Constant-sign solutions of a system of Volterra integral equations

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Abstract

We consider the following system of Volterra integral equations:

\[ u_i(t) = \int_0^t g_i(t,s) \left[ f_i(s, u_1(s), u_2(s), \ldots, u_n(s)) + h_i(s, u_1(s), u_2(s), \ldots, u_n(s)) \right] ds, \quad t \in [0, T], \ 1 \leq i \leq n \]

and some of its particular cases that arise from physical problems. Criteria are offered for the existence of one and more constant-sign solutions \( u = (u_1, u_2, \ldots, u_n) \) of the system in \( (C[0, T])^n \). We say \( u \) is of constant sign if for each \( 1 \leq i \leq n \), \( \theta_i u_i(t) \geq 0 \) for all \( t \in [0, T] \), where \( \theta_i \in \{1, -1\} \) is fixed. Examples are also included to illustrate the results obtained.

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1. Introduction

In this paper we shall consider the system of Volterra integral equations

\[ u_i(t) = \int_0^t g_i(t,s) \left[ f_i(s, u_1(s), u_2(s), \ldots, u_n(s)) + h_i(s, u_1(s), u_2(s), \ldots, u_n(s)) \right] ds, \quad t \in [0, T], \ 1 \leq i \leq n \]  

(1.1)

and some of its special cases. The nonlinearities \( h_i(t, u_1, u_2, \ldots, u_n) \) can be singular at \( t = 0 \) and \( u_j = 0, \ j \in \{1, 2, \ldots, n\} \).

Throughout, let \( u = (u_1, u_2, \ldots, u_n) \). We are interested in establishing the existence of one and more solutions \( u \) of the system (1.1) (and its particular cases) in \( (C[0, T])^n = C[0, T] \times C[0, T] \times \cdots \times C[0, T] \) \( (n \) times). Moreover, we are concerned with constant-sign solutions \( u \), by which we mean \( \theta_i u_i(t) \geq 0 \) for all \( t \in [0, T] \) and \( 1 \leq i \leq n \),

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where \( \theta_i \in \{1, -1\} \) is fixed. Note that positive solution is a special case of constant-sign solution when \( \theta_i = 1 \) for \( 1 \leq i \leq n \). Recently, Agarwal, O’Regan and Wong [1–4] have been interested in the existence of constant-sign solutions of systems of Fredholm integral equations. We shall tackle the existence of constant-sign solutions of (1.1) in Section 2.

The system (1.1) when \( h_i = 0, 1 \leq i \leq n \) reduces to

\[
\begin{align*}
\begin{cases}
\quad u_i(t) = \int_0^t g_i(t, s) f_i(s, u_1(s), u_2(s), \ldots, u_n(s))ds, 
\quad t \in [0, T], \quad 1 \leq i \leq n.
\end{cases}
\end{align*}
\]

This equation when \( n = 1 \) has received a lot of attention in the literature [5–11], since it arises in real-world problems. For example, astrophysical problems (e.g., the study of the density of stars) give rise to the Emden differential equation

\[ y'' - t^p y^q = 0, \quad t \in [0, T], \quad y(0) = y'(0) = 0 \quad \text{where} \quad p \geq 0 \quad \text{and} \quad 0 < q < 1 \]

which reduces to (1.2) \( |n| = 1 \) when \( g_1(t, s) = (t - s)^p \) and \( f_1(t, y) = y^q \). Other examples occur in nonlinear diffusion and percolation problems (see [6,7] and the references cited therein), and here we get (1.2) where \( g_i \) is a convolution kernel, i.e.,

\[
\begin{align*}
\begin{cases}
\quad u_i(t) = \int_0^t g_i(t - s) f_i(s, u_1(s), u_2(s), \ldots, u_n(s))ds, 
\quad t \in [0, T], \quad 1 \leq i \leq n.
\end{cases}
\end{align*}
\]

In particular, Bushell and Okrasiński [6] investigated a special case of the above system given by

\[
y(t) = \int_0^t (t - s)^{\gamma - 1} f(y(s))ds, \quad t \in [0, T]
\]

where \( \gamma > 1 \). To generalize their problem and also to illustrate the usefulness of the results obtained for (1.1), we shall consider the system

\[
\begin{align*}
\begin{cases}
\quad u_i(t) = \int_0^t (t - s)^{\gamma - 1} f_i(s, u_1(s), u_2(s), \ldots, u_n(s))ds, 
\quad t \in [0, T], \quad 1 \leq i \leq n
\end{cases}
\end{align*}
\]

where \( \gamma > 1 \). The systems (1.2) and (1.3) will be discussed in Section 3, the results presented are general variations of some of those found in [5–11].

On the other hand, when \( f_i = 0, 1 \leq i \leq n \), the system (1.1) reduces to

\[
\begin{align*}
\begin{cases}
\quad u_i(t) = \int_0^t g_i(t, s) h_i(s, u_1(s), u_2(s), \ldots, u_n(s))ds, 
\quad t \in [0, T], \quad 1 \leq i \leq n
\end{cases}
\end{align*}
\]

This is a system of singular Volterra equations as the nonlinearities \( h_1(t, u_1, u_2, \ldots, u_n) \) can be singular at \( t = 0 \) and \( u_j = 0, j \in \{1, 2, \ldots, n\} \). There are only a handful of papers on singular Volterra equations in the literature, the reader may refer to [4,12–14] and the references cited therein. Note that the technique employed in [4] is entirely different from the present work. We shall establish existence theorems for (1.4) in Section 4.

In Sections 3 and 4, examples are also presented to illustrate the usefulness of the results obtained.

2. Existence results for (1.1)

Our main tool is Krasnosel’skii’s fixed point theorem which we state as follows.

**Theorem A** ([15]). Let \( B = (B, \| \cdot \|) \) be a Banach space, and let \( C \subset B \) be a cone in \( B \). Assume \( \Omega_1, \Omega_2 \) are open subsets of \( B \) with \( 0 \in \Omega_1, \Omega_1 \subset \Omega_2 \), and let \( S : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C \) be a continuous and completely continuous operator such that, either

(a) \( \| Su \| \leq \| u \|, u \in C \cap \partial \Omega_1, \) and \( \| Su \| \geq \| u \|, u \in C \cap \partial \Omega_2, \) or

(b) \( \| Su \| \geq \| u \|, u \in C \cap \partial \Omega_1, \) and \( \| Su \| \leq \| u \|, u \in C \cap \partial \Omega_2. \)

Then, \( S \) has a fixed point in \( C \cap (\overline{\Omega_2} \setminus \Omega_1). \) \( \square \)
Let the Banach space $B = (C[0, T])^n$ be equipped with the norm
\[ \|u\| = \max_{1 \leq i \leq n} \sup_{t \in [0, T]} |u_i(t)| = \max_{1 \leq i \leq n} |u_i|_0 \]
where we let $|u_i|_0 = \sup_{t \in [0, T]} |u_i(t)|$, $1 \leq i \leq n$. Define a cone in $B$ as
\[ C_a = \{ u \in B \mid \text{for each } 1 \leq i \leq n, \theta_i u_i(t) \geq 0 \text{ for } t \in [0, T], \theta_i u_i(t) \text{ is nondecreasing on } [0, T], \text{ and } \theta_i u_i(t) \geq a(t)\|u\| \text{ for } t \in [0, T] \}. \tag{2.1} \]
Of course here we assume $0 \leq a(t) \leq 1$ for $t \in [0, T]$. Also, for $u \in C_a$, we have $|u_i|_0 = \theta_i u_i(T)$, $1 \leq i \leq n$ and $\|u\| = \max_{1 \leq i \leq n} |u_i|_0 = \max_{1 \leq i \leq n} \theta_i u_i(T)$.

To begin our discussion, let the operator $S: B \to B$ be defined by
\[ Su(t) = (S_1 u(t), S_2 u(t), \ldots, S_n u(t)), \quad t \in [0, T] \tag{2.2} \]
where
\[ S_i u(t) = \int_0^t g_i(t, s)[f_i(s, u(s)) + h_i(s, u(s))]ds, \quad t \in [0, T], \quad 1 \leq i \leq n. \tag{2.3} \]
Clearly, a fixed point of the operator $S$ is a solution of the system (1.1). Further, a fixed point of the operator $S$ in $C_a$ is a constant-sign solution of (1.1). Since a solution $u$ of (1.1) satisfies $u_i(0) = 0$, $1 \leq i \leq n$, from (2.1) we must have $a(0) = 0$ if we require $u$ to be in $C_a$. Thus, from now on we assume that $a(0) = 0$ and $0 < a(t) \leq 1$, $t \in (0, T]$. More conditions on $a(t)$ will be presented later.

In our first result that follows, we consider (1.1) where the nonlinearities $h_i(t, u_1, u_2, \ldots, u_n)$ can be singular at $t = 0$ and $u_j = 0$, $j \in \{1, 2, \ldots, n\}$.

**Theorem 2.1.** Let $1 \leq p < \infty$ be a constant and $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$ be fixed. Assume

(C1) for each $1 \leq i \leq n$,
\[ \int_0^t |g_i(t, s) - g_i(t', s)|^pds \to 0 \quad \text{as } t \to t' \]
where $t^* = \min\{t, t'\}$;

(C2) for each $1 \leq i \leq n$ and any $t, t' \in [0, T]$,
\[ g_i(t, s) - g_i(t', s) \geq 0, \quad a.e. s \in [0, t], \quad g_i(t, s) > 0, \quad t \in (0, T], \quad a.e. s \in [0, t], \]
\[ g_i'(s) \equiv g_i(t, s) \in L^p[0, t] \text{ for each } t \in [0, T], \quad \sup_{t \in [0, T]} \int_0^t |g_i'(s)|^pds < \infty; \]

(C3) for each $1 \leq i \leq n$ and any $t_1, t_2$ satisfying $0 < t_1 \leq t_2 \leq T$,
\[ g_i(t_2, s) - g_i(t_1, s) \geq 0, \quad a.e. s \in [0, t_1]; \]

(C4) for each $1 \leq i \leq n$, $\theta_i f_i : [0, T] \times \prod_{j=1}^n [0, \infty)_j \to [0, \infty)_j$ and $\theta_i h_i : (0, T] \times \prod_{j=1}^n [0, \infty)_j \to [0, \infty)$ are continuous, where
\[ [0, \infty)_j = \begin{cases} [0, \infty), & \theta_j = 1 \\ (-\infty, 0], & \theta_j = -1 \end{cases} \]
and $(0, \infty)_j$ is similarly defined; also
\[ \theta_i f_i(t, u) + \theta_i h_i(t, u) > 0, \quad (t, u) \in (0, T] \times \prod_{j=1}^n (0, \infty)_j; \]
moreover, $\theta_i f_i$ is 'nondecreasing' and $\theta_i h_i$ is 'nonincreasing' in the sense that if $c \leq \theta_j u_j \leq d$ for some $j \in \{1, 2, \ldots, n\}$, then
\[ \theta_i f_i(t, u_1, \ldots, u_j, \ldots, u_n) \leq \theta_i f_i(t, u_1, \ldots, \theta_j d, \ldots, u_n), \quad t \in [0, T] \]
and
\[\theta_i h_i(t, u_1, \ldots, u_j, \ldots, u_n) \leq \theta_i h_i(t, u_1, \ldots, \theta_j c, \ldots, u_n), \quad t \in (0, T);\]

(C5) there exists a function \( a \in C[0, T] \) with \( a(0) = 0 \) and \( 0 < a(t) \leq 1, \quad t \in (0, T) \) such that the following holds for each \( 1 \leq i \leq n \) and any \( R > 0 \),
\[
\int_0^T g_i(t, s)[\theta_i f_i(s, \theta_1 R a(s), \theta_2 R a(s), \ldots, \theta_n R a(s)) + \theta_i h_i(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)]ds
\geq a(t) \cdot \max_{1 \leq j \leq n} \int_0^T g_j(T, s)[\theta_j f_j(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)
+ \theta_j h_j(s, \theta_1 R a(s), \theta_2 R a(s), \ldots, \theta_n R a(s))]ds, \quad t \in [0, T];
\]

(C6) for each \( 1 \leq i \leq n \) and any \( R > 0 \), if \( p > 1 \) then
\[
\int_0^T |f_i(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)|^p ds < \infty
\]
and
\[
\int_0^T |h_i(s, \theta_1 R a(s), \theta_2 R a(s), \ldots, \theta_n R a(s))|^p ds < \infty;
\]
if \( p = 1 \), then
\[
\text{ess sup}_{t \in [0, T]} |f_i(t, \theta_1 R, \theta_2 R, \ldots, \theta_n R)| < \infty
\]
and
\[
\text{ess sup}_{t \in [0, T]} |h_i(t, \theta_1 R a(t), \theta_2 R a(t), \ldots, \theta_n R a(t))| < \infty;
\]

(C7) there exists \( \alpha > 0 \) such that for each \( 1 \leq i \leq n \),
\[
\int_0^T g_i(T, s)[\theta_i f_i(s, \theta_1 \alpha a(s), \theta_2 \alpha a(s), \ldots, \theta_n \alpha a(s))]ds \leq \alpha;
\]

(C8) there exists \( \beta (\neq \alpha) > 0 \) such that for each \( 1 \leq i \leq n \),
\[
\int_0^T g_i(T, s)[\theta_i f_i(s, \theta_1 \beta a(s), \theta_2 \beta a(s), \ldots, \theta_n \beta a(s)) + \theta_i h_i(s, \theta_1 \beta, \theta_2 \beta, \ldots, \theta_n \beta)]ds \geq \beta.
\]

Then, the system (1.1) has at least one constant-sign solution \( u \in (C[0, T])^n \) such that
\[(a) \ \alpha \leq \|u\| \leq \beta \text{ and } \theta_i u_i(t) \geq a(t)\alpha, \quad t \in [0, T], \quad 1 \leq i \leq n \text{ if } \alpha < \beta;
(b) \ \beta \leq \|u\| \leq \alpha \text{ and } \theta_i u_i(t) \geq a(t)\beta, \quad t \in [0, T], \quad 1 \leq i \leq n \text{ if } \beta < \alpha.
\]

Proof. We shall employ Theorem A. Without any loss of generality, let \( \beta < \alpha \). Define
\[
\Omega_\alpha = \{u \in B \mid \|u\| < \alpha\} \quad \text{and} \quad \Omega_\beta = \{u \in B \mid \|u\| < \beta\}.
\]

We shall show that the operator \( S : C_a \cap (\overline{\Omega_\alpha} \setminus \Omega_\beta) \to C_a \) is continuous and completely continuous, where \( C_a \) is defined in (2.1) and \( a \) is as in (C5). First, we shall prove that
\[
\text{for } t \in (0, T) \text{ and any } R > 0, \quad 0 \leq \theta_i h_i(t, u(t)) \leq \theta_i h_i(t, u(t)) \leq \theta_i h_i(t, u(t)) \leq \theta_i h_i(t, \theta_1 \alpha a(t), \theta_2 \alpha a(t), \ldots, \theta_n \alpha a(t)) + \theta_i h_i(t, \theta_1 \beta a(t), \theta_2 \beta a(t), \ldots, \theta_n \beta a(t)).
\]
Using (2.5) we find
\[
\int_0^T |f_i(s, u(s)) + h_i(s, u(s))|^q \, ds \\
\leq \int_0^T \left[ \theta_i f_i(s, \theta_1 \alpha, \theta_2 \alpha, \ldots, \theta_n \alpha) + \theta_i h_i(s, \theta_1 \beta \alpha(s), \theta_2 \beta \alpha(s), \ldots, \theta_n \beta \alpha(s)) \right]^q \, ds \\
\leq 2^{q-1} \int_0^T \left[ |f_i(s, \theta_1 \alpha, \theta_2 \alpha, \ldots, \theta_n \alpha)|^q + |h_i(s, \theta_1 \beta \alpha(s), \theta_2 \beta \alpha(s), \ldots, \theta_n \beta \alpha(s))|^q \right] \, ds \\
< \infty 
\] (2.6)
where we have also applied (C6) in the last inequality. Now, for \( t, t' \in [0, T], t' < t \), we employ Hölder’s inequality and (2.5) to obtain for each \( 1 \leq i \leq n \),
\[
|S_i u(t) - S_i u(t')| \\
\leq \int_0^{t'} |g_i(t, s) - g_i(t', s)| \cdot |f_i(s, u(s)) + h_i(s, u(s))| \, ds + \int_{t'}^t |g_i(t, s)| \cdot |f_i(s, u(s)) + h_i(s, u(s))| \, ds \\
\leq \left[ \left( \int_0^{t'} |g_i(t, s) - g_i(t', s)|^p \, ds \right)^{\frac{1}{p}} + \left( \int_{t'}^t |g_i(t, s)|^p \, ds \right)^{\frac{1}{p}} \right] \left( \int_0^T |f_i(s, u(s)) + h_i(s, u(s))|^q \, ds \right)^{\frac{1}{q}}. 
\]
Then, in view of (2.6), (C1) and (C2), it follows that
\[
|S_i u(t) - S_i u(t')| \to 0 \quad \text{as} \quad t \to t', \quad 1 \leq i \leq n. 
\] (2.7)
This proves (2.4).

Next, we shall check that
\[
S : C_a \cap (\overline{C_a} \setminus \Omega_\beta) \to C_a \quad \text{is well defined.} \tag{2.8}
\]
Once again let \( u \in C_a \cap (\overline{C_a} \setminus \Omega_\beta) \). Noting (C1) and (2.5), we obtain for each \( 1 \leq i \leq n \),
\[
\theta_i(S_i u)(t) = \int_0^t g_i(t, s)[\theta_i f_i(s, u(s)) + \theta_i h_i(s, u(s))] \, ds \geq 0, \quad t \in [0, T]. \tag{2.9}
\]
Now, let \( t_1, t_2 \in [0, T] \) with \( t_1 < t_2 \). Then, for each \( 1 \leq i \leq n \),
\[
\theta_i(S_i u)(t_2) - \theta_i(S_i u)(t_1) = \int_0^{t_1} [g_i(t_2, s) - g_i(t_1, s)] \cdot [\theta_i f_i(s, u(s)) + \theta_i h_i(s, u(s))] \, ds \\
+ \int_{t_1}^{t_2} g_i(t_2, s)[\theta_i f_i(s, u(s)) + \theta_i h_i(s, u(s))] \, ds \\
\geq 0 \tag{2.10}
\]
where we have used (C3), (C1) and (2.5) in the last inequality. Hence, \( \theta_i(S_i u) \) is nondecreasing on \([0, T]\). What remains is to show that \( \theta_i(S_i u)(t) \geq a(t) \|u\| \), \( t \in [0, T] \) for each \( 1 \leq i \leq n \). Since \( u \in C_a \cap (\overline{C_a} \setminus \Omega_\beta) \), there exists \( R \in [\beta, \alpha] \) such that
\[
\|u\| = R \quad \text{and} \quad 0 < a(t) R \leq \theta_i u_i(t) \leq R, \quad t \in (0, T], \quad 1 \leq i \leq n. \tag{2.11}
\]
From (C1) and (C4), it is clear that for \( t \in [0, T] \) and \( 1 \leq i \leq n \),
\[
\theta_i(S_i u)(t) \geq \int_0^t g_i(t, s)[\theta_1 f_i(s, \theta_1 R a(s), \theta_2 R a(s), \ldots, \theta_n R a(s)) \cdot \theta_i h_i(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)] \, ds. \tag{2.12}
\]
Again, using (C1) and (C4), we get for each \( 1 \leq j \leq n \),
\[
\theta_j(S_j u)(T) \leq \int_0^T g_j(T, s)[\theta_j f_j(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R) + \theta_j h_j(s, \theta_1 R a(s), \theta_2 R a(s), \ldots, \theta_n R a(s))] \, ds. \tag{2.13}
\]
Since \( \theta_j(S_j u) \) is nondecreasing, using (2.13) we obtain
\[
\|Su\| = \max_{1 \leq j \leq n} |S_j u|_0 = \max_{1 \leq j \leq n} \theta_j(S_j u)(T)
\leq \max_{1 \leq j \leq n} \int_0^T g_j(T, s)[\theta_j f_j(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R) + \theta_j h_j(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s))]ds \equiv A.
\] (2.14)
Applying (2.14) in (2.12), we get for \( t \in [0, T] \) and \( 1 \leq i \leq n \),
\[
\theta_i(S_i u)(t) \geq \int_0^t g_i(t, s)[\theta_i f_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s)) + \theta_i h_i(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)]ds \cdot \frac{\|Su\|}{A}
\]
where we have used (C5) in the last inequality. This completes the proof of (2.8).

Next, we shall show that
\[
S : C_a \cap (\overline{\Omega_a} \setminus \Omega_\beta) \to C_a \text{ is compact.} \tag{2.15}
\]
Once again let \( u \in C_a \cap (\overline{\Omega_a} \setminus \Omega_\beta) \). Using the monotonicity of \( \theta_i(S_i u), \) (2.13), Hölder’s inequality, (C1) and (C6), we obtain for \( t \in [0, T] \) and \( 1 \leq i \leq n \),
\[
\theta_i(S_i u)(t) \leq \theta_i(S_i u)(T)
\leq \int_0^T g_i(T, s)[\theta_i f_i(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R) + \theta_i h_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s))]ds
\leq \|g_i^T\|_p \left( \int_0^T [\theta_i f_i(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R) + \theta_i h_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s))]^q ds \right)^{\frac{1}{q}}
\leq 2^{-q-1} \|g_i^T\|_p \left( \int_0^T [\theta_i f_i(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)]^q + [\theta_i h_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s))]^q ds \right)^{\frac{1}{q}}
\equiv A_0 < \infty.
\]
Thus, \( S_i(C_a \cap (\overline{\Omega_a} \setminus \Omega_\beta)) \) is uniformly bounded. Moreover, (2.7) guarantees the continuity of \( S_i u \). Hence, the compactness of \( S : C_a \cap (\overline{\Omega_a} \setminus \Omega_\beta) \to C_a \) follows from the Arzéla–Ascoli theorem. Having established (2.4), (2.8) and (2.15), we have shown that \( S : C_a \cap (\overline{\Omega_a} \setminus \Omega_\beta) \to C_a \) is continuous and completely continuous.

We claim that (i) \( \|Su\| \leq \|u\| \) for \( u \in C_a \cap \partial \Omega_a \), and (ii) \( \|Su\| \geq \|u\| \) for \( u \in C_a \cap \partial \Omega_\beta \). To verify (i), let \( u \in C_a \cap \partial \Omega_\beta \). Then,
\[
\|u\| = \alpha \quad \text{and} \quad 0 < a(t)\alpha \leq \theta_i u_i(t) \leq \alpha, \quad t \in (0, T], \quad 1 \leq i \leq n.
\]
Applying (C1) and (C4), we obtain (2.13) \( |_{R=\alpha} \) and hence noting (C7) we get
\[
\|Su\| = \max_{1 \leq j \leq n} |S_j u|_0 = \max_{1 \leq j \leq n} \theta_j(S_j u)(T)
\leq \max_{1 \leq j \leq n} \int_0^T g_j(T, s)[\theta_j f_j(s, \theta_1 \alpha, \theta_2 \alpha, \ldots, \theta_n \alpha) + \theta_j h_j(s, \theta_1 \alpha \alpha(s), \theta_2 \alpha \alpha(s), \ldots, \theta_n \alpha \alpha(s))]ds
\leq \alpha = \|u\|.
\]
Next, to prove (ii), let \( u \in C_a \cap \partial \Omega_\beta \). So
\[
\|u\| = \beta \quad \text{and} \quad 0 < a(t)\beta \leq \theta_i u_i(t) \leq \beta, \quad t \in (0, T], \quad 1 \leq i \leq n.
\]
Now \( \|Su\| = \theta_k(S_k u)(T) \) for some \( k \in \{1, 2, \ldots, n\} \). Thus, using (2.12) \( |_{R=\beta} \) and (C8) we find
\[
\|Su\| = \theta_k(S_k u)(T)
\geq \int_0^T g_k(T, s)[\theta_k f_k(s, \theta_1 \beta \alpha(s), \theta_2 \beta \alpha(s), \ldots, \theta_n \beta \alpha(s)) + \theta_k h_k(s, \theta_1 \beta, \theta_2 \beta, \ldots, \theta_n \beta)]ds
\geq \beta = \|u\|.
\]
Having obtained (i) and (ii), it follows from Theorem A that $S$ has a fixed point $u \in C_a \cap (\overline{\Omega_a} \setminus \Omega_\beta)$. Therefore, conclusion (b) follows immediately. □

**Remark 2.1.** If (C2) is changed to
(C2)' for each $1 \leq i \leq n$ and any $t$, $t' \in [0, T]$,
\[
\int_0^T |g_i(t, s) - g_i(t', s)|^p \, ds + \int_{t'}^{t^*} |g_i(t^{**}, s)|^p \, ds \to 0 \quad \text{as } t \to t',
\]
where $t^* = \min\{t, t'\}$ and $t^{**} = \max\{t, t'\}$,
then automatically we have $\sup_{t \in [0, T]} \int_0^T |g_i(t)|^p \, ds < \infty$ which appears in (C1).

**Remark 2.2.** In (C7) if we have strict inequality instead, i.e.,
\[
\int_0^T g_i(T, s)[\theta_i f_i(s, \theta_1 \alpha, \theta_2 \alpha, \ldots, \theta_n \alpha) + \theta_i h_i(s, \theta_1 \alpha a(s), \theta_2 \alpha a(s), \ldots, \theta_n \alpha a(s))] \, ds < \alpha,
\]
then from the latter part of the proof of Theorem 2.1 we see that a fixed point $u$ of $S$ must satisfy $\|u\| \neq \alpha$. Similarly, if the inequality in (C8) is strict, i.e.,
\[
\int_0^T g_i(T, s)[\theta_i f_i(s, \theta_1 \beta a(s), \theta_2 \beta a(s), \ldots, \theta_n \beta a(s)) + \theta_i h_i(s, \theta_1 \beta, \theta_2 \beta, \ldots, \theta_n \beta)] \, ds > \beta,
\]
then a fixed point $u$ of $S$ must fulfill $\|u\| \neq \beta$. Hence, with strict inequalities in (C7) and (C8), the conclusion of Theorem 2.1 becomes: the system (1.1) has at least one constant-sign solution $u \in (C[0, T])^n$ such that

(a) $\alpha < \|u\| < \beta$ and $\theta_i u_i(t) > \alpha(t) \alpha$, $t \in (0, T]$, $1 \leq i \leq n$ if $\alpha < \beta$;
(b) $\beta < \|u\| < \alpha$ and $\theta_i u_i(t) > \alpha(t) \beta$, $t \in (0, T]$, $1 \leq i \leq n$ if $\beta < \alpha$.

The next result generalizes Theorem 2.1 and gives the existence of multiple constant-sign solutions of (1.1).

**Theorem 2.2.** Let $1 \leq p < \infty$ be a constant and $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$ be fixed. Assume (C1)–(C6) hold. Let (C7) be satisfied for $\alpha = \alpha_\ell$, $\ell = 1, 2, \ldots, k$, and (C8) be satisfied for $\beta = \beta_\ell$, $\ell = 1, 2, \ldots, m$.

(a) If $m = k + 1$ and $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k < \beta_{k+1}$, then (1.1) has (at least) $2k$ constant-sign solutions $u^1, \ldots, u^{2k-1} \in (C[0, T])^n$ such that
\[
\beta_1 \leq \|u^1\| \leq \alpha_1 \leq \|u^2\| \leq \beta_2 \leq \cdots \leq \alpha_k \leq \|u^{2k}\| \leq \beta_{k+1}.
\]
(b) If $m = k$ and $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k$, then (1.1) has (at least) $2k - 1$ constant-sign solutions $u^1, \ldots, u^{2k-1} \in (C[0, T])^n$ such that
\[
\beta_1 \leq \|u^1\| \leq \alpha_1 \leq \|u^2\| \leq \beta_2 \leq \cdots \leq \beta_k \leq \|u^{2k-1}\| \leq \alpha_k.
\]

**Proof.** In (a) and (b), we just apply Theorem 2.1 repeatedly. □

**Remark 2.3.** Suppose in Theorem 2.2 we have some strict inequalities in (C7) and (C8), say, involving $\alpha_i$ and $\beta_j$ for some $i \in \{1, 2, \ldots, k\}$ and some $j \in \{1, 2, \ldots, m\}$. Then, noting Remark 2.2, those inequalities in the conclusion involving $\alpha_i$ and $\beta_j$ will also be strict.

### 3. Existence results for (1.2) and (1.3)

Using the results obtained in Section 2, we are now ready to discuss systems (1.2) and (1.3) that arise from real-world applications. In particular, we shall obtain more specific conditions concerning the existence of $a(t)$ in (C5). Our first result is for the system (1.2).

**Theorem 3.1.** Let $1 \leq p < \infty$ be a constant and $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\theta_i \in \{1, -1\}$, $1 \leq i \leq n$ be fixed. Suppose (C1)–(C3) and (C4)$_{h_i=0}$ hold. Further, assume
(C9) there exists \( N > 0 \) such that for each \( 1 \leq i \leq n \),
\[
g_i(t, s) \geq N > 0, \quad t \in (0, T], \text{ a.e. } s \in [0, r];
\]
(C10) for each \( 1 \leq i \leq n \) and any \((t, u) \in [0, T] \times \prod_{j=1}^n [0, \infty)\),
\[
r(t)w_1(|u_1|)w_2(|u_2|) \cdots w_n(|u_n|) \leq \theta_i f_i(t, u)
\leq \rho(t)w_1(|u_1|)w_2(|u_2|) \cdots w_n(|u_n|)
\]
where \( \rho, r : [0, T] \to [0, \infty) \), \( r(t) > 0 \) for a.e. \( t \in [0, T] \), \( r \) is continuous, and for \( 1 \leq j \leq n \), \( w_j : [0, \infty) \to [0, \infty) \) is continuous, \( w_j(c) > 0 \) for \( c > 0 \), \( w_j(cd) \geq w_j(c)w_j(d) \) for \( c, d > 0 \);

(C11) the function \( J : [0, \infty) \to [0, \infty) \) defined by
\[
J(y) = \int_0^y \frac{dx}{w_1(x)w_2(x) \cdots w_n(x)}
\]
satisfies
\[
J^{-1}\left( \frac{N}{Q} \int_0^t r(s)ds \right) \leq 1, \quad t \in [0, T]
\]
where \( Q = \max_{1 \leq j \leq n} \int_0^T g_j(T, s) \rho(s)ds \).

Let \( a(t) = J^{-1}\left( \frac{N}{Q} \int_0^t r(s)ds \right) \), and let \((C6)|_{h_i=0}, (C7)|_{h_i=0} \) and \((C8)|_{h_i=0} \) hold. Then, the system (1.2) has at least one constant-sign solution \( u \in (C[0, T])^n \) such that
(a) \( \alpha \leq \|u\| \leq \beta \) and \( \theta_i u_i(t) \geq a(t)\alpha, t \in [0, T], 1 \leq i \leq n \) if \( \alpha < \beta \);
(b) \( \beta \leq \|u\| \leq \alpha \) and \( \theta_i u_i(t) \geq a(t)\beta, t \in [0, T], 1 \leq i \leq n \) if \( \beta < \alpha \).

**Proof.** Clearly, Theorem 2.1 is applicable if we can show that (C5) is satisfied. To begin, notice the inequality in (C5) reduces to

\[
\int_0^t g_i(t, s)\theta_i f_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s))ds \\
\geq a(t) \cdot \max_{1 \leq j \leq n} \int_0^T g_j(T, s)\theta_j f_j(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)ds, \quad t \in [0, T], \ 1 \leq i \leq n
\]
or

\[
\frac{\int_0^t g_i(t, s)\theta_i f_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s))ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T, s)\theta_j f_j(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)ds} \geq a(t), \quad t \in [0, T], \ 1 \leq i \leq n.
\] (3.1)

Using (C10) and (C9), it is clear that

\[
\frac{\int_0^t g_i(t, s)\theta_i f_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s))ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T, s)\theta_j f_j(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)ds} \\
\geq \frac{\int_0^t g_i(t, s)r(s)w_1(Ra(s))w_2(Ra(s)) \cdots w_n(Ra(s))ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T, s)\rho(s)w_1(R)w_2(R) \cdots w_n(R)ds} \\
\geq \frac{\int_0^t g_i(t, s)r(s)w_1(a(s))w_2(a(s)) \cdots w_n(a(s))ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T, s)\rho(s)ds} \\
\geq \frac{N}{Q} \int_0^t r(s)w_1(a(s))w_2(a(s)) \cdots w_n(a(s))ds.
\]

Now, (3.1) is satisfied if we can find some \( a \in C[0, T] \) with \( a(0) = 0, 0 < a(t) \leq 1, t \in (0, T] \), and such that

\[
a(t) = \frac{N}{Q} \int_0^t r(s)w_1(a(s))w_2(a(s)) \cdots w_n(a(s))ds.
\] (3.2)
We claim that (3.2) is satisfied if
\[ a(t) = J^{-1}\left(\frac{N}{Q} \int_0^t r(s)ds\right). \]  
(3.3)

In fact, from (3.3) we have \( J(a(t)) = \frac{N}{Q} \int_0^t r(s)ds \), or
\[ \int_0^{a(t)} \frac{dx}{w_1(x)w_2(x) \cdots w_n(x)} = \frac{N}{Q} \int_0^t r(s)ds. \]

Next, the above equation is the same as
\[ \int_0^t \frac{a'(s)ds}{w_1(a(s))w_2(a(s)) \cdots w_n(a(s))} = \frac{N}{Q} \int_0^t r(s)ds \]
which upon differentiation gives
\[ a'(t) = \frac{N}{Q} r(t)w_1(a(t))w_2(a(t)) \cdots w_n(a(t)). \]

Integrating the above from 0 to \( t \) then yields (3.2). Thus, (3.2) is satisfied if \( a(t) \) is defined by (3.3), moreover this \( a \in C[0, T] \) fulfills \( a(0) = 0 \) and \( 0 < a(t) \leq 1, t \in (0, T) \) (see (C11)).

We have shown that the condition (C5) is satisfied and so Theorem 2.1 is applicable to the system (1.2). \( \square \)

By using Theorem 3.1 repeatedly, we obtain the existence of multiple constant-sign solutions of (1.2).

**Theorem 3.2.** Let \( 1 \leq p < \infty \) be a constant and \( q \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( \theta_i \in \{1, -1\}, 1 \leq i \leq n \) be fixed. Assume (C1)–(C3), (C4)|\( h_i=0 \), (C9)–(C11) and (C6)|\( h_i=0 \) hold. Let \( a(t) = J^{-1}\left(\frac{N}{Q} \int_0^t r(s)ds\right) \). Let (C7)|\( h_i=0 \) be satisfied for \( \alpha = \alpha_{\ell}, \ell = 1, 2, \ldots, k, \) and (C8)|\( h_i=0 \) be satisfied for \( \beta = \beta_{\ell}, \ell = 1, 2, \ldots, m. \)

(a) If \( m = k + 1 \) and \( 0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k < \beta_{k+1} \), then (1.2) has (at least) \( 2k \) constant-sign solutions \( u^1, \ldots, u^{2k} \in \{C[0, T]\}^n \) such that
\[ \beta_1 \leq \|u^1\| \leq \alpha_1 \leq \|u^2\| \leq \beta_2 \leq \cdots \leq \alpha_k \leq \|u^{2k}\| \leq \beta_{k+1}. \]

(b) If \( m = k \) and \( 0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k \), then (1.2) has (at least) \( 2k - 1 \) constant-sign solutions \( u^1, \ldots, u^{2k-1} \in \{C[0, T]\}^n \) such that
\[ \beta_1 \leq \|u^1\| \leq \alpha_1 \leq \|u^2\| \leq \beta_2 \leq \cdots \leq \beta_k \leq \|u^{2k-1}\| \leq \alpha_k. \] \( \square \)

**Remark 3.1.** Remarks similar to those of Remarks 2.1–2.3 also hold for Theorems 3.1 and 3.2.

The next example illustrates an application of Theorem 3.1.

**Example 3.1.** Consider the system (1.2) where
\[ f_i(t, u) = (u_1u_2 \cdots u_n)^{k_i}, \quad 1 \leq i \leq n. \]  
(3.4)

Here \( 0 \leq k_i < 1, 1 \leq i \leq n \) is fixed and \( k^* = k_1 + k_2 + \cdots + k_n < 1 \). Assume that \( g_i, 1 \leq i \leq n \) satisfies (C1)–(C3) and (C9).

Let \( \theta_i = 1, 1 \leq i \leq n \). Clearly, conditions (C4)|\( h_i=0 \) and (C6)|\( h_i=0 \) are satisfied. Next, in (C10) we can pick
\[ \rho = r = 1 \quad \text{and} \quad w_i(x) = x^{k_i}, \quad 1 \leq i \leq n. \]  
(3.5)

Thus, in (C11) the function \( J : [0, \infty) \to [0, \infty) \) reduces to
\[ J(y) = \int_0^y \frac{dx}{w_1(x)w_2(x) \cdots w_n(x)} = \int_0^y \frac{dx}{x^{k^*}} = \frac{y^{1-k^*}}{1-k^*}. \]

It follows that
\[ J^{-1}(z) = [(1 - k^*)z]^{\frac{1}{1-k^*}}. \]
Now, for \( t \in [0, T] \),
\[
a(t) = J^{-1} \left( \frac{N}{Q} \int_0^t r(s) \, ds \right) = J^{-1} \left( \frac{N}{Q} t \right) = \left[ (1 - k^*) \frac{N}{Q} t \right]^{\frac{1}{1 - k^*}} \\
\leq (1 - k^*)^{\frac{1}{1 - k^*}} \left[ \frac{N}{Q} T \right]^{\frac{1}{1 - k^*}} \leq 1
\]
where we have used the fact that \( Q = \max_{1 \leq j \leq n} \int_0^T g_j(T, s) \, ds \geq NT \) in the last inequality. Hence, (C11) is fulfilled.

If in addition \((C7)|_{h_i=0}\) and \((C8)|_{h_i=0}\) are satisfied, then by Theorem 3.1 the system \((1.2)\) with \((3.4)\) has at least one positive solution in \((C[0, T])^n\). \( \square \)

Next, we shall consider the system \((1.3)\) where \( \gamma > 1 \). With \( g_i(t, s) = (t - s)^{-\gamma}, 1 \leq i \leq n, \) it is clear that \((C1) - (C3)\) hold with \( p = 1 \).

**Theorem 3.3.** Let \( \theta_i \in \{1, -1\}, 1 \leq i \leq n \) be fixed. Let \( \delta > 1 \) be such that \( \frac{1}{\gamma} + \frac{1}{\delta} = 1 \). Suppose \((C4)|_{h_i=0}\) and \((C10)\) hold. Further, assume

\((C12)\) the function \( K : [0, \infty) \to [0, \infty) \) defined by
\[
K(y) = \int_0^y \left[ \frac{x}{w_1(x)w_2(x) \ldots w_n(x)} \right]^{\frac{1}{\delta}} - 1 \, dx
\]
satisfies
\[
K^{-1} \left( c \int_0^t r(s) \, ds \right) \leq 1, \quad t \in [0, T]
\]
where
\[
c = \delta Q_0^{\frac{1}{\gamma}} \left[ \sup_{t \in [0, T]} \int_0^t (t - s)^{\frac{1}{\gamma}} r(s) \, ds \right]^{\frac{1}{\gamma}} \quad \text{and} \quad Q_0 = \int_0^T (T - s)^{-\gamma - 1} \rho(s) \, ds.
\]
Let \( a(t) = K^{-1} \left( c \int_0^t r(s) \, ds \right), \) and let \((C6)|_{(h_i=0, p=1)}\),

\((C13)\) there exists \( \alpha > 0 \) such that for each \( 1 \leq i \leq n \),
\[
\int_0^T (T - s)^{-\gamma - 1} \theta_i f_i(s, \theta_1 \alpha, \theta_2 \alpha, \ldots, \theta_n \alpha) \, ds \leq \alpha
\]
and

\((C14)\) there exists \( \beta(\neq \alpha) > 0 \) such that for each \( 1 \leq i \leq n \),
\[
\int_0^T (T - s)^{-\gamma - 1} \theta_i f_i(s, \theta_1 \alpha(s), \theta_2 \alpha(s), \ldots, \theta_n \alpha(s)) \, ds \geq \beta
\]
be satisfied. Then, the system \((1.3)\) has at least one constant-sign solution \( u \in (C[0, T])^n \) such that

(a) \( \alpha \leq \|u\| \leq \beta \) and \( \theta_i u_i(t) \geq a(t) \alpha, t \in [0, T], 1 \leq i \leq n \) if \( \alpha < \beta \);

(b) \( \beta \leq \|u\| \leq \alpha \) and \( \theta_i u_i(t) \geq a(t) \beta, t \in [0, T], 1 \leq i \leq n \) if \( \beta < \alpha \).

**Proof.** Once again, Theorem 2.1 is applicable if we can prove that \((C5)\) is satisfied. To begin, the inequality in \((C5)\) reduces to \((3.1)\), or equivalently
\[
\frac{\int_0^T (T - s)^{-\gamma - 1} \theta_j f_j(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s)) \, ds}{\max_{1 \leq j \leq n} \int_0^T (T - s)^{-\gamma - 1} \theta_j f_j(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R) \, ds} \geq a(t), \quad t \in [0, T], 1 \leq i \leq n.
\] (3.6)
Using (C10), we find
\[
\left\{ \begin{array}{l}
\int_0^t (t-s)^{\gamma-1} \theta_i f_j(s, \theta_i Ra(s)) \, ds \\
\max_{1 \leq j \leq n} \int_0^T (T-s)^{\gamma-1} \theta_j f_j(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R) \, ds \\
\geq \frac{\int_0^t (t-s)^{\gamma-1} r(s) w_1(a(s)) w_2(a(s)) \cdots w_n(a(s)) \, ds}{\int_0^T (T-s)^{\gamma-1} \rho(s) \, ds} \\
= \frac{1}{Q_0} \int_0^t (t-s)^{\gamma-1} r(s) w_1(a(s)) w_2(a(s)) \cdots w_n(a(s)) \, ds.
\end{array} \right.
\]
Hence, (3.6) is satisfied if we can find some \( a \in C[0, T] \) with \( a(0) = 0, 0 < a(t) \leq 1, t \in (0, T] \), and such that
\[
\frac{1}{Q_0} \int_0^t (t-s)^{\gamma-1} r(s) w_1(a(s)) w_2(a(s)) \cdots w_n(a(s)) \, ds \geq a(t).
\]
(3.7)
Let
\[
a(t) = K^{-1} \left( c \int_0^t r(s) \, ds \right).
\]
(3.8)
We shall show that \( a(t) \) defined by (3.8) is the unique solution of the initial value problem
\[
\left\{ \begin{array}{l}
a'(t) = c \cdot r(t)[a(t)]^{1-\frac{j}{n}} [w_1(a(t)) w_2(a(t)) \cdots w_n(a(t))]^{\frac{j}{n}} \\
a(0) = 0.
\end{array} \right.
\]
(3.9)
and this \( a(t) \) satisfies (3.7).
First, we integrate (3.9) from 0 to \( t \) to get
\[
\int_0^t a'(s) \, ds = \int_0^t c \cdot r(s)[a(s)]^{1-\frac{j}{n}} [w_1(a(s)) w_2(a(s)) \cdots w_n(a(s))]^{\frac{j}{n}} \, ds
\]
or equivalently
\[
\int_0^a(t) \left[ \frac{x}{w_1(x)w_2(x) \cdots w_n(x)} \right]^{\frac{j}{n}} \, dx = c \int_0^t r(s) \, ds
\]
and so \( K(a(t)) = c \int_0^t r(s) \, ds \). This shows that \( a(t) \) defined by (3.8) is the unique solution of (3.9) with \( a(t) > 0, t \in (0, T] \).
Next, from (3.9) we have
\[
[a(t)]^{\frac{j}{n}} - a(t) = c \cdot r(t)[w_1(a(t)) w_2(a(t)) \cdots w_n(a(t))]^{\frac{j}{n}}
\]
which on integrating yields
\[
\int_0^t x^{\frac{j}{n}} - 1 \, dx = c \int_0^t r(s)[w_1(a(s)) w_2(a(s)) \cdots w_n(a(s))]^{\frac{j}{n}} \, ds.
\]
By direct computation and an application of Hölder’s inequality, we find
\[
a(t) = \left( \frac{c}{\delta} \right)^{\delta} \left( \int_0^t r(s)[w_1(a(s)) w_2(a(s)) \cdots w_n(a(s))]^{\frac{j}{n}} \, ds \right)^{\frac{1}{\delta}}
\]
\[
= \left( \frac{c}{\delta} \right)^{\delta} \left( \int_0^t (t-s)^{\gamma-1} r(s) w_1(a(s)) w_2(a(s)) \cdots w_n(a(s)) \, ds \right)^{\frac{j}{n}} \cdot (t-s)^{\frac{1}{n}} \left[ r(s) \right]^{\frac{1}{n}} \, ds \right)^{\frac{1}{\delta}}
\]
\[
\leq \left( \frac{c}{\delta} \right)^{\delta} \left( \left[ \int_0^t (t-s)^{\gamma-1} r(s) w_1(a(s)) w_2(a(s)) \cdots w_n(a(s)) \, ds \right]^{\frac{j}{n}} \left[ \int_0^t (t-s)^{\gamma-1} r(s) \, ds \right]^{\frac{1}{n}} \right)^{\frac{1}{\delta}}
\]
\[
= \left( \frac{c}{\delta} \right)^{\delta} \left( \int_0^t (t-s)^{\gamma-1} r(s) w_1(a(s)) w_2(a(s)) \cdots w_n(a(s)) \, ds \right)^{\frac{j}{n}} \left( \int_0^t (t-s)^{\frac{1}{n}} r(s) \, ds \right)^{\frac{1}{n-1}}.
\]
Substituting the constant $c$ into the above inequality gives
\[
a(t) \leq Q_0^{-1} \left[ \sup_{t \in [0,T]} \int_0^t (t-s)^{1-\gamma} r(s) ds \right]^{1-\delta} \left( \int_0^t (t-s)^{\gamma-1} r(s) w_1(a(s)) w_2(a(s)) \cdots w_n(a(s)) ds \right)
\times \left( \int_0^t (t-s)^{\frac{\gamma}{\delta}} r(s) ds \right)^{\delta-1}
\leq \frac{1}{Q_0} \int_0^t (t-s)^{\gamma-1} r(s) w_1(a(s)) w_2(a(s)) \cdots w_n(a(s)) ds
\]
which is exactly (3.7).

Therefore, (3.6) is satisfied if $a(t)$ is defined by (3.8), further this $a \in C[0,T]$ fulfills $a(0) = 0$ and $0 < a(t) \leq 1$, $t \in (0,T)$ (see (C12)). Hence, the condition (C5) is satisfied and we can apply Theorem 2.1 to the system (1.3). □

By using Theorem 3.3 repeatedly, we obtain the existence of multiple constant-sign solutions of (1.3).

**Theorem 3.4.** Let $\theta_i \in \{1,-1\}, 1 \leq i \leq n$ be fixed. Let $\delta > 1$ be such that $\frac{1}{\gamma} + \frac{1}{\delta} = 1$. Assume (C4)$_{\theta_i=0}$, (C10), (C12) and (C6)$_{\theta_i=0,p=1}$ hold. Let $a(t) = K^{-1} \left( \int_0^t r(s) ds \right)$. Let (C13) be satisfied for $\alpha = \alpha_\ell$, $\ell = 1,2,\ldots,k$, and (C14) be satisfied for $\beta = \beta_\ell$, $\ell = 1,2,\ldots,m$.

(a) If $m = k+1$ and $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k < \beta_k$, then (1.3) has (at least) 2\k constant-sign solutions $u^1, \ldots, u^{2\k} \in (C[0,T])^n$ such that

\[
\beta_1 \leq \|u^1\| \leq \alpha_1 \leq \|u^2\| \leq \beta_2 \leq \cdots \leq \alpha_k \leq \|u^{2\k}\| \leq \beta_{k+1}.
\]

(b) If $m = k$ and $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k$, then (1.3) has (at least) 2\k constant-sign solutions $u^1, \ldots, u^{2\k-1} \in (C[0,T])^n$ such that

\[
\beta_1 \leq \|u^1\| \leq \alpha_1 \leq \|u^2\| \leq \beta_2 \leq \cdots \leq \alpha_k \leq \|u^{2\k-1}\| \leq \beta_k.
\]

**Remark 3.2.** Remarks similar to those of Remarks 2.1–2.3 also hold for Theorems 3.3 and 3.4.

The next example illustrates an application of Theorem 3.3.

**Example 3.2.** Consider the system (1.3) where
\[
f_i(t,u) = (u_{1i} u_{2i} \cdots u_{ni})^{k_i}, \quad 1 \leq i \leq n.
\]
Here $0 \leq k_i < 1, 1 \leq i \leq n$ is fixed and $k^* = k_1 + k_2 + \cdots + k_n < 1$. Assume that
\[
\gamma > 1 \quad \text{such that there exists } \delta \text{ with } \delta > \gamma \quad \text{and} \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1
\]
and
\[
\left( 1 - k^* \right)^{\left( \frac{\delta}{\delta - 1} \right)^{\frac{1}{\gamma}}} \leq \frac{1}{\gamma^{\frac{1}{k^*}}} \leq 1.
\]

Fix $\theta_i = 1$, $1 \leq i \leq n$. Clearly, conditions (C4)$_{\theta_i=0}$ and (C6)$_{\theta_i=0,p=1}$ are satisfied. Next, in (C10) we can pick $r$ and $w_i$, $1 \leq i \leq n$ as in (3.5).

Thus, in (C12) the function $K : [0, \infty) \to [0, \infty)$ reduces to
\[
K(y) = \int_0^y \left[ \frac{x}{w_1(x) w_2(x) \cdots w_n(x)} \right]^{\frac{1}{\gamma}} \frac{1}{x} \, dx = \int_0^y \left( \frac{x}{x^{k^*}} \right)^{\frac{1}{\gamma}} \frac{1}{x} \, dx = \frac{\delta}{1 - k^*} y^{\frac{1}{\gamma} - \frac{k^*}{\delta}}.
\]

It follows that
\[
K^{-1}(z) = \left( \frac{1 - k^*}{\delta} z \right)^{\frac{1}{\gamma} - \frac{k^*}{\delta}}.
\]
Moreover, by direct computation we have

\[ Q_0 = \frac{T^\gamma}{\gamma}, \quad \sup_{t \in [0, T]} \int_0^t (t - s)^{1-\gamma} r(s) \, ds = \frac{\delta - 1}{\delta - \gamma} T^{\delta-\gamma} \]

and so

\[ c = \delta Q_0^{-\frac{1}{\gamma}} \left[ \sup_{t \in [0, T]} \int_0^t (t - s)^{1-\gamma} r(s) \, ds \right]^{-\frac{1}{\gamma}} = \frac{\delta}{T} \left( \frac{\delta - \gamma}{\delta - 1} \right)^{\frac{1}{\gamma}} \gamma^{\frac{1}{\gamma}}. \]

Now, for \( t \in [0, T] \),

\[ a(t) = K^{-1} \left( c \int_0^t r(s) \, ds \right) = K^{-1}(ct) = \left( \frac{1 - k^*}{\delta} ct \right)^{\frac{1}{1-\gamma}} \leq \left( \frac{1 - k^*}{\delta} cT \right)^{\frac{1}{1-\gamma}} = \left[ (1 - k^*) \left( \frac{\delta - \gamma}{\delta - 1} \right)^{\frac{1}{\gamma}} \right]^{\frac{1}{1-\gamma}} \gamma^{\frac{1}{1-\gamma}}. \]

Hence, (C12) is satisfied in view of (3.12).

If in addition (C13) and (C14) are satisfied, then by Theorem 3.3 the system (1.3) with (3.10)–(3.12) has at least one positive solution in \((C[0, T])^n\).

As a specific case, consider the system

\[
\begin{align*}
&u_1(t) = \int_0^t (t - s)^{0.5} [u_1(s)u_2(s)]^{0.04} \, ds, \quad t \in [0, 2] \\
&u_2(t) = \int_0^t (t - s)^{0.5} [u_1(s)u_2(s)]^{0.06} \, ds, \quad t \in [0, 2].
\end{align*}
\]

(3.13)

Here \( n = 2, T = 2, \gamma = 1.5, \delta = 3, k_1 = 0.04, k_2 = 0.06 \) and \( k^* = k_1 + k_2 = 0.1 < 1 \).

The condition (3.11) is fulfilled. Further, by direct computation we see that (3.12) is satisfied and also \( a(t) = 0.057817r^{10} \). Next, the conditions (C13) and (C14) respectively reduce to

\[ \alpha^{2k_j} \int_0^2 (2 - s)^{0.5} \, ds \leq \alpha, \quad j = 1, 2 \]

and

\[ \beta^{2k_j} \left( \frac{1 - k^*}{\delta} c \right)^{\frac{20k_j}{\delta}} \int_0^2 (2 - s)^{0.5} s^{\frac{20k_j}{\delta}} \, ds \geq \beta, \quad j = 1, 2 \]

which are satisfied if

\[ \alpha \geq 2.0560 \quad \text{and} \quad \beta \leq 1.1614. \] (3.14)

Hence, from the earlier analysis we conclude that (3.13) has at least one positive solution \( u \in (C[0, 2])^2 \) with

\[ \beta \leq \|u\| \leq \alpha \quad \text{and} \quad u_i(t) \geq (0.057817r^{10})^\frac{10}{\gamma} \beta, \quad t \in [0, 2], \ i = 1, 2. \] (3.15)

Noting the ranges in (3.14), it follows from (3.15) that

\[ 1.1614 \leq \|u\| \leq 2.0560 \quad \text{and} \quad u_i(t) \geq (0.057817r^{10})^\frac{10}{\gamma} (1.1614), \quad t \in [0, 2], \ i = 1, 2. \]
4. Existence results for (1.4)

We shall now tackle the singular system (1.4) using the results obtained in Section 2. Here the nonlinearities $h_i(t,u_1,u_2,\ldots,u_n)$ can be singular at $t=0$ and $u_j=0$, $j \in \{1,2,\ldots,n\}$. In particular, we shall develop more specific conditions that actually construct an explicit function $a(t)$ in (C5).

**Theorem 4.1.** Let $1 \leq p < \infty$ be a constant and $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\theta_i \in \{1,-1\}$, $1 \leq i \leq n$ be fixed. Suppose (C1)–(C3) and (C4)$_{f_i=0}$ hold. Further, assume

(C15) for each $1 \leq i \leq n$ and any $(t,u) \in (0,T) \times \prod_{j=1}^n (0,\infty)$,

$$
\eta(t)b_1(|u_1|)b_2(|u_2|)\cdots b_n(|u_n|) \leq \kappa(t)b_1(|u_1|)b_2(|u_2|)\cdots b_n(|u_n|)
$$

where $\kappa, \eta : (0,T) \to [0,\infty)$, $\eta(t) > 0$ for a.e. $t \in [0,T]$, $\eta \in L^1[0,T]$, and for $1 \leq j \leq n$, $b_j : (0,\infty) \to (0,\infty)$ is continuous, $b_j(c) > 0$ for $c > 0$, $b_j(c) = b_j(c)b_j(d)$ for $c,d > 0$;

(C16) the function $b : (0,\infty) \to (0,\infty)$ defined by

$$
b(x) = b_1(x)b_2(x)\cdots b_n(x)
$$

is nonincreasing and $b \left( \frac{1}{x} \right) = \frac{1}{b(x)}$ for $x > 0$;

(C17) the function $F : (0,\infty) \to (0,\infty)$ defined by

$$
F(x) = \frac{1}{xb(x)}
$$

is nonincreasing;

(C18) for each $1 \leq i \leq n$,

$$
g_i(t,s) \geq g_0(t,s), \quad t \in [0,T], \ a.e. \ s \in [0,t]
$$

where $g_0(t,s) > 0$ for $t \in (0,T)$, $a.e. \ s \in [0,t]$;

(C19) $\int_0^T \left[ \kappa(s)b \left( \int_0^s g_0(s,\tau)\eta(\tau)d\tau \right) \right]^q \eta(s)ds < \infty$ if $p > 1$, and

$$
\text{ess sup}_{t \in [0,T]} \kappa(t)b \left( \int_0^t g_0(t,s)\eta(s)ds \right) < \infty \quad \text{if } p = 1;
$$

(C20) there exists $\alpha > 0$ such that for each $1 \leq i \leq n$,

$$
b(\alpha)b(L_0) \int_0^T g_i(T,s)\kappa(s)b \left( \int_0^s g_0(s,\tau)\eta(\tau)d\tau \right) ds \leq \alpha
$$

where the constant $L_0 > 0$ is defined by

$$
L_0 = F^{-1} \left( \max_{1 \leq j \leq n} \int_0^T g_j(T,s)\kappa(s)b \left( \int_0^s g_0(s,\tau)\eta(\tau)d\tau \right) ds \right);
$$

(C21) there exists $\beta(\neq \alpha) > 0$ such that for each $1 \leq i \leq n$,

$$
b(\beta) \int_0^T g_i(T,s)\eta(s)ds \geq \beta.
$$

Let $a(t) = L_0 \int_0^t g_0(t,s)\eta(s)ds$. Then, the system (1.4) has at least one constant-sign solution $u \in (C[0,T])^n$ such that

(a) $\alpha \leq \|u\| \leq \beta$ and $\theta_i u_i(t) \geq a(t)\alpha, t \in [0,T], 1 \leq i \leq n$ if $\alpha < \beta$;

(b) $\beta \leq \|u\| \leq \alpha$ and $\theta_i u_i(t) \geq a(t)\beta, t \in [0,T], 1 \leq i \leq n$ if $\beta < \alpha$.

**Proof.** We shall apply Theorem 2.1, thus we shall show that (C5)–(C8) are satisfied.

To begin, notice the inequality in (C5) reduces to

$$
\frac{\int_0^t g_i(t,s)\theta_i h_i(s,\theta_1 R, \theta_2 R, \ldots, \theta_n R)ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T,s)\theta_j h_j(s,\theta_1 R a(s), \theta_2 R a(s), \ldots, \theta_n R a(s))ds} \geq a(t), \quad t \in [0,T], 1 \leq i \leq n.
$$

(4.1)
Using (C15), (C18) and (C16), we get

\[
\frac{\int_0^t g_i(t, s)\theta_i h_i(s, \theta_1 R, \theta_2 R, \ldots, \theta_n R)ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T, s)\theta_j h_j(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s))ds} \\
\geq \frac{\int_0^t g_i(t, s)\eta(s)b_1(R)b_2(R)\cdots b_n(R)ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T, s)\kappa(s)b_1(Ra(s))b_2(Ra(s))\cdots b_n(Ra(s))ds} \\
= \frac{\int_0^t g_i(t, s)\eta(s)ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T, s)\kappa(s)b_1(a(s))b_2(a(s))\cdots b_n(a(s))ds} \\
\geq \frac{\int_0^t g_0(t, s)\eta(s)ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T, s)\kappa(s)b(a(s))ds}.
\]

Clearly, (4.1) is satisfied if we can find some \(a \in C[0, T] \) with \(a(0) = 0\), \(0 < a(t) \leq 1\), \(t \in (0, T]\), and such that

\[
a(t) = \frac{\int_0^t g_0(t, s)\eta(s)ds}{\max_{1 \leq j \leq n} \int_0^T g_j(T, s)\kappa(s)b(a(s))ds} = \frac{1}{\ell} \int_0^t g_0(t, s)\eta(s)ds \tag{4.2}
\]

where \(\ell = \max_{1 \leq j \leq n} \int_0^T g_j(T, s)\kappa(s)b(a(s))ds\). To solve (4.2), noting \(b(xy) = b(x)b(y)\) and \(b(\frac{1}{x}) = \frac{1}{b(x)}\) for \(x, y > 0\), we get from (4.2)

\[
b \left( \int_0^t g_0(t, s)\eta(s)ds \right) = b(\ell) \cdot b(a(t)).
\]

Multiply the above relation by \(g_j(T, t)\kappa(t)\), then integrate from 0 to \(T\), and then follow by taking maximum over \(j\) yields

\[
\max_{1 \leq j \leq n} \int_0^T g_j(T, t)\kappa(t)b \left( \int_0^t g_0(t, s)\eta(s)ds \right) dt = b(\ell) \cdot \max_{1 \leq j \leq n} \int_0^T g_j(T, t)\kappa(t)b(a(t))dt \\
= b(\ell) \cdot \ell \\
= \frac{1}{F(\ell)} = F \left( \frac{1}{\ell} \right)
\]

where the last equality follows from the property of \(b\). Hence, we have

\[
\frac{1}{\ell} = F^{-1} \left( \max_{1 \leq j \leq n} \int_0^T g_j(T, t)\kappa(t)b \left( \int_0^t g_0(t, s)\eta(s)ds \right) dt \right) = L_0
\]

which upon substituting into (4.2) gives

\[
a(t) = L_0 \int_0^t g_0(t, s)\eta(s)ds. \tag{4.3}
\]

From (4.3), it is clear that \(a(0) = 0\) and \(a(t) > 0\), \(t \in (0, T]\).

It remains to show that this \(a(t) \leq 1\) for \(t \in (0, T]\). Since \(b\) and \(F^{-1}\) are nonincreasing (note that \(F\) is nonincreasing), together with \(\eta \leq \kappa\) and \(g_0(t, s) \leq g_j(t, s) \leq g_j(T, s), \quad t \in [0, T], \) a.e. \(s \in [0, t] \), \(1 \leq j \leq n\) (from (C18) and (C3)), we find

\[
L_0 = F^{-1} \left( \max_{1 \leq j \leq n} \int_0^T g_j(T, s)\kappa(s)b \left( \int_0^s g_0(s, \tau)\eta(\tau)d\tau \right) ds \right)
\]
Of course conditions (C19)–(C21) can be replaced by (C6)–(C8) with \(a = 0\) and \(a(t)\) defined in (4.3) fulfills (4.2) with \(a(0) = 0\) and \(0 < a(t) \leq 1, t \in (0, T]\). Thus, (C5) is satisfied.

Next, to check that (C6) is fulfilled, we apply (C15) and (4.3) to get

\[
|h_i(s, \theta_1 Ra(s), \theta_2 Ra(s), \ldots, \theta_n Ra(s))| \leq \kappa(s)b_1(R)b_1(a(s))b_2(R)b_2(a(s)) \cdots b_n(R)b_n(a(s))
\]

\[
= \kappa(s)b(R)b(a(s)) = \kappa(s)b(R)b(L_0)\left(\int_0^{T} g_0(s, \tau)\eta(\tau)\,d\tau\right).
\]

It is then clear that (C19) guarantees (C6).

Finally, to see that (C7) and (C8) are satisfied, we employ (C15) and (4.3) to obtain

\[
\int_0^{T} g_i(T, s)\theta_i h_i(s, \theta_1 a(s), \theta_2 a(s), \ldots, \theta_n a(s))\,ds
\]

\[
\leq \int_0^{T} g_i(T, s)\kappa(s)b(\alpha)b(a(s))\,ds = \int_0^{T} g_i(T, s)\kappa(s)b(\alpha)b(L_0)b\left(\int_0^{T} g_0(s, \tau)\eta(\tau)\,d\tau\right)\,ds
\]

and

\[
\int_0^{T} g_i(T, s)\theta_i h_i(s, \theta_1 \beta, \theta_2 \beta, \ldots, \theta_n \beta)\,ds \geq \int_0^{T} g_i(T, s)\eta(s)b(\beta)\,ds.
\]

It is now clear that (C20) and (C21) guarantee (C7) and (C8) respectively.

We have shown that (C5)–(C8) are satisfied, hence Theorem 2.1 can be applied to the system (1.4) and the conclusion is immediate. \(\square\)

**Remark 4.1.** Of course conditions (C19)–(C21) can be replaced by (C6)–(C8) with \(f_i = 0\) and \(a(t)\) defined in (4.3), which, in fact, are weaker conditions compared to (C19)–(C21). However, (C19)–(C21) are easier to verify than (C6)–(C8).

By using Theorem 4.1 repeatedly, we obtain the existence of multiple constant-sign solutions of (1.4).

**Theorem 4.2.** Let \(1 \leq p < \infty\) be a constant and \(q\) be such that \(\frac{1}{p} + \frac{1}{q} = 1\). Let \(\theta_i \in \{1, -1\}, 1 \leq i \leq n\) be fixed. Assume (C1)–(C3), (C4)\(|f_i| = 0\) and (C15)–(C19) hold. Let (C20) be satisfied for \(\alpha = \alpha_\ell, \ell = 1, 2, \ldots, k,\) and (C21) be satisfied for \(\beta = \beta_\ell, \ell = 1, 2, \ldots, m\).
(a) If \( m = k + 1 \) and \( 0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k < \beta_{k+1} \), then (1.4) has (at least) \( 2k \) constant-sign solutions \( u^1, \ldots, u^{2k} \in (C[0, T])^n \) such that
\[
\beta_1 \leq \|u^1\| \leq \alpha_1 \leq \|u^2\| \leq \beta_2 \leq \cdots \leq \alpha_k \leq \|u^{2k}\| \leq \beta_{k+1}.
\]
(b) If \( m = k \) and \( 0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k \), then (1.4) has (at least) \( 2k - 1 \) constant-sign solutions \( u^1, \ldots, u^{2k-1} \in (C[0, T])^n \) such that
\[
\beta_1 \leq \|u^1\| \leq \alpha_1 \leq \|u^2\| \leq \beta_2 \leq \cdots \leq \alpha_k \leq \|u^{2k-1}\| \leq \beta_k.
\]

**Remark 4.2.** Remarks similar to those of Remarks 2.1–2.3 also hold for Theorems 4.1 and 4.2.

We shall illustrate the usefulness of Theorem 4.1 with the following example.

**Example 4.1.** Consider the system (1.4) where
\[
g_i(t, s) = (t - s)^{\gamma - 1} \quad \text{and} \quad h_i(t, u) = (u_1 u_2 \cdots u_n)^{-k_i}, \quad 1 \leq i \leq n
\]
where \( \gamma > 1, 0 \leq k_1 < 1, 1 \leq i \leq n \) is fixed and \( k^* = k_1 + k_2 + \cdots + k_n < 1 \).

Fix \( \theta_i = 1, 1 \leq i \leq n \). Clearly, condition (C1) is satisfied with \( p = \infty \), (C2), (C3) and (C4)|\( f_i = 0 \) are also fulfilled. Next, in (C15) and (C18) we can pick
\[
\eta = \kappa = 1, \quad b_i(x) = x^{-k_i}, \quad 1 \leq i \leq n \quad \text{and} \quad g_0(t, s) = (t - s)^{\gamma - 1}.
\]

Then, (C16) and (C17) are satisfied with
\[
b(x) = x^{-k^*}, \quad F(x) = x^{-(1 - k^*)} \quad \text{and} \quad F^{-1}(z) = z^{-\frac{1}{1 - k^*}}.
\]

Thus, by direct computation we have
\[
L_0 = F^{-1} \left( \int_0^T (T - s)^{\gamma - 1} \left( \frac{s^\gamma}{\gamma} \right)^{-k^*} ds \right) = \gamma^{-\frac{k^*}{1 - k^*}} \left( \int_0^T (T - s)^{\gamma - 1} s^{-\gamma k^*} ds \right)^{-\frac{1}{1 - k^*}}
\]
and
\[
a(t) = \frac{L_0}{\gamma} t^{\gamma - 1}.
\]

Next, condition (C19)|\( q = 1 \) reduces to the inequality
\[
\frac{T^{1 - \gamma k^*}}{1 - \gamma k^*} < \infty
\]
which is satisfied if
\[
1 < \gamma < \frac{1}{k^*}.
\]

Moreover, conditions (C20) and (C21) are simplified to
\[
\left( \frac{\gamma}{L_0} \right)^{k^*} \int_0^T (T - s)^{\gamma - 1} s^{-\gamma k^*} ds \leq \alpha^{1 + k^*}
\]
and
\[
\frac{T^\gamma}{\gamma} \geq \beta^{1 + k^*}
\]
respectively.

Hence, if (4.8)–(4.10) are satisfied, then we can conclude from Theorem 4.1 that the system (1.4) with (4.4) and (4.8) has at least one positive solution in \( (C[0, T])^n \).
As a specific case, consider the system

\[
\begin{align*}
    u_1(t) &= \int_0^t (t-s)^{0.5} [u_1(s)u_2(s)]^{-k_1} ds, & t \in [0, 3] \\
    u_2(t) &= \int_0^t (t-s)^{0.5} [u_1(s)u_2(s)]^{-k_2} ds, & t \in [0, 3]
\end{align*}
\]

(4.11)

where \(0 \leq k_1, k_2 < 1\) are such that \(k_1 + k_2 = 0.5\).

Here \(n = 2, T = 3, \gamma = 1.5\) and \(k^* = k_1 + k_2 = 0.5 < 1\). From (4.7) we compute that \(a(t) = 0.0077229t^{3}\). Note that (4.8) is fulfilled. Further, by direct computation we see that (4.9) and (4.10) are satisfied if

\[
\alpha \geq 19.532 \quad \text{and} \quad \beta \leq 2.2894.
\]

Hence, from the earlier analysis we conclude that (4.11) has at least one positive solution \(u \in (C[0, 2])^2\) with

\[
\beta \leq \|u\| \leq \alpha \quad \text{and} \quad u_i(t) \geq (0.0077229t^{3})\beta, \quad t \in [0, 3], \ i = 1, 2.
\]

(4.13)

Noting the ranges in (4.12), it follows from (4.13) that

\[
2.2894 \leq \|u\| \leq 19.532 \quad \text{and} \quad u_i(t) \geq (0.0077229t^{3})(2.2894), \quad t \in [0, 3], \ i = 1, 2. \quad \Box
\]

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