On Infeasibility of Systems of Convex Analytic Inequalities

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Submitted by E. S. Lee

Received October 13, 1998

In this paper we address the problem of the infeasibility of systems defined by convex analytic inequality constraints. In particular, we investigate properties of irreducible infeasible sets and provide an algorithm that identifies a set of all constraints \((K)\) that may affect the feasibility status of the system after some perturbation of the right-hand sides. We analyze properties of the irreducible sets, as well as infeasibility sets in connection with the set \(K\), showing in particular that every infeasible system contains an inconsistent subsystem of cardinality not greater than the number of variables plus one.

The results presented in this paper are generalizations of a theory developed for the systems of quadratic and linear inequality constraints.

1. INTRODUCTION

We consider the convex region

\[
\mathcal{R} = \{ x \in \mathbb{R}^n \mid f_i(x) \leq 0, i \in I \}
\]  

where \( f_i(x): \mathbb{R}^n \to \mathbb{R}, i \in I \cup \{0\}, I = \{1, \ldots, m\} \), are real convex analytic functions, or more general, faithfully convex functions.

The region \(\mathcal{R}\) represents the feasible region of the convex programming problem. Convex programming problem has a long history. Early works include classical monographs by Fiacco and McCormick [10] and Rockafellar [23]. Algorithms for the solution of convex programs [10, 14] include interior point methods [9, 18], with particularly many algorithms devised for the solutions of the quadratically constrained quadratic convex programmes [2, 3, 12, 17, 19]. In [18] Nemirovski and Nesterov developed theory that applies to more general convex problems. Due to the diversity of results, it is not possible to list them all.
During formulation of the convex programming problem, particularly if it consists of a large number of constraints and variables, it is often difficult to determine whether or not the system is consistent. To the best of our knowledge, there are no known simple and efficient techniques to determine whether the model involving nonlinear constraints is correctly defined, that is, whether the system is feasible (see [6, 7, 10, 13]). Traditionally the problem of determining whether the system (1.1) is consistent has been handled by methods devised to identify an initial feasible point. They usually require the solution of some nonlinear problem which still has the same structure as the original problem, and contains one more constraint and variable.

In linear programs a common approach to testing infeasibility relies either on the identification of an irreducible infeasible subset of constraints (IIS), that is, the set of constraints that is infeasible, but for which any proper subset of constraints is feasible or on identifying the infeasibility set (IN), i.e., a subset of constraints whose removal will transform the system into a feasible one.

The importance and some aspects of detecting of irreducible infeasible sets and infeasibility sets has been discussed in [4, 5, 21]. Another important observation in detecting infeasibility is that there are usually constraints that do not have any impact on the consistency of the system, regardless of the values of the right-hand sides. This aspect of the infeasibility analysis has been investigated in [21] in relation to the systems of quadratic convex inequality constraints and, in particular, linear systems.

In this paper we extend the results obtained in [21] to the more general regions defined by faithfully convex functions. In particular, in Section 2 we present a method to identify a maximal subset of constraints that may have an impact on the feasibility status of the system after possible perturbation of the right-hand sides. Following the terminology introduced in [21], we call the latter set "killing constraints." The method is given in the form of the algorithm, which at each iteration requires the identification of implicit equality constraints in a homogeneous linear system. Since the implicit equalities can be detected in a finite number of simplex steps and since each iteration of the algorithm reduces the number of constraints and the number of variables, the algorithm terminates in a finite number of steps. In particular we show that the algorithm can be solved in $O(mn^2 \min\{m, n\})$ operations.

In Section 3 we investigate properties of the irreducible infeasible sets, which are expressed in terms of the implicit equalities in the corresponding linear system. We show that every irreducible infeasible set, as well as the infeasibility sets, are subsets of the set of killing constraints. The above property has the potential to reduce the number of constraints in the
process of determining of the above subsets. Finally, we extend the result on irreducible infeasible subsets proved by Chvatal for linear systems in
[8], and later in [21] for quadratic systems, by showing that any system of faithfully convex inequality constraints contains an irreducible infeasible subset of cardinality not greater than \( n + 1 \).

As indicated in [21], another possible applications of the set of killing constraints seems to be related to the sensitivity analysis, in particular, to the problems of the “distance to the ill-posedness” and of the condition measure of a linear system, which were investigated by Renegar [22] and Freund and Vera [11]. Some results on the distance to the ill-posedness and the condition measure would remain true, if these notions were extended not only to the set of quadratic inequality constraints but, more generally, to the set of convex inequality constraints.

Similar to case of quadratic systems, the algorithm to detect the set of killing constraints is implementable, and is based upon the method to solve linear programs with homogeneous constraints [1, 25], although we have not yet tested its practicality. The mentioned properties of the set of killing constraints indicate that the detection of this set may be of interest from both the theoretical and practical points of view.

2. THE KILLING CONSTRAINTS

It is well known that if a convex analytic function is constant along some half-line with the direction \( s \), then it is constant along any line with this direction. The set of vectors with the latter property forms what is called constancy space of \( f(x) \), which is denoted by \( D_{f}^{-} \) [15, 23]. A vector \( s \) is called a direction of recession \( f(x) \) if for every \( x \) the function \( f(x + ts) \) is a nondecreasing function of \( t \) [23]. Since any proper, convex, lower semicontinuous function is a closed function, then Theorem 8.6 in [23] implies that if \( f(x + ts) \) is nonincreasing for even one \( x \in R^{n} \), then it is nonincreasing for every \( x \). The set of all directions of recession of \( f(x) \) forms a convex cone called the cone of recession of \( f(x) \), denoted by \( 0^{+}f \).

The constancy space \( D_{f}^{-} \) of \( f(x) \) may be defined in terms of the set \( 0^{+}f \) [23] as

\[
D_{f}^{-} = \{ y \in R^{n} | y \in 0^{+}f \wedge -y \in 0^{+}f \}.
\]

Rockafellar [24] has shown that every convex analytic function \( f \) can be represented in the form

\[
f(x) = F(Bx + c) + \langle a, x \rangle - d,
\]
where $B \in R^{p \times n}$, $c \in R^p$, $a \in R^n$, $d \in R$, where $F$ is a strictly convex analytic function. Based on the observation that

$$D_f^* = \mathcal{N} \left( \begin{pmatrix} B \\ a^T \end{pmatrix} \right),$$

(where $\mathcal{N}(\cdot)$ denotes the null space of the matrix $(\cdot)$), Wolkowicz [15, 26] developed an algorithm to determine the matrix $B$ and the vector $a$.

Following the result in [24], we assume that $f_i(x)$, $i \in I$ are given in the form

$$f_i(x) = F_i(B_ix + c_i) + \langle a_i, x \rangle - d_i,$$

where $B_i \in R^{p_i \times n}$, $c_i \in R^{p_i}$, $a_i \in R^n$, $d_i \in R$, where $F_i$ is a strictly convex analytic function.

Some of the constraints in (1.1) may be in particular linear. We assume that if $f_i(x)$, $i \in I$, is a nonlinear function, then the Hessian matrix of $f_i(x)$ is uniformly bounded away from zero along a half-line with the direction of recession, in the sense that

$$\forall x_0 \in R^{p_i}, \quad \forall \mathbf{s} \in 0^+ f, \quad B_i \mathbf{s} \neq 0 \Rightarrow \exists t_0 > 0, \quad M_i(s, x_0) = \inf_{t \geq t_0} (B_i s)^T \nabla^2 F_i(x_0 + tB_i s) B_i s > 0$$

(2.2)

It follows immediately that if the function $f_i(x)$ satisfies condition (2.2), then it is affine along a line with the direction vector $s \in 0^+ f$. On the other hand, for any direction vector $s$ along which the function $f_i$ is affine, we have $B_i s = 0$ and therefore the condition (2.2) holds by default.

COROLLARY 2.1. If the convex function $f(x) = F(Bx + c) + \langle a, x \rangle - d$ is a polynomial of $n$ variables, then it satisfies condition (2.2).

Proof. It follows from the fact that the condition (2.2) is equivalent to the condition that the function is affine along any direction of recession. On the other hand, it is well known [23] that polynomial functions have the latter property.

THEOREM 2.1. If the convex analytic functions $f_i$, $i \in I$, satisfy conditions (2.2), then the region $\mathcal{R}$ is unbounded if and only if there exists a nonzero vector $s$ satisfying the following system:

$$B_i s = 0, \quad \forall i \in I$$

$$\langle a_i, s \rangle \leq 0, \quad \forall i \in I.$$
Proof. It follows from Theorem 8.4 [23] that \( \mathcal{R} \) is unbounded iff it contains a half-line. We will show that the half-line \( x(t) = x_0 + ts, \ t \geq 0, \ x_0 \in \mathcal{R} \) is in \( \mathcal{R} \) if and only if \( s \) satisfies conditions (2.3). Let us suppose first that conditions (2.3) hold. Then, for \( x_0 \in \mathcal{R} \),

\[
 f_i(x(t)) = F_i(c_i + B_i(x_0 + ts)) + \langle a_i, x_0 + ts \rangle - d_i \\
 \leq F_i(c_i + B_i x_0) + \langle a_i, x_0 \rangle - d_i \\
 = f_i(x_0) \leq 0, \quad \forall i \in I, \quad \forall t \geq 0.
\]

Let us now suppose that \( x(t) = x_0 + ts \in \mathcal{R}, \ \forall t \geq 0 \). This implies that

\[
 f_i(x(t)) = F_i(c_i + B_i(x_0 + ts)) + \langle a_i, x_0 + ts \rangle - d_i \\
 = f_i(x_0) + \langle \nabla f_i(x_0), B_is \rangle t + (B_is)^T \nabla^2 F_i(c_i + B_i(x_0 + ts))B_is t^2 \\
 \leq 0, \quad \forall i \in I, \quad \forall t \geq 0,
\]

where \( \hat{i} \in [0,t] \). If for some \( k \in I \), \( B_k s \neq 0 \), then, by assumption (2.2) there exist \( \bar{M}_k > 0 \) and \( t_0 > 0 \), such that for \( \hat{x}_0 = x_0 + t_0 s \)

\[
 f_k(\hat{x}_0) + \langle \nabla f_k(\hat{x}_0), B_k s \rangle t + \bar{M}_k t^2 \\
 \leq f_k(\hat{x}_0) + \langle \nabla f_k(\hat{x}_0), B_k s \rangle t + (B_k s)^T \nabla^2 F_k(c_k + B_k(\hat{x}_0 + \hat{i}s))B_k s t^2 \\
 \leq 0,
\]

\( \forall t \geq t_0 \), where \( \hat{i} \in [t_0, t] \). Since \( \lim_{t \to \infty} [\langle \nabla f_k(\hat{x}_0), B_k s \rangle t + \bar{M}_k t^2] = \infty \), then the latter inequality implies that \( B_k s = 0, \ \forall i \in I \). Therefore the inequality (2.4) reduces to

\[
 f_i(c_i + B_i x_0) + \langle a_i, x_0 \rangle - d_i + \langle a_i, st \rangle \\
 = f_i(x_0) + \langle a_i, s \rangle t \leq \langle a_i, s \rangle t \leq 0, \quad \forall i \in I, \quad \forall t \geq 0.
\]

The latter inequalities hold if and only if \( \langle a_i, s \rangle \leq 0, \ \forall i \in I \), which completes the proof of the theorem. \( \blacksquare \)

Corollary 2.2. If the convex analytic functions \( f_i(x), \ i \in I \cup \{0\} \) satisfy condition (2.2), then the function \( f_0(x) \) is unbounded from below along a half-line in \( \mathcal{R} \) if and only if there exists a vector \( s \) satisfying the following conditions

\[
 \langle a_0, s \rangle < 0, \\
 B_i s = 0, \quad \forall i \in I \cup \{0\}, \\
 \langle a_i, s \rangle \leq 0, \quad \forall i \in I.
\]
Proof. The backward part of the proof follows immediately. To prove the forward part, let us assume that \( f_0(x) \) is unbounded along a half-line \( x(t) = x_0 + ts, t \geq 0 \). Since \( x(t) \subset \mathcal{R} \), then by Theorem 2.1, conditions (2.3) are satisfied. If \( B_0s \neq 0 \), then by assumption (2.2) there exist \( M_0 > 0 \), and \( t_0 > 0 \), such that \[
 f_0(x_0) + \langle \nabla f_0(x_0), B_0s \rangle t + M_0 t^2 \leq f_0(x(t)), \quad \forall t \geq t_0.
\]

The left-hand side of the latter inequality increases to \( +\infty \), when \( t \to \infty \), which contradicts the assumption that \( f_0(x(t)) \to \infty \), and proves that \( B_0s = 0 \). Now it follows from the equation \( f_0(x(t)) = f_0(x_0) + \langle a_0, s \rangle t \) that \( \langle a_0, s \rangle < 0 \), which completes the proof of the corollary.

The Algorithm A given below identifies the set of all killing constraints in the system (1.1). We begin with a more formal definition of this set and some auxiliary results given in Lemma 2.1.

**Definition 2.1.** We say that the \( k \)th inequality in the system (1.1) belongs to the set \( K \) of killing constraints \( (k \in K) \), if there exist values \( b_k' > -\infty \), and \( b_i > -\infty, i \in I \), such that the system

\[
 f_i(x) \leq b_i, \quad i \in I,
\]
is infeasible and the system

\[
 f_i(x) \leq b_i, \quad i \in I \setminus \{k\},
\]

\[
 f_k(x) \leq b_k',
\]
is feasible or conversely.

**Definition 2.2.** An inequality \( \langle a_i, s \rangle \leq 0 \) in the system (2.3) is an implicit equality if \( \langle a_i, s \rangle = 0 \) for all \( s \) satisfying (2.3).

**Lemma 2.1.** If in the system (2.3) there are no implicit equalities, then the region \( \mathcal{R} \) is nonempty with \( \text{int}(\mathcal{R}) \neq \emptyset \), and the system (1.1) has no killing constraints.

**Proof.** Suppose that there are no implicit equalities in (2.3). Then there exists \( s^* \), such that

\[
 B_is^* = 0, \quad \forall i \in I,
\]

\[
 \langle a_i, s^* \rangle < 0, \quad \forall i \in I.
\]

Thus for any half-line \( x(t) = x_0 + s^*t, t \geq 0 \), with the direction vector \( s^* \) there exists \( t_0 \), such that \( x(t) \subset \text{int}(\mathcal{R}) \), for \( t \geq t_0 \), which completes the proof.
From Lemma 2.1 it follows that implicit equalities can be involved in causing the infeasibility of the systems of convex analytic inequalities. Algorithm A identifies the largest possible subsystem of the system (2.3), which contains only implicit equalities. We will show that this subsystem corresponds to the set of killing constraints in (1.1).

**ALGORITHM A.**

**Step 1.** \( k := 1, \ I_{\text{imp}}^0 := I \).

**Step 2.** Find the set \( I_{\text{imp}}^k \) of all implicit equalities in the system

\[
B_i s = 0, \quad \forall i \in I_{\text{imp}}^{k-1},
\]

\[
\langle a_i, s \rangle \leq 0, \quad \forall i \in I_{\text{imp}}^{k-1}.
\]

If \( I_{\text{imp}}^k = \emptyset \), terminate the algorithm with the message that \( \mathcal{R} \) is nonempty and \( K = 0 \).

If \( I_{\text{imp}}^k = I_{\text{imp}}^{k-1} \), terminate the algorithm with \( K = I_{\text{imp}}^k \).

**Step 3.** Set \( k := k + 1 \). Go to step 2.

We note that a similar algorithm has been proposed in [21] to determine the set of killing constraints in quadratically constrained convex regions. It follows directly from the algorithm that each iteration (except the terminating one) detects at least one nonkilling constraint. This implies that the algorithm terminates in at most \((m - |K| + 1)\) iterations.

Also, note that if \( K = I \) (which happens, for instance, when the set \( \mathcal{R} \) is bounded), then Algorithm A terminates in the first iteration, which indicates that all inequalities are implicit inequalities. Another extreme situation takes place when there are no implicit equalities in the system, which was considered in Lemma 2.1. In this case Algorithm A terminates in the first iteration as well.

The properties of the set \( K \) indicate that it would be interesting to identify a class of systems for which the cardinality of this set is significantly lower than the cardinality of \( I \). In the corollary below we show some relationship between cardinality of \( K \), and boundedness of the set \( \mathcal{R} \).

**Corollary 2.3.** If \( |K| < |I| \), then the set \( \mathcal{R} \) is either empty or unbounded. Furthermore, there do not exist such values \( b_i, i \in I \), that the set

\[
\mathcal{R}_b = \{ x | f_i(x) \leq b_i, i \in I \},
\]

is nonempty and bounded.

**Proof.** If \( \mathcal{R} \) is bounded and nonempty, Theorem 2.1 implies that the system (2.3) has no nonzero solution. This, on the other hand, implies that
in the first step of Algorithm A, all inequalities are implicit equalities, that is, $K = I$, which is a contradiction. The proof of the remaining part of the corollary follows from the fact that neither conditions of Theorem 2.1 nor the outcome of Algorithm A are dependent on the vector $b$ of the right-hand sides in the system $f_i(x) \leq b_i, i \in I$.

Let $A(I_{imp}^k)$ denote the matrix whose first columns are the vectors $a_i, i \in I_{imp}^k$ and whose remaining columns are the columns of the matrices $B_i, i \in I_{imp}^k$. The column space of $A(I_{imp}^k)$ is denoted by $C(A(I_{imp}^k))$ and the orthogonal complement of $C(A(I_{imp}^k))$ is the nullspace of $A^T(I_{imp}^k)$, which is denoted by $N(A^T(I_{imp}^k))$. Let us denote

$$\mathcal{B}(I_{imp}^k, p) = \{p + s \mid s \in C(A(I_{imp}^k))\}$$

and

$$\mathcal{B}(I_{imp}^k) = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, i \in I_{imp}^k\}.$$

**Lemma 2.2.** If Algorithm A terminates in step 2 with the set of indices $I_{imp}^k$, then

$$\mathcal{B}(I_{imp}^k, p) = \mathcal{B}(I_{imp}^k) \cap \mathcal{B}(I_{imp}^k, p)$$

is bounded for all $p \in \mathcal{B}(I_{imp}^k)$.

**Proof.** We use the method of contradiction. Suppose that $\mathcal{B}(I_{imp}^k, p)$ is unbounded. This implies that $\mathcal{B}(I_{imp}^k)$ is unbounded in the affine space $\mathcal{B}(I_{imp}^k, p)$, so that there exists a nonzero $s \in C(A(I_{imp}^k))$ such that

$$x(p, s) \in \mathcal{B}(I_{imp}^k) \cap \mathcal{B}(I_{imp}^k, p)$$

where $x(p, s) = p + ts, t \geq 0$.

Furthermore, it follows from Theorem 2.1 that $B_i s = 0$ and $\langle a_i, s \rangle \leq 0, \forall i \in I_{imp}^k$. Now termination in step 2 implies that $s \in N(A^T(I_{imp}^k))$, which is a contradiction since $s \neq 0$ and $s \in C(A(I_{imp}^k))$. Thus $\mathcal{B}(I_{imp}^k, p)$ is bounded.

In the theorem below we will give the proof of Algorithm A.

**Theorem 2.2.** If Algorithm A terminates with the message that $I_{imp}^k = I_{imp}^{k-1}$, then $K = I_{imp}^{k-1}$. In particular, if $I_{imp}^{k-1} = \emptyset$, then the set $\mathcal{B}$ is nonempty.

**Proof.** Let us suppose that Algorithm A terminates with the message that $I_{imp}^k = I_{imp}^{k-1}$. We will prove first that $K \subset I_{imp}^k$. Let us assume that the system (1.1) is feasible and that the system

$$f_i(x) \leq 0, \quad \forall i \in I \setminus \{j\},$$

$$f_j(x) \leq b_j,$$

for some $j \in K$ and some $b_j < 0$ is infeasible. Let us suppose that $j \notin I_{imp}^k$. 
Let \( \hat{x} \) be a solution to the system
\[
    f_i(x) \leq 0, \quad \forall i \in I \setminus \{j\}.
\]
Since \( j \not\in I^{k}_{\text{imp}} \), then \( \hat{x} \) is also a solution to the system
\[
    f_i(x) \leq 0, \quad \forall i \in I^{k}_{\text{imp}}.
\]
Since Algorithm A terminated with the index set \( I^{k}_{\text{imp}} = I^{k-1}_{\text{imp}} \), then for each index \( \kappa \in I^{k-2}_{\text{imp}} \setminus I^{k-1}_{\text{imp}} \), there exists a vector \( s^\kappa_k \) with \( \langle a_{\kappa}, s^\kappa_k \rangle \leq 0 \), \( j \in I^{k-2}_{\text{imp}} \), \( j \neq \kappa \), and \( B_j s^\kappa_k = 0 \), \( j \in I^{k-2}_{\text{imp}} \).

Define
\[
    s_k = \sum_{\kappa \in I^{k-2}_{\text{imp}} \setminus I^{k-1}_{\text{imp}}} s^\kappa_k.
\]
It follows that there exists a \( \tau_k \geq 0 \), such that \( x_{k-1} = \hat{x} + \tau_k s_k \) satisfies the system
\[
    f_i(x) \leq 0, \quad \forall i \in I^{k-2}_{\text{imp}}, \quad \text{if} \quad j \not\in I^{k-2}_{\text{imp}},
\]
or the system
\[
    f_i(x) \leq 0, \quad \forall i \in I^{k-2}_{\text{imp}} \setminus \{j\},
\]
\[
    f_j(x) \leq b_j, \quad \text{if} \quad j \in I^{k-2}_{\text{imp}}.
\]
We repeat the process until a solution to the system (2.7) is obtained. This nevertheless contradicts the earlier assumption that the system (2.7) is infeasible. This proves that \( K \subset I^{k}_{\text{imp}} \).

To prove that \( I^{k}_{\text{imp}} \subset K \), we will show that for every \( j \in I^{k}_{\text{imp}} \), the constraint \( f_j(x) \leq 0 \) may cause the infeasibility of \( \mathcal{R} \) by an appropriate change of the right-hand side. Let us assume that \( j \in I^{k}_{\text{imp}} \) and that (1.1) is feasible. Let us consider the problem
\[
    \begin{align*}
    \min & \quad f_j(x) \\
    \text{s.t.} & \quad f_i(x) \leq 0, \quad \forall i \in I^{k}_{\text{imp}} \setminus \{j\}. 
    \end{align*} \tag{2.8}
\]
The cone of recession of the problem (2.8) is given by the set of solutions to the system
\[
    B_i s = 0, \quad \forall i \in I^{k}_{\text{imp}},
\]
\[
    \langle a_i, s \rangle \leq 0, \quad \forall i \in I^{k}_{\text{imp}}. \tag{2.9}
\]
Since Algorithm A terminated in the \( k \)th step with \( I_{\text{imp}}^k = I_{\text{imp}}^{k-1} \), then every inequality in the system (2.9) is an implicit equality. Therefore, the cone of recession of the problem (2.8), denoted by \( C \), satisfies

\[
C = \bigcap_{i \in I_{\text{imp}}^k} D_{f_i}^C,
\]

where \( D_{f_i}^C \) denotes constancy space of \( f_i \). Now, Corollary 27.3.3 in [23] implies that the infimum of \( f_j(x) \) on the set \( f_j(x) \leq 0, i \in I_{\text{imp}}^k \setminus \{j\} \) is attained. It follows that if the point \( \bar{x} \) is a solution to the problem (2.8), then the system

\[
f_j(x) \leq 0, \quad i \in I_{\text{imp}}^k \setminus \{j\},
\]

\[
f_j(x) \leq f_j(\bar{x}) - \epsilon,
\]

is infeasible, \( \forall \epsilon > 0 \). Thus the system (2.7), is infeasible for \( b_j = f_j(\bar{x}) - \epsilon \). Consequently \( I_{\text{imp}}^k \subseteq K \), which completes the proof of the theorem.

Let \( W \) denote a matrix whose columns form an orthonormal basis for \( C[A] \), where \( A = [a_j, B_j, j \in I_{\text{imp}}^k] \), where \( I_{\text{imp}}^k \) is the index set obtained as a result of the termination of Algorithm A.

**Corollary 2.4.** If Algorithm A terminates after \( k \) iterations with the index set \( I_{\text{imp}}^k = I_{\text{imp}}^{k-1} \), then the system (1.1) is feasible iff the system

\[
f_i(W\xi) \leq 0, \quad i \in I_{\text{imp}}^k
\]

(2.10)

is feasible, where

\[
|I_{\text{imp}}^k| \leq m - k + 1 \text{ and } \xi \in R^N, \text{ with } N \leq n - k + 1.
\]

(2.11)

**Proof.** Theorem 2.2 and the definition of the set of the killing constraints imply that the feasibility of the system (1.1) is equivalent to the feasibility of the system

\[
f_i(x) \leq 0, \quad i \in I_{\text{imp}}^k.
\]

(2.12)

Furthermore, if the system (2.10) is satisfied for some \( \xi \), then the system (2.12) is satisfied for \( x = W\xi \). Now suppose that the inequalities (2.12) hold for some \( x_0 \in R^n \). Since \( x_0 \) can be represented as \( x_0 = v_0 + W\xi_0 \), for some \( v_0 \in \mathcal{N}(A^T) \) and some \( \xi_0 \), this implies that the inequality (2.10) holds for \( \xi = \xi_0 \). To prove that the first of the inequalities in (2.11) holds, we observe that each nonterminating iteration of step 3 has \( I_{\text{imp}}^l \subsetneq I_{\text{imp}}^{l-1} \) (\( l = 1, \ldots, k - 1 \)), which implies that \( |I_{\text{imp}}^k| \leq m - k + 1 \). To prove that
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In $n - k + 1$, we need to show that

$$\text{rank}(C(A^l)) < \text{rank}(C(A^{l-1})), $$

where $A^l = [a_l, B_l, \tau \in I_{\text{imp}}^l]$, for $0 \leq l \leq k - 1$. Suppose that $k \in I_{\text{imp}}^{l-1} \setminus I_{\text{imp}}^l$, so that $\langle a_k, s \rangle \leq 0$ is not an implicit equality in the system

$$B_i s = 0, \quad \forall i \in I_{\text{imp}}^{l-1},$$

$$\langle a_i, s \rangle \leq 0, \quad \forall i \in I_{\text{imp}}^{l-1}.$$ 

Thus there exists a vector $s$ with $\langle a_k, s \rangle < 0$, $B_j s = 0$ and $\langle a_j, s \rangle = 0$ for $j \in I_{\text{imp}}^l$. Thus, $a_k \not\in C(A^l)$, which completes the proof. 

Corollary 2.4 implies that Algorithm A terminates after at most $\min\{m, n - 1\}$ iterations with the set of killing constraints, which has an equivalent representation in the space of the dimension not higher than $n - k + 1$, where $k$ is the actual number of steps. The implementation of step 3 of Algorithm A requiring one to detect the implicit equality in the system of linear constraints has been discussed in [21], where it has been shown that the algorithm can be solved in either $O((nm)^2)$ or $O(mn^3)$ operations.

3. IRREDUCIBLE INFEASIBLE SETS

We will use the symbol $IIS$ (irreducible infeasible subset) with respect to the system of convex inequality constraints with the same meaning that has been used by other authors [4, 5] with respect to the system of the linear inequality constraints or in [21] with respect to quadratic constraints. Precisely, $IIS$ denotes the subset of constraints in (1.1), which is infeasible but for which any proper subset of constraints is feasible.

Algorithm B below, which is a straightforward extension of the deletion filtering algorithm for linear programs [5], identifies the irreducible infeasible set of the system (1.1). If $m$ is significantly larger than $|I_{\text{imp}}^k|$, then constraints in the set $I_{\text{imp}}^k$ obtained by the application of Algorithm A can be considered as an input set of Algorithm B identifying an $IIS$.

ALGORITHM B (extension of deletion filtering algorithm [5]).

1. Let $IIS = I_{\text{imp}}^k$
2. Determine whether or not the system

$$f_i(x) \leq 0, \quad i \in IIS \setminus \{\tau\}$$

is feasible.
3. If NO then $IIS := IIS \setminus \{\tau\}$ and go to step 2; if YES then go to step 2 with the next $\tau \in IIS$.

The proof of Algorithm B is contained in the lemma below.

**Lemma 3.1.** If the system (1.1) is infeasible, then the output of Algorithm B will contain exactly one IIS.

**Proof.** Proof is similar to the proof of the Lemma 3.3 in [21].

The lemma below states some properties of the irreducible infeasible subsets of constraints (IIS). These are expressed in terms of the implicit equalities in the corresponding linear system.

**Theorem 3.1.** Let IIS denote the irreducible subset of the system (1.1). Then

(a) Every inequality in the system

\[
B_\alpha s = 0, \quad \forall i \in IIS, \\
\langle a_i, s \rangle \leq 0, \quad \forall i \in IIS,
\]

is an implicit equality.

(b) Any IIS belongs to $K$, that is,

\[
\bigcup \alpha IIS_\alpha \subset K.
\]

(c) There exist $b_i \leq 0$, $i \in K$, such that for the system

\[
f_i(x) \leq b_i, \quad i \in K,
\]

\[
f_i(x) \leq 0, \quad i \in I \setminus \{K\}
\]

we have $\bigcup_{\alpha \in \Delta_\rho} IIS_\alpha = K$.

(d) If $B_\xi s = 0$, $\forall i \in IIS$, then for all $k \in IIS$, the system

\[
\langle a_i, s \rangle \leq 0, \quad \forall i \in IIS \setminus \{k\}
\]

contains an inequality that is not an implicit equality.

**Proof.** (a) Suppose that the system (3.1) contains a constraint which is not an implicit equality, say, with the index $k$. Therefore there exists $\hat{s}$, such that

\[
B_\alpha \hat{s} = 0, \quad \forall i \in IIS,
\]

\[
\langle a_i, \hat{s} \rangle \leq 0, \quad \forall i \in IIS \setminus \{k\},
\]

\[
\langle a_k, \hat{s} \rangle < 0.
\]
It follows from the definition of the set $IIS$ that the system
\[ f_i(x) \leq 0, \quad i \in IIS \setminus \{k\}, \]
is feasible. Thus for $\hat{x}$ satisfying the latter system, there exists $\hat{t} \geq 0$, such that
\[
f_i(\hat{x} + \hat{t}\hat{s}) = F_i(B_i(\hat{x} + \hat{t}\hat{s}) + c_i) + \langle a_i, \hat{x} + \hat{t}\hat{s} \rangle - d_i
\]
\[ = f_i(\hat{x}) + \langle a_i, \hat{s} \rangle t \leq \langle a_i, \hat{s} \rangle t \leq 0, \quad \forall i \in IIS, \]
$\forall \hat{t} \geq \hat{t}$, which contradicts the definition of the set $IIS$.

(b) Let $IIS_{a_0}$ be an arbitrary irreducible infeasible set. By part (a) of this theorem, for every $j \in IIS_{a_0}$, the $j$th inequality in the system
\[ B_i s = 0, \quad i \in IIS_{a_0}, \]
\[ \langle a_i, s \rangle \leq 0, \quad i \in IIS_{a_0} \]
is an implicit equality. Note that the latter implies that for any $S \subset I$, such that $IIS_{a_0} \subset S$, we have also that $\forall j \in IIS_{a_0}$, the $j$th inequality in the system
\[ B_i s = 0, \quad i \in S, \]
\[ \langle a_i, s \rangle \leq 0, \quad i \in S \]
is an implicit equality. This implies that in each subsequent $l$th step of Algorithm A ($l = 1, 2, \ldots, k$), $IIS_{a_0} \subset \text{Imp}^l_{I_0}$, and therefore $IIS_{a_0} \subset K$. Because $a_0$ was arbitrarily chosen, then
\[ \bigcup_{\alpha} IIS_{\alpha} \subset K. \]

(c) Let us consider the following procedure to determine updated vector of the right-hand sides of the system (1.1).

1. Let $IIS := K$ and $i_1 \in K \setminus \bigcup_{\alpha \in \Delta} IIS_{\alpha}$.
2. Determine whether or not the system $f_i(x) \leq 0, \; i \in IIS \setminus \{\tau\}, \; \tau \neq i_1$, is feasible.
3. If NO, then $IIS := IIS \setminus \{\tau\}$ and go to step 2 with the next $\tau$ ($\tau \neq i_1$).
4. If YES, then go to step 2 with the next $\tau$ ($\tau \neq i_1$).
5. If all $\tau \in K \setminus \{i_1\}$ were examined, then determine whether the system
\[ f_i(x) \leq 0, \quad i \in IIS \]
is feasible. If NO, then go to step 6. If YES, then find a solution \( x_{i_1} \) to the problem

\[
\begin{align*}
\text{min} & \quad f_i(x) \\
\text{s.t.} & \quad f_i(x) \leq 0, \quad i \in IIS.
\end{align*}
\]

Set \( b_{i_1} = f_i(x_{i_1}) - \epsilon \), for some \( \epsilon > 0 \).

6. Repeat the above steps for the next index \((i_2, i_3, \ldots)\) in the set \( K \setminus \bigcup_{\alpha \in \Delta} IIS_{\alpha} \). It follows that the vector of the right-hand sides \( b \) obtained as an output of the above procedure has a property that \( \bigcup_{\alpha \in \Delta} IIS_{\alpha} = K \).

(d) First note that the region determined by the inequalities

\[
\langle a_i, x \rangle \leq d_i, \quad i \in IIS \setminus \{k\}
\]

is nonempty. Infeasibility of \( IIS \) implies that the problem

\[
\begin{align*}
\text{min} & \quad \langle a_k, x \rangle \\
\text{subject to} & \quad \langle a_i, x \rangle \leq d_i, \quad i \in IIS \setminus \{k\},
\end{align*}
\]

is bounded below and that its minimum is achieved at some point \( x_k^* \), and that

\[
\langle a_k, x_k^* \rangle > d_k.
\]

We will show that at any optimal solution \( x_k^* \) to the problem (3.3)–(3.4), all constraints in (3.4) are active. To this end let us suppose that the opposite is true, i.e., there exists \( \tau \in IIS \setminus \{k\} \) that \( \{i \in IIS \setminus \{k\} | \langle a_i, x_k^* \rangle = d_i\} = IIS \setminus \{k, \tau\} \).

Since the \( \tau \)th constraint is not active at \( x_k^* \), then the optimal solution \( x_{k\tau}^* \) to the problem

\[
\begin{align*}
\text{min} & \quad \{\langle a_k, x \rangle | \langle a_i, x \rangle \leq d_i, \quad i \in IIS \setminus \{k, \tau\}\},
\end{align*}
\]

has a property

\[
\langle a_k, x_k^* \rangle = \langle a_k, x_{k\tau}^* \rangle.
\]

On the other hand, since \( IIS \setminus \{\tau\} \) is feasible, it follows that \( \langle a_k, x_{k\tau}^* \rangle \leq d_k \), which along with (3.6) contradicts (3.5). This proves that all constraints in (3.4) are active at the optimal solution \( x_k^* \). Let us now assume that all inequalities in (3.2) are implicit equalities. Since all constraints of \( IIS \setminus \{k\} \) are active at \( x_k^* \), and any inequality in (3.4) is an implicit equality, it
follows that the region defined by (3.4) is a linear manifold. Since the constraints are linear, this implies that there exists a proper subset of $IIS \setminus \{k\}$, say, $J$, such that a feasible region defined by the constraints with indices $J$ has a property that any solution optimal to the problem (3.3)–(3.4) is also optimal to the problem

$$\min\{\langle a_k, x \rangle \mid \langle a_i, x \rangle \leq d_i, i \in J\}.$$ 

The latter fact, along with the inequality (3.5), implies that the system of constraints with indices $J \cup \{k\}$ is infeasible, contradicting the definition of an IIS.

We note that the property proved in part (d) of the Theorem 3.1 is not valid for nonlinear functions, including quadratic convex constraints. The proof of the property analogous to the one given in part (d), but stated for quadratic systems in [21], was incorrect.

We will show in Theorem 3.3 below that the set $K$ is related to one of the fundamental theorems for a class of convex functions.

**Theorem 3.2** (Bohnenblust–Karlin–Shapley [16]). Let $\Gamma$ be a nonempty compact convex set in $\mathbb{R}^n$ and let $f_i, i \in I$ be a set of functions which are convex and lower semicontinuous on $\Gamma$. If $f_i(x) \leq 0, i \in I$ has no solutions on $\Gamma$, then there exists $p \in R^n_+$ such that

$$\sum_{i=1}^{m} p_i f_i(x) > 0, \quad \forall x \in \Gamma.$$ 

**Theorem 3.3.** Assume that $\mathcal{R}$ defined in (1.1) is empty. Let $p \in R^n_+$ be such that

$$\sum_{i=1}^{m} p_i f_i(x) > 0, \quad \forall x \in \mathbb{R}^n. \quad (3.7)$$ 

Then

(i) $I_p = \{j \mid p_j > 0, j \in I\} \subset K$.

(ii) $\exists p \in R^n_+, \text{ satisfying } (3.7)$, such that $I_p = \bigcup_{\alpha \in \Delta} IIS_{\alpha}$, where \{IIS_{\alpha}, \alpha \in \Delta\} is the indexed family of all irreducible infeasible subsets of (1.1).

**Proof.**

(i) The proof will be by contradiction. Let us assume that $\bar{k}$ is the smallest index such that

$$\left( I_{imp}^{\bar{k}} \setminus I_{imp}^{\bar{k}+1} \right) \cap I_p \neq \emptyset,$$
that is

\[ \left( I_{\text{imp}}^{k-1} \setminus I_{\text{imp}}^{k} \right) \cap I_{p} = \emptyset. \]

Let us suppose that Algorithm A terminates with index set \( I_{\text{imp}}^{k} = I_{\text{imp}}^{k-1} \). This implies that \( \forall l \in I_{\text{imp}}^{k-2} \setminus I_{\text{imp}}^{k-1} \) there exists a vector \( s_{l}^{k} \) with \( \langle a_{l}, s_{l}^{k} \rangle \leq 0 \), \( j \in I_{\text{imp}}^{k-2} \), \( j \neq l \) and \( B_{j}s_{l}^{k} = 0 \) \( j \in I_{\text{imp}}^{k-2} \). Let \( x_{k} \in \mathbb{R}^{n} \) be arbitrary but fixed. Let us define

\[ s_{k}^{l} = \sum_{l \in I_{\text{imp}}^{k-2} \setminus I_{\text{imp}}^{k-1}} s_{l}^{k}. \]

It follows that there exists \( \sigma_{k} \geq 0 \), such that \( x_{k-1} = x_{k} + \sigma s_{k} \) satisfies the system \( f_{i}(x) < 0 \), \( i \in I_{\text{imp}}^{k-2} \setminus I_{\text{imp}}^{k-1} \) for \( \forall \sigma \geq \sigma_{k} \). We have obviously \( f_{i}(x_{k}) = f_{i}(x_{k-1}) \), \( \forall i \in K \). We repeat the above process to obtain a solution \( x_{i} \) where \( x_{k} = x_{k+1} + \sigma_{k+1} s_{k+1} \) to the system

\[ f_{i}(x) < 0, \quad \forall i \in I_{\text{imp}}^{k} \setminus I_{\text{imp}}^{k+1}, \]

for some \( \sigma_{k+1} \geq 0 \). We clear have

\[ f_{i}(x_{k}) \leq 0, \quad \forall i \in I_{\text{imp}}^{k} \setminus K, \]

and

\[ f_{i}(x_{k}) = f_{i}(x_{k}), \quad \forall i \in K. \]

Therefore the latter equation yields for any \( \sigma \geq \sigma_{k+1} \)

\[
\sum_{i \in K \cap I_{p}} p_{i} f_{i}(x_{k+1} + \sigma s_{k+1}) + \sum_{i \in (I_{\text{imp}}^{k} \setminus K) \cap I_{p}} p_{i} f_{i}(x_{k+1} + \sigma s_{k+1}) = \sum_{i \in K \cap I_{p}} p_{i} f_{i}(x_{k}) + \sum_{i \in (I_{\text{imp}}^{k} \setminus K) \cap I_{p}} p_{i} (F_{i}(c_{i} + B_{i}(x_{k+1} + \sigma s_{k+1}))) + \langle a_{i}, x_{k+1} + \sigma s_{k+1} \rangle - d_{i}. \tag{3.8}
\]

Since

\[ B_{i}s_{k+1} = 0, \]

\[ \langle a_{i}, s_{k+1} \rangle \leq 0, \quad \forall i \in \left( I_{\text{imp}}^{k} \setminus K \right) \cap I_{p} \]

and for \( i_{0} \in (I_{\text{imp}}^{k} \setminus I_{\text{imp}}^{k+1}) \cap I_{p} \), \((I_{\text{imp}}^{k} \setminus I_{\text{imp}}^{k+1}) \cap I_{p} \subset (I_{\text{imp}}^{k} \setminus K) \cap I_{p} \), we have \( \langle a_{i_{0}}, s_{k+1} \rangle < 0 \), then expression in (3.8) approaches to \(-\infty\), as \( \sigma \to \infty \), which contradicts the equation (3.7). This completes the proof of part (i) of the theorem.
(ii) By Theorem 3.2, for every \( IIS_\alpha, \alpha \in \Delta \), there exists \( p^\alpha \in R^n_+ \), such that
\[
\sum_{i \in IIS_\alpha} p_i^\alpha f_i(x) > 0, \quad \forall x \in R^n.
\] (3.9)

Since any proper subset of \( IIS_\alpha \) is feasible, then it follows that \( p_i^\alpha > 0, \forall i \in IIS_\alpha \). Now let \( p_i^{\alpha_i}, \alpha_j \in \Delta \), be such that \( p_i^{\alpha_i} > 0, \forall i \in IIS_{\alpha_j} \), and inequality (3.9) holds with \( \alpha = \alpha_j \). This implies that the vector \( p = \sum_{\alpha_j \in \Delta} p_i^{\alpha_j} \) satisfies
\[
\sum_{i \in \cup_{\alpha_j \in \Delta} IIS_{\alpha_j}} p_i f_i(x) > 0, \quad \forall x \in R^n,
\]
and consequently \( I_p = \cup_{\alpha_j \in \Delta} IIS_{\alpha_j} \). \( \square \)

An interesting relationship between the infeasibility set \((IN)\) and the irreducible infeasible sets in linear systems has been proved in Theorem 4 in [4], which states that \( IN \) is an infeasibility set if and only if \( IN \) contains a member of each irreducible infeasible set of \( I \). This property may be generalized in a straightforward way to convex inequality constraints, which is stated in the lemma below.

**Lemma 3.2.** Let the system (1.1) be infeasible. A subset \( IN \) is an infeasibility set if and only if \( IN \) contains an element of each irreducible infeasible set of \( I \).

**Proof.** Proof is analogous to the proof of the Theorem 4 in [4]. \( \square \)

We observe that Lemma 3.2 remains true for arbitrary, not necessarily convex, functions \( f_i, i \in I \), since presence of any \( IIS \) in any subsystem of the system (1.1) means that the subsystem is infeasible.

**Corollary 3.1.** Let \( \{IN_\alpha, \alpha \in \Omega\} \) be an indexed family of the infeasibility sets. Then \( \cup_{\alpha \in \Omega} IN_\alpha \subseteq K \).

**Proof.** The proof follows from the fact that none of the constraints in the set \( I \setminus K \) has an impact on the infeasibility status neither of the system (1.1) nor of its proper subsystem, so \( (I \setminus K) \cap \cup_{\alpha \in \Omega} IN_\alpha = \emptyset \). \( \square \)

Now we will consider a problem of the upper bound for the minimal cardinality of the irreducible infeasible sets. Chvatal [8] proved that every inconsistent system of linear inequalities in \( n \) variables contains as inconsistent subsystem of at most \( n + 1 \) inequalities. In the theorem below we will extend this result to the systems of convex analytic inequality constraints.
In the proof of this theorem we will use the results stated in the following two lemmas.

**Lemma 3.3** [20]. *Constraint kth is an implicit equality in the system (1.1), if and only if*

\[ \forall x_1, x_2 \in \mathcal{R}, \quad s = x_2 - x_1 \in \mathcal{N}\left( \begin{pmatrix} B_k \\ a_k^T \end{pmatrix} \right). \]

**Proof.** Suppose first that \( \forall x_1, x_2 \in \mathcal{R}, \)

\[ s = x_2 - x_1 \in \mathcal{N}\left( \begin{pmatrix} B_k \\ a_k^T \end{pmatrix} \right). \]

Therefore

\[ f_k(x_2) = F_k(B_k x_1 + c_k) + \langle a_k, x_1 \rangle - d_k = f_k(x_1) = 0, \]

so that constraint \( k \) is an implicit equality. Now suppose that constraint \( k \) is an implicit equality. Therefore we have \( f_k(x_2) = f_k(x_1) = 0, \forall x_1, x_2 \in \mathcal{R}. \)

Since \( \mathcal{R} \) is convex, we also have that \( x_1 + \lambda s \in \mathcal{R}, \) for \( s = x_2 - x_1, \)

\( 0 \leq \lambda \leq 1. \) Thus for all \( 0 \leq \lambda \leq 1, \) we have \( f_k(x_2) = f_k(x_1) = f_k(x_1 + \lambda s) = 0. \)

Therefore

\[
\begin{align*}
  f_k(x_1 + \lambda s) &= F_k(B_k(x_1 + \lambda s) + c_k) + \langle a_k, x_1 + \lambda s \rangle - d_k \\
  &= F_k(B_k x_1 + \lambda B_k s + c_k) + \lambda \langle a_k, s \rangle + \langle a_k, x_1 \rangle - d_k = 0,
\end{align*}
\]

(3.10)

for every \( \lambda \in [0, 1]. \) For given vectors \( x_1 \) and \( s \) the expression on the left-hand side of the latter equation is a function of \( \lambda \) (\( \lambda \in [0, 1] \)). If \( B_k s \neq 0, \) then \( f_k(x_1 + \lambda s) \) is a strictly convex function of \( \lambda. \) Therefore, there are at most two different roots of the equation \( f_k(x_1 + \lambda s) = 0, \) which contradicts (3.10). So \( B_k s = 0. \) Now, let us suppose that \( \langle a_k, s \rangle \neq 0. \) Therefore the equation (3.10) is a linear function of \( \lambda \) that is satisfied for exactly one value of \( \lambda, \) which again contradicts the earlier conclusion that (3.10) is satisfied for every \( \lambda \in [0, 1]. \) Therefore \( \langle a_k, s \rangle = 0, \) which ends the proof of the Lemma 3.3. \]

**Lemma 3.4** [20]. *Let \( \hat{x} \in \mathcal{R} \) and suppose that constraint \( k \) is an implicit equality. Then*

\[\mathcal{R} = \mathcal{R}^k := \{ x \in \mathbb{R}^n \mid f_i(x) \leq 0, \ i \in I \setminus \{ k \}, \ A^k x = b^k \}.\]
where

\[ A^k = \begin{bmatrix} B_k \\ a_k^T \end{bmatrix}, \quad b^k = \begin{bmatrix} B_k \hat{x} \\ \langle a_k, \hat{x} \rangle \end{bmatrix}. \]

**Proof.** We will show that \( \mathcal{R} \subset \mathbb{R}^k \) and \( \mathbb{R}^k \subset \mathcal{R} \). Let \( x \in \mathcal{R} \) and \( s \) be such that \( x = \hat{x} + s \). Since constraint \( k \) is an implicit equality it follows from Lemma 3.3 that \( B_k s = 0 \) and \( \langle a_k, s \rangle = 0 \). Thus, \( A^k(\hat{x} + s) = A^k \hat{x} = b^k \), and \( x \in \mathbb{R}^k \) so that \( \mathcal{R} \subset \mathbb{R}^k \). Now let \( x = \hat{x} + s \in \mathbb{R}^k \) and note that \( A^k(\hat{x} + s) = b^k \) implies that \( A^k s = 0 \), i.e., that \( B_k s = 0 \) and \( \langle a_k, s \rangle = 0 \).

We have

\[
\begin{align*}
f_k(x) & = f_k(\hat{x} + s) = F_k(B_k(\hat{x} + s) + c_k) + \langle a_k, \hat{x} + s \rangle - d_k \\
& = F_k(B_k \hat{x} + c_k) + \langle a_k, \hat{x} \rangle - d_k = 0.
\end{align*}
\]

Thus, \( x \in \mathcal{R} \) and \( \mathbb{R}^k \subset \mathcal{R} \).

**Theorem 3.4.** *If the system (1.1) is inconsistent, then there exists an infeasible subset of constraints in (1.1) of cardinality not greater than \( n + 1 \), that is,

\[
\min |IIS| \leq n + 1.
\]

**Proof.** Suppose that the statement in the theorem is not true. This implies that every infeasible subset of constraints contains at least \( n + 2 \) constraints. Without losing generality, let us assume that the minimal infeasible subsystem of constraints contains exactly \( n + 2 \) constraints, with indices \( I_n = \{1, \ldots, n + 2\} \). Given that this subsystem is minimal and therefore irreducible, removing the constraint with index, e.g., 1 will cause the system of constraints with indices \( I_n \setminus \{1\} \) to be feasible. Let us consider the problem

\[
\begin{align*}
\min & \quad f_i(x) \\
\text{subject to:} & \quad x \in \mathcal{R}_i = \{ x \mid f_i(x) \leq 0, \in I_n \setminus \{1\} \}.\quad (3.11)
\end{align*}
\]

The problem (3.11) is bounded below because the set \( \mathcal{R}_i \cap \{ x \mid f_i(x) \leq 0 \} \) is empty. Let \( I(x^*) = \{ i \mid f_i(x^*) = 0 \} \), where \( x^* \) is an optimal solution to the problem (3.11). If \( |I(x^*)| \leq n \), then the result follows, since the system of constraints with indices \( I(x^*) \cap \{1\} \) is infeasible. So we can assume that \( |I(x^*)| = n + 1 \). We consider two possibilities:

(a) the region \( \mathcal{R}_i \) has a nonempty interior and

(b) \( \dim(\mathcal{R}_i) < n \).
When the first case takes place, we use the Kuhn–Tucker necessary conditions [10], which ensure that nonnegative numbers $u_i$ exist, such that not more than $n$ of them are strictly positive, which satisfy

$$-\nabla f_i(x^*) = \sum_{i=2}^{n+2} u_i \nabla f_i(x^*)$$

(3.12)

$$u_i f_i(x^*) = 0, \quad i = 2, \ldots, n + 2.$$  

Let $U = \{i \mid u_i > 0, i \in I_n \setminus \{1\}\}$. Since $f_i(x^*) \leq 0$, $i = 2, \ldots, n + 2$, then conditions (3.12), by the Kuhn–Tucker sufficiency theorem [10], imply that $x^*$ is also an optimal solution to the problem

$$\min \quad f_1(x)$$

subject to: $f_i(x) \leq 0, \quad i \in U.$

The latter fact, along with the inequality $f_i(x^*) > 0$, implies that the constraints with the indices in the set $U \cup \{1\}$ form an inconsistent system. Inequality $|U \cup \{1\}| \leq n + 1$ concludes the proof of the theorem for case (a).

Now consider case (b), that is, when $\dim(\mathcal{R}_1) < n$. Let us suppose that the set $J_1$ is a complete set of implicit equalities in the system of convex inequalities with indices $I_n \setminus \{1\}$. Lemma 3.4 implies that replacing all inequalities in the set $J_1$ by the corresponding linear equations will result in a system of inequalities that form a region having a nonempty relative interior (in the linear manifold spanned by that system of equations). Making a change of variables eliminates this system of linear equations and reduces the problem (3.11) to an equivalent problem in lower $N$-dimensional space ($N < n$), which satisfies condition (a). Moreover, the new system still satisfies the assumptions imposed on the system (1.1). So, by applying to the new system the part of the proof that was provided under assumption (a), we obtain that there exists an inconsistent subset of constraints of cardinality not greater than $N + 1$. It is clear that the constraints in the system (1.1), which correspond to this inconsistent subsystem, also form an inconsistent subsystem and its cardinality does not exceed $n$. This ends the proof of the theorem. ■

**Example 3.1.** We apply Algorithm A to the system

$$f_1(x) = \frac{1}{4} \langle x, B_1 x \rangle^2 + \langle a_1, x \rangle - d_1 \leq 0,$$

$$f_3(x) = \frac{1}{3} \langle x, B_3 x \rangle^{3/2} + \langle a_3, x \rangle - d_3 \leq 0,$$

$$f_i(x) = \langle a_i, x \rangle - d_i \leq 0, \quad i = 2, 4,$$
with \( m = 4, n = 4 \), \( B_2 = B_4 = 0 \).

\[
B_1 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}, \quad B_3 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}.
\]

\( a_1^T = (1, 1, 1, 2), \quad a_2^T = (-1, 1, 0, 0), \quad a_3^T = (1, 1, 1, 4), \quad a_4^T = (1, 1, 0, 0), \quad d_1 = -6, \quad d_2 = 8, \quad d_3 = d_4 = 15. \)

We start the algorithm with \( k = 1 \) and \( I_{\text{imp}}^0 = \{1, 2, 3, 4\} \). Consider the system (2.6) with \( I_{\text{imp}}^{k-1} = I_{\text{imp}}^0 \). We denote \( s = ((s)_1, (s)_2, (s)_3, (s)_4)^T \). From \( B_1s = B_3s = 0 \) we get \( (s)_3 = -(s)_1 - (s)_2 \). The latter equality allows us to observe that inequalities

\[
\langle a_2, s \rangle \leq 0, \quad \langle a_3, s \rangle \leq 0 \quad \text{and} \quad \langle a_4, s \rangle \leq 0
\]

are the only implicit equalities. The inequality \( a_1^Ts \leq 0 \) is not an implicit equality, which can be verified using the vector \( s_1^T = (-1, -1, 2, 0)^T \). So the algorithm identifies \( I_{\text{imp}}^1 = \{2, 3, 4\} \). Now consider the system (2.6) with \( k = 2 \). The set of the implicit equalities is given by \( I_{\text{imp}}^2 = \{2, 4\} \). The vector \( s = (-1, -1, 3, -1)^T \) indicates that \( \langle a_3, s \rangle \leq 0 \) is not an implicit equality. Finally, we consider (2.6) with \( k = 3 \) and obtain \( I_{\text{imp}}^3 = I_{\text{imp}}^2 \), which terminates the algorithm with \( K = \{2, 4\} \) as a set of killing constraints. Therefore, the feasibility of the original system is equivalent to the feasibility of the following reduced system

\[
\begin{align*}
-x_1 + x_2 & \leq 8 \\
x_1 - x_2 & \leq 15.
\end{align*}
\] (3.13)

Note that if either the value \( d_2 \) is changed from \( d_2 = 8 \) to \( d_2 < -15 \) or the value \( d_4 \) is decreased from \( d_4 = 15 \) to \( d_4 < -8 \), then the original system becomes infeasible. By Corollary 2.4, where \( W = (-1/\sqrt{2}, 1/\sqrt{2}) \), the system (3.13) is equivalent to the following double inequality

\( -15/\sqrt{2} \leq \xi \leq 8/\sqrt{2} \). Finally, the problem of checking the feasibility of the original system with four variables and four constraints, including two nonlinear convex constraints, has been replaced by a system of two linear inequality constraints in a single variable. We note that although the system considered in this example is feasible, any infeasible system obtained by the perturbation of the right-hand sides has an easily identified irreducible infeasible set and infeasibility sets. They are respectively equal: \( \text{IIS} = \{2, 4\}, \text{IN}_1 = \{2\}, \text{IN}_2 = \{4\}; \) all being subsets of \( K \).

We investigated the performance of Algorithm A on several other systems other than quadratic convex inequalities. The results obtained for these examples seem to be promising, particularly for a large sparse...
systems of convex inequalities, which suggests that the results obtained in this paper deserve more study from the computational point of view. Clearly, all results obtained in the paper are applicable in particular to quadratic convex or linear systems.

4. CONCLUSIONS

In this paper we presented a method for detecting constraints that do not affect the infeasibility of the region defined by faithfully convex inequality constraints. Detection of these constraints gives a new system with the reduced number of constraints and variables that is infeasible iff the original system is infeasible. We also provide some results on irreducible infeasible sets and infeasibility sets. We prove that infeasibility sets as well as irreducible infeasible sets are subsets of a set of killing constraints, and we show that any system of convex analytic inequality constraints contains an infeasible subsystem of cardinality not greater than \( n + 1 \), which generalizes a similar result proved by Chvatal in [8] for linear systems.

REFERENCES