_{n+1}}{n+1}-\frac{a_n}{n}=\int_0^{\pi}\left[\frac{1}{n+1}\,\frac{\sin(n+1)\phi}{\sin\phi}-\frac{1}{n}\,\frac{\sin n\phi}{\sin\phi}\right]d\alpha(\phi).$$

By Lemma 2 we have

$$\left|\frac{a_{n+1}}{n+1}-\frac{a_n}{n}\right|\leqslant\int_0^{\pi}\frac{(2n+1)}{6}\left(2-\frac{\sin 2\phi}{\sin\phi}\right)d\alpha(\phi)=\frac{2n+1}{6}(2-a_2).$$

The function $f(z)=z(1-2z\cos\phi+z^2)^{-1}\in S^*(0)$ has

$$\lim_{\phi\to 0}\left|\left(\frac{a_{n+1}}{n+1}-\frac{a_n}{n}\right)\right|\cdot(2-a_2)^{-1}=\frac{2n+1}{6},$$

and

$$\lim_{\phi\to 0}(n-a_n)\cdot(2-a_2)^{-1}=\frac{n(n^2-1)}{6}.$$

The inequalities (4.8) were established in another way by Leeman [5] and again later by Krzyż and Złotkiewicz [4]. However, the inequalities (4.8) are implied by the stronger inequalities (4.7). From (4.7)

$$-\frac{(2n-1)}{6}(2-a_2)\leqslant\frac{a_n}{n}-\frac{a_{n-1}}{n-1},$$

$$-\frac{(2n-3)}{6}(2-a_2)\leqslant\frac{a_{n-1}}{n-1}-\frac{a_{n-2}}{n-2},$$

$$\cdots$$

$$-\frac{3}{6}(2-a_2)\leqslant\frac{a_2}{2}-1.$$

Addition gives

$$-\frac{(2-a_2)}{6}[3+5+\cdots+(2n-1)]\leqslant\frac{a_n}{n}-1$$

or

$$n-a_n\leqslant\frac{n(n^2-1)}{6}(2-a_2). \tag{4.11}$$

With only a slight modification of this proof of (4.11) one can show that the inequalities (4.2), which hold for $S\in S^*(0)$, imply the inequalities

$$|n-a_n|\leqslant\frac{n(n^2-1)}{6}|2-a_2|,\qquad n=3,4,... \tag{4.12}$$

established by Hummel [3]. For a generalization of (4.12) to β-spiral-like functions see Robertson [12].

A similar result for another class of functions is given in the following theorem.

THEOREM 10. *Let $P(z)=1+\sum_{n=1}^{\infty}p_nz^n$ be analytic in $D\{z:|z|<1\}$, and let $P(z)$ have a positive real part in D. Then $|p_{n+1}-p_n|\leqslant(2n+1)|2-p_1|$, $n=1,2,...,$ and $||p_{n+1}|-|p_n||\leqslant(2n+1)(2-|p_1|)$, $n=1,2,....$ The factor $(2n+1)$ cannot be replaced by a smaller one.*

Proof. We take $p(z)=P(z)=1+\sum_{n=1}^{\infty}p_nz^n$, as in Theorem 10, and let

$$q(z)=\frac{1-z^2}{z}-\frac{(1-z)^2}{z}P(z)=(2-p_1)+\sum_{n=1}^{\infty}q_nz^n.$$

By Lemma 1 $q(z)$ is analytic and $\mathscr{Re}\ q(z)\geqslant0$ in D. Then

$$|q_n|\leqslant2\,\mathscr{Re}\,(2-p_1),\qquad n=1,2,....$$

From the identity

$$\frac{1+z}{z}-\frac{(1-z)}{z}P(z)=\frac{q(z)}{1-z}$$

we have

$$(1-z)\left[2-p_1+\sum_{n=2}^{\infty}(2-p_n)z^{n-1}\right]$$
$$=\left[1+\sum_{n=1}^{\infty}z^n\right]\left[2-p_1+\sum_{n=1}^{\infty}q_nz^n\right].$$

$p_n - p_{n+1} =$ the coefficient of $z^n = 2 - p_1 + q_1 + \cdots + q_n$.

$$\begin{aligned}|p_{n+1}-p_n| &\leqslant |2-p_1| + 2n\,\mathscr{Re}\,(2-p_1)\\ &\leqslant (2n+1)\,|2-p_1|, \qquad n=1,2,\ldots. \end{aligned}\tag{4.13}$$

Since $P(z)$ may be replaced by $P(e^{i\phi}z)$ for arbitrary real ϕ we have for $p_1e^{i\phi}\geqslant 0$

$$\begin{aligned}||p_{n+1}|-|p_n|| &\leqslant |p_{n+1}e^{i\phi}-p_n| \leqslant (2n+1)\,|2-p_1e^{i\phi}|\\ &= (2n+1)(2-|p_1|). \end{aligned}\tag{4.14}$$

The example $P(z)=(1-z^2)(1-2z\cos\phi+z^2)^{-1}$ has $\mathscr{Re}\ P(z)>0$ in D and shows for small ϕ that the factor $(2n+1)$ in the theorem cannot be replaced by a smaller one.

5. Some Applications

The property of the class $\mathscr{S}$ provided in Theorem 6 leads with the help of Theorem 1 to an application to functions convex of order 1/2.

THEOREM 11. *Let $K(z)=z+\sum_{n=2}^{\infty}k_nz^n$ be analytic and convex of order 1/2 in $D\{z:|z|<1\}$. Then all the partial sums $K_n(z)=z+k_2z^2+\cdots+k_nz^n$, $n=1,2,\ldots$ are univalent and close-to-convex with respect to the convex $K(z)$. For any $n>1$ and $\alpha<1/2$ there exists a $K(z)$, convex of order α, and a $K_n(z)$ that is not univalent in D.*

Proof. Let $S(z)=\sum_{k=1}^{\infty}b_kz^k\in S^*(1/2)$, $S_n(z)=\sum_{k=1}^{n}b_kz^k$, $b_1=1$. For $0<\rho<1$ we have $S(\rho z)/\rho\in S^*(1/2)$ and Theorem 6 gives

$$\mathscr{Re}\left[(1-z)\frac{S(\rho z)}{\rho z}e^{-i\arg S(\rho)}\right]>0 \qquad (z\in D),$$

or equivalently for $0<|z|<1$

$$\left|e^{-i\arg S(\rho)}\cdot\frac{S(\rho z)}{\rho}-\frac{z}{1-z}\right|<\left|e^{-i\arg S(\rho)}\cdot\frac{S(\rho z)}{\rho}+\frac{z}{1-z}\right|.$$

It follows that the function $[e^{-i\arg S(\rho)} \cdot S(\rho z)/\rho - z/(1-z)]$ is quasi-subordinate [10] to $[e^{-i\arg S(\rho)} \cdot S(\rho z)/\rho + z/(1-z)]$. Consequently we have the coefficient inequalities [10]

$$\sum_{k=1}^{n} |e^{-i\arg S(\rho)} \cdot b_k \rho^{k-1} - 1|^2 \leqslant \sum_{k=1}^{n} |e^{-i\arg S(\rho)} \cdot b_k \rho^{k-1} + 1|^2,$$

$$\mathscr{R}e \left[e^{-i\arg S(\rho)} \cdot \sum_{k=1}^{n} b_k \rho^k\right] \geqslant 0, \qquad \mathscr{R}e\, [e^{-i\arg S(\rho)} \cdot S_n(\rho)] \geqslant 0.$$

Replacing $S(\rho z)/\rho$ by $e^{-i\theta} \cdot S(\rho z e^{i\theta})/\rho \in S^*(1/2)$, we have for $z = \rho e^{i\theta}$ that $\mathscr{R}e\, [e^{-i\arg S(z)} \cdot S_n(z)] \geqslant 0$, or $\mathscr{R}e\, [S_n(z)/S(z)] \geqslant 0$, $|z| = \rho > 0$, since $S(z) \neq 0$ for $0 < |z| < 1$. For $z = 0$ we define $S_n(z)/S(z)$ to be 1. Hence $\mathscr{R}e\, [S_n(z)/S(z)] \geqslant 0$ for $z \in D$. Since $S_n(z)/S(z)$ is not the constant zero we have $\mathscr{R}e\, [S_n(z)/S(z)] > 0$, $z \in D$.

The example $S(z) = z/(1-z)$, $S_n(z)/S(z) = 1 - z^n$, shows that we cannot have $\mathscr{R}e\, [S_n(z)/S(z)] > \delta_n > 0$ where δ_n is a constant independent of $S(z)$.

If we now let $K(z) = z + \sum_{k=2}^{\infty} k_n z^n$ be convex of order 1/2 in D and let $S(z) = zK'(z) \in S^*(1/2)$ then

$$\begin{aligned} \mathscr{R}e\, [K_n'(z)/K'(z)] &= \mathscr{R}e \left[\frac{z + 2k_2 z^2 + \cdots + nk_n z^n}{S(z)}\right] \\ &= \mathscr{R}e\, [S_n(z)/S(z)] > 0 \end{aligned}$$

$(z \in D)$ where $K_n(z)$ and $S_n(z)$ are the nth partial sums of $K(z)$ and $S(z)$, respectively. Thus $K_n(z)$ is close-to-convex relative to $K(z)$. For another proof see Ruscheweyh and Sheil-Small [13].

The example $K(z) = (1 - (1-z)^{2\alpha - 1})/2\alpha - 1$ is convex of order α, and has no partial sum $K_n(z)$, $n > 1$, that is univalent in D for $\alpha < 1/2$. This follows since $n|k_n| > 1$ and so $K_n'(z)$ has a zero in D.

THEOREM 12. *Let $K(z) = z + \sum_{n=2} c_n z^n$ be convex in D. Then in D*

$$\left| K(z) - \sum_{k=1}^{n-1} c_k z^k - \frac{c_n z^n}{1-z} \right| \leqslant n|z|^{n-1} \left| K(z) - \frac{z}{1-z} \right|,$$

$$n = 2, 3, \ldots, c_1 = 1.$$

Proof. Since $K(z)$ is convex in D then for $|z| < 1$, $|u| < 1$, by a result of Ruscheweyh and Sheil-Small [13]

$$G(z) = z \left[\frac{K(z) - K(u)}{z - u}\right] \frac{u}{K(u)} \in S^*(1/2), \qquad G(z) = \sum_{k=1}^{\infty} C_k(u)\, z^k$$

where $C_1(u) = 1$, and for $k > 1$

$$C_k(u) = \frac{u}{K(u)}\left[\frac{K(u)}{u^k} - \frac{1}{u^{k-1}} - \frac{c_2}{u^{k-2}} - \cdots - \frac{c_{k-1}}{u}\right].$$

Since $G(z) \in S^*(1/2)$ we have by Corollary 2

$$|C_{n+1}(u) - C_n(u)| \leqslant n\,|1 - C_2(u)|,$$

and this inequality may be written in the form

$$\left|K(u) - \sum_{k=1}^{n-1} c_k u^k - \frac{c_n u^n}{1-u}\right| \leqslant n\,|u|^{n-1}\left|K(u) - \frac{u}{1-u}\right|, \qquad |u| < 1,$$

where $c_1 = 1$.

COROLLARY 4. *Let $K(z) = z + \sum_{n=2}^{\infty} c_n z^n$ be convex in D. Then*

$$\left|\sum_{k=n}^{\infty} (c_n - c_k)\, z^k\right| \leqslant n\,|z|^{n-1}\left|\sum_{k=1}^{\infty} (1 - c_k)\, z^k\right|, \qquad c_1 = 1,$$

$$n = 2, 3, \ldots.$$

THEOREM 13. *Let $K(z) = z + \sum_{n=2}^{\infty} c_n z^n$ be convex in D. Then for $n = 2, 2, \ldots$, $m = 1, 2, \ldots$*

$$\sum_{k=n+1}^{n+m} |c_n - c_k|^2 \leqslant n^2 \sum_{k=2}^{m+1} |1 - c_k|^2.$$

Proof. $\sum_{k=n}^{\infty} (c_n - c_k)\, z^{k-n}$ is quasi-subordinate to $\sum_{k=1}^{\infty} n(1 - c_k)\, z^{k-1}$ and the inequalities of Theorem 13 follow [10].

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