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An equitable partition for a distance-regular graph of negative type

Štefko Miklavič

Department of Mathematics and Computer Science, Faculty of Education, University of Primorska, 6000 Koper, Slovenia

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Abstract

Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Assume Γ has classical parameters (d, b, α, β) with $b < -1$.

We investigate the extent to which Γ is 1-homogeneous in the sense of Nomura. We show that either Γ is 1-homogeneous, or else Γ has a certain equitable partition of its vertex set which involves $4d - 1$ cells.

Keywords: Distance-regular graph; Q-polynomial; Classical parameters; Kite-free

1. Introduction

Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Then Γ is said to have *classical parameters* (d, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \qquad (1 \leqslant i \leqslant d), \tag{1}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad (0 \leqslant i \leqslant d-1), \tag{2}$$

E-mail address: miklavic@pef.upr.si.

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where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}. \tag{3}$$

In this case b is an integer and $b \notin \{0, -1\}$. We say that Γ has *negative type* whenever Γ has classical parameters (d, b, α, β) such that $b < -1$.

For the rest of this introduction assume Γ has negative type. We are interested in the extent to which Γ is 1-homogeneous in the sense of Nomura [4]. If the intersection number a_1 of Γ is 0, then Γ is 1-homogeneous by Miklavič [3]. For the rest of this introduction assume $a_1 \neq 0$.

Let $V\Gamma$ denote the vertex set of Γ and fix adjacent vertices $x, y \in V\Gamma$. We define

$$D^i_j = D^i_j(x, y) = \{z \in V\Gamma | \partial(x, z) = i \quad \text{and} \quad \partial(y, z) = j\} \quad (0 \leqslant i, j \leqslant d).$$

For each i $(1 \leqslant i \leqslant d)$ and for $z \in D^i_i$, define $\sigma(z) := |\{w \in D^1_1 | \partial(z, w) = i - 1\}|$. It turns out that Γ is kite-free in the sense of Terwilliger [6]; using this we find $\sigma(z) \in \{0, 1\}$. For $j \in \{0, 1\}$ we define $D^i_i(j) = \{z \in D^i_i | \sigma(z) = j\}$.

Assume for the moment that Γ is a near polygon [4]. In this case $D^i_i(0) = \emptyset$ $(1 \leqslant i \leqslant d)$. Moreover Γ is 1-homogeneous by Nomura [4].

Next assume Γ is not a near polygon. In this case we will show

(i) The sets $D^{i-1}_i,\ D^i_{i-1},\ D^i_i(1)$ $(1 \leqslant i \leqslant d)$ and the sets $D^i_i(0)$ $(2 \leqslant i \leqslant d)$ are all nonempty.
(ii) The partition of $V\Gamma$ into the sets $D^i_{i-1},\ D^{i-1}_i,\ D^i_i(1)$ $(1 \leqslant i \leqslant d)$ and $D^i_i(0)$ $(2 \leqslant i \leqslant d)$ is equitable.
(iii) The corresponding parameters of the equitable partition in (ii) are independent of the choice of x, y.

We will prove (ii) using Terwilliger's "balanced set" characterization of the Q-polynomial property [5]. We will prove (iii) by displaying explicit formulae for the corresponding parameters in terms of the intersection numbers of Γ. The results (i)–(iii) are the main results of the paper.

Our paper is organized as follows. In Section 2, we review basic definitions and concepts about distance-regular graphs. In Section 3, we discuss the 1-homogeneous property and the Q-polynomial property. In Section 4 we discuss kites and parallelograms. We prove our main results in Sections 5 and 6.

2. Preliminaries

In this section, we review some definitions and basic concepts of distance-regular graphs. See the book of Brouwer et al. [1] for more background information.

Throughout this paper, Γ will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $V\Gamma$, edge set $E\Gamma$, path-length distance function ∂, and diameter $d := \max\{\partial(x, y) | x, y \in V\Gamma\}$. For $x \in V\Gamma$ and for an integer i define $\Gamma_i(x)$ to be the set of vertices of Γ at distance i from x. We abbreviate $\Gamma(x) := \Gamma_1(x)$. The graph

Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \leqslant h, i, j \leqslant d)$ and all $x, y \in V\Gamma$ with $\partial(x, y) = h$, the number

$$p_{ij}^h := |\Gamma_i(x) \cap \Gamma_j(y)| \tag{4}$$

is independent of x, y. The constants p_{ij}^h $(0 \leqslant h, i, j \leqslant d)$ are known as the *intersection numbers* of Γ. For notational convenience define $c_i := p_{1,i-1}^i$ $(1 \leqslant i \leqslant d)$, $a_i := p_{1i}^i$ $(0 \leqslant i \leqslant d)$, $b_i := p_{1,i+1}^i$ $(0 \leqslant i \leqslant d-1)$, $k_i := p_{ii}^0$ $(0 \leqslant i \leqslant d)$ and $c_0 = b_d = 0$. We observe $a_0 = 0$ and $c_1 = 1$. Moreover,

$$c_i + a_i + b_i = k \quad (0 \leqslant i \leqslant d), \tag{5}$$

where $k := k_1$. From now on we assume Γ is distance-regular with diameter $d \geqslant 3$.

In the following two lemmas we cite some well-known facts about the intersection numbers; see, for example, [1, p. 127, 134].

Lemma 2.1. *Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Then for all integers h, i, j $(0 \leqslant h, i, j \leqslant d)$ the following* (i), (ii) *hold.*

(i) *If one of h, i, j is greater than the sum of the other two then $p_{ij}^h = 0$.*
(ii) *If one of h, i, j is equal to the sum of the other two then $p_{ij}^h \neq 0$.*

Lemma 2.2. *Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Then the following* (i)–(iii) *hold.*

(i) $k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$ $(0 \leqslant i \leqslant d)$,
(ii) $p_{i,i-1}^1 = \frac{c_i k_i}{k}$ $(1 \leqslant i \leqslant d)$,
(iii) $p_{ii}^1 = \frac{a_i k_i}{k}$ $(0 \leqslant i \leqslant d)$.

Lemma 2.3. *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), $d \geqslant 3$. Then the following* (i), (ii) *hold.*

(i) *If $b < -1$ then $\alpha \neq 0$.*
(ii) *For each integer i $(1 \leqslant i \leqslant d)$,*

$$a_i - a_1 c_i = -\alpha(1 + b + a_1) \begin{bmatrix} i \\ 1 \end{bmatrix} \begin{bmatrix} i-1 \\ 1 \end{bmatrix}.$$

Proof. (i) If $b < -1$ and $\alpha = 0$ then $c_2 = 1 + b < 0$, a contradiction.

(ii) Straightforward from (1)–(3) and (5). $\square$

Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. We recall the Bose–Mesner algebra of Γ. For each i $(0 \leqslant i \leqslant d)$ let A_i denote the matrix with rows and columns indexed by $V\Gamma$, and x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \qquad (x, y \in V\Gamma). \tag{6}$$

We call A_i the ith *distance matrix* of Γ. We observe

$$A_0 = I, \tag{7}$$

$$A_0 + A_1 + \cdots + A_d = J, \tag{8}$$

$$A_i^t = A_i \quad (0 \leqslant i \leqslant d), \tag{9}$$

where J is the all 1's matrix, and where A_i^t denotes the transpose of A_i. Furthermore, we have

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \quad (0 \leqslant i, j \leqslant d). \tag{10}$$

By (7)–(10), the matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative semi-simple $\mathbb{R}$-algebra *M*, known as the *Bose–Mesner algebra*. By Godsil [2, Theorem 12.2.1], the algebra *M* has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |V\Gamma|^{-1} J, \tag{11}$$

$$E_i E_j = \delta_{ij} E_i \quad (0 \leqslant i, j \leqslant d), \tag{12}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{13}$$

$$E_i^t = E_i \quad (0 \leqslant i \leqslant d). \tag{14}$$

The $E_0, E_1, \ldots, E_d$ are known as the *primitive idempotents* of Γ and E_0 is the *trivial* idempotent. It is well known that *M* is also closed for the entrywise multiplication.

Set $A := A_1$ and define the real numbers θ_i $(0 \leqslant i \leqslant d)$ by

$$A = \sum_{i=0}^{d} \theta_i E_i. \tag{15}$$

Then $AE_i = E_i A = \theta_i E_i$ $(0 \leqslant i \leqslant d)$ and $\theta_0 = k$. The scalars $\theta_0, \theta_1, \ldots, \theta_d$ are distinct, since *A* generates *M* [1, p. 128]. The $\theta_0, \theta_1, \ldots, \theta_d$ are known as the *eigenvalues* of Γ.

For notational convenience we identify $V\Gamma$ with the standard orthonormal basis in the Euclidean space $V, \langle , \rangle$, where $V = \mathbb{R}^{|V\Gamma|}$ (column vectors), and where $\langle , \rangle$ is the dot product

$$\langle u, v \rangle = u^t v \quad (u, v \in V).$$

Observe *M* acts on *V* by left multiplication. The Euclidean space $V, \langle , \rangle$ is known as the *standard module* of Γ.

In the following lemma we cite some well-known results about primitive idempotents.

Lemma 2.4 (*Terwilliger [5, Lemma 1.1]*). *Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Pick any $\theta, \theta_0^*, \theta_1^*, \ldots, \theta_d^* \in \mathbb{R}$, and set*

$$E := |V\Gamma|^{-1} \sum_{i=0}^{d} \theta_i^* A_i. \tag{16}$$

Then the following (i)–(iii) *are equivalent*:

(i) θ *is an eigenvalue of* Γ *and* E *is the associated primitive idempotent*.
(ii) *For all* $x, y \in V\Gamma$,

$$\langle Ex, Ey\rangle = |V\Gamma|^{-1}\theta_i^* \text{ whenever } \partial(x, y) = i$$

and

$$\sum_{z\in\Gamma(x)} Ez = \theta Ex.$$

(iii) *The intersection numbers of* Γ *satisfy*

$$c_i\theta_{i-1}^* + a_i\theta_i^* + b_i\theta_{i+1}^* = \theta\theta_i^* \quad (0 \leqslant i \leqslant d)$$

and $\theta_0^* = \operatorname{rank} E$.

If (i)–(iii) hold, we call the sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ the *dual eigenvalue sequence* associated with θ, E. The sequence is *trivial* whenever $E = E_0$ (in which case $\theta_0^* = \theta_1^* = \cdots = \theta_d^* = 1$).

3. The 1-homogeneous property and the Q-polynomial property

In this section, we discuss the 1-homogeneous property and the Q-polynomial property of distance-regular graphs. We begin with a definition.

Definition 3.1. Let Γ denote a distance-regular graph with diameter $d \geqslant 3$ and let x, y denote adjacent vertices in $V\Gamma$. For all integers i and j we define $D_j^i = D_j^i(x, y)$ by

$$D_j^i = \Gamma_i(x) \cap \Gamma_j(y).$$

We observe $D_j^i = \emptyset$ unless $0 \leqslant i, j \leqslant d$. Moreover $|D_j^i| = p_{ij}^1$ $(0 \leqslant i, j \leqslant d)$.

Lemma 3.2. *Let* Γ *denote a distance-regular graph with diameter* $d \geqslant 3$ *and let* x, y *denote adjacent vertices in* $V\Gamma$. *Then, with the reference to Definition* 3.1, *the following* (i), (ii) *hold*.

(i) *For all* i, j $(0 \leqslant i, j \leqslant d)$ *if* $|i - j| > 1$ *then* $D_j^i = \emptyset$. *If* $|i - j| = 1$ *then* $D_j^i \neq \emptyset$.
(ii) *For each* i $(0 \leqslant i \leqslant d)$ *we have* $D_i^i = \emptyset$ *if and only if* $a_i = 0$.

Proof. Immediate from Lemmas 2.1 and 2.2. □

An *equitable partition* of a graph is a partition $\pi = \{C_1, C_2, \ldots, C_s\}$ of its vertex set into nonempty cells, so that for all i, j $(1 \leqslant i, j \leqslant s)$ the number c_{ij} of neighbours, which a vertex in the cell C_i has in the cell C_j, is independent of the choice of the vertex in C_i. We call the c_{ij} the *corresponding parameters*.

Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Then Γ is said to be *1-homogeneous*, whenever for all pairs x, y of adjacent vertices, the partition of $V\Gamma$ given by $\{D^i_j(x, y) \mid 0 \leqslant i, j \leqslant d,\ D^i_j(x, y) \neq \emptyset\}$ is equitable, and moreover the corresponding parameters are independent of the choice of x, y.

Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. The *Krein parameters* q^h_{ij} $(0 \leqslant h, i, j \leqslant d)$ of Γ are defined by

$$E_i \circ E_j = |V\Gamma|^{-1} \sum_{h=0}^{d} q^h_{ij} E_h \quad (0 \leqslant i, j \leqslant d), \tag{17}$$

where $\circ$ denotes entrywise multiplication. We say Γ is *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents), whenever for all distinct integers i, j $(0 \leqslant i, j \leqslant d)$,

$$q^1_{ij} \neq 0 \text{ if and only if } |i - j| = 1.$$

Let E denote a nontrivial primitive idempotent of Γ. We say Γ is *Q-polynomial with respect to E* whenever an ordering $E_0, E_1 = E, \ldots, E_d$ of the primitive idempotents of Γ exists, with respect to which Γ is Q-polynomial.

We have the following lemma about Q-polynomial distance-regular graphs.

Lemma 3.3 (*Brouwer et al. [1, Theorem 8.1.1]*). *Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ denote the associated dual eigenvalue sequence. Suppose Γ is Q-polynomial with respect to E. Then $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are mutually distinct.*

The following result of Terwilliger will play a crucial role in our investigation.

Lemma 3.4 (*Terwilliger [5, Theorem 3.3]*). *Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Let E denote a nontrivial primitive idempotent of Γ and let $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ denote the associated dual eigenvalue sequence. Then the following* (i), (ii) *are equivalent*:

(i) *Γ is Q-polynomial with respect to E.*
(ii) *$\theta^*_0 \neq \theta^*_i$ $(1 \leqslant i \leqslant d)$, and for all integers h, i, j $(1 \leqslant h \leqslant d)$, $(0 \leqslant i, j \leqslant d)$ and for all vertices $x, y \in V\Gamma$ with $\partial(x, y) = h$ the following holds*:

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} Ez - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} Ez \in \operatorname{span}\{Ex - Ey\}.$$

Suppose (i), (ii) *hold. Then for all integers h, i, j $(1 \leqslant h \leqslant d)$, $(0 \leqslant i, j \leqslant d)$ and for all vertices $x, y \in V\Gamma$ with $\partial(x, y) = h$,*

$$\sum_{z \in \Gamma_i(x) \cap \Gamma_j(y)} Ez - \sum_{z \in \Gamma_j(x) \cap \Gamma_i(y)} Ez = p^h_{ij} \frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_h} (Ex - Ey). \tag{18}$$

Lemma 3.5 (*Brouwer et al. [1, Corollary 8.4.2]*). *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), $d \geqslant 3$. Then the following* (i)–(iii) *hold.*

(i) *The scalar $\theta = \left[{d-1 \atop 1}\right](\beta - \alpha) - 1$ is an eigenvalue of Γ.*
(ii) *Γ is Q-polynomial with respect to θ.*
(iii) *Let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the dual eigenvalue sequence associated with θ. Then there exist real numbers γ, δ such that*

$$\theta_i^* = \gamma \begin{bmatrix} d-i \\ 1 \end{bmatrix} + \delta \quad (0 \leqslant i \leqslant d).$$

4. Kites and parallelograms

In this section we recall kites and parallelograms and review some of their properties. Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. We recall the notion of a kite in Γ.

Pick an integer i $(2 \leqslant i \leqslant d)$. By a *kite of length i* (or *i-kite*) in Γ we mean a 4-tuple *xyzu* of vertices of Γ such that x, y and z are mutually adjacent, and $\partial(u, x) = i$, $\partial(u, y) = \partial(u, z) = i - 1$. We say Γ is *kite-free* whenever Γ has no kites of any length. We have the following result about kite-free distance-regular graphs.

Theorem 4.1 (*Terwilliger [6, Theorem 2.12] and Weng [7, Theorem 2.6]*). *Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geqslant 3$ and intersection number $a_1 \neq 0$. Then the following* (i) *and* (ii) *are equivalent.*

(i) *Γ has classical parameters (d, b, α, β), and either $b < -1$, or Γ is a dual polar graph or a Hamming graph.*
(ii) *Γ has no kites of any length.*

Let Γ denote a distance-regular graph with diameter $d \geqslant 3$. Pick an integer i $(2 \leqslant i \leqslant d)$. By a *parallelogram of length i* (or *i-parallelogram*) in Γ, we mean a 4-tuple *xyzu* of vertices in $V\Gamma$ such that $\partial(x, y) = \partial(z, u) = 1$, $\partial(x, u) = i$, and $\partial(x, z) = \partial(y, z) = \partial(y, u) = i - 1$.

Theorem 4.2. *Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geqslant 3$ and intersection number $a_1 \neq 0$. Then the following* (i) *and* (ii) *are equivalent.*

(i) *Γ has no kites of any length.*
(ii) *Γ has no parallelogram of any length.*

Proof. (i) $\rightarrow$ (ii) By Theorem 4.1, Γ has classical parameters (d, b, α, β). Pick an integer i $(2 \leqslant i \leqslant d)$. Let x, y, u denote vertices of Γ such that $\partial(x, y) = 1$, $\partial(x, u) = i$ and $\partial(y, u) = i - 1$. Define

$$f_i := |\{z \mid z \in V\Gamma, xyzu \text{ is an } i\text{-parallelogram}\}|$$

and

$$e_i := |\{z \mid z \in V\Gamma, xyzu \text{ is an } i\text{-kite}\}|.$$

By Weng [8, Lemma 7.3], we have

$$f_i = b^{i-2} e_i.$$

Since $e_i = 0$ by the assumption, we obtain $f_i = 0$ as well.

(ii) → (i) This follows from Weng [8, Lemma 6.12]. □

Let Γ denote a kite-free distance-regular graph. Pick an integer i $(1 \leqslant i \leqslant d)$. With reference to Definition 3.1, for $z \in D_i^i$ define $\sigma(z) := |\Gamma_{i-1}(z) \cap D_1^1|$. Observe in this case that $\sigma(z) \in \{0, 1\}$; otherwise Γ has an i-kite or a 2-kite. This allows us to make the following definition.

Definition 4.3. Let Γ denote a kite-free distance-regular graph with diameter $d \geqslant 3$ and intersection number $a_1 \neq 0$. Pick an integer i $(1 \leqslant i \leqslant d)$. Then with reference to Definition 3.1, for $j \in \{0, 1\}$ we define $D_i^i(j) = \{z \in D_i^i \mid \sigma(z) = j\}$. We observe $D_i^i = D_i^i(1) \cup D_i^i(0)$. We further observe $D_1^1(1) = D_1^1$ and $D_1^1(0) = \emptyset$.

Lemma 4.4. *Let Γ denote a kite-free distance-regular graph with diameter $d \geqslant 3$ and intersection number $a_1 \neq 0$. Then with reference to Definitions* 3.1 *and* 4.3 *the following* (i) *and* (ii) *hold.*

(i) $\partial(u, z) = 1$ *for all distinct* $u, z \in D_1^1$.
(ii) *For any integer i $(2 \leqslant i \leqslant d)$ we have $\partial(u, z) = i$ for all $u \in D_1^1$ and all $z \in D_i^i(0)$.*

Proof. (i) If u and z are nonadjacent, then $zxyu$ is a 2-kite, a contradiction.

(ii) Observe that $\partial(u, z) \in \{i, i + 1\}$. If $\partial(u, z) = i + 1$, then the 4-tuple $uxyz$ is an $(i + 1)$-kite, a contradiction. □

5. The main result I

Let Γ denote a Q-polynomial kite-free distance-regular graph with diameter $d \geqslant 3$ and intersection number $a_1 \neq 0$. In this section we show some basic properties of the partition of the $V\Gamma$ given by the nonempty sets D_i^{i-1}, D_{i-1}^i, $D_i^i(1)$ $(1 \leqslant i \leqslant d)$ and $D_i^i(0)$ $(2 \leqslant i \leqslant d)$. We will be discussing the following situation.

Definition 5.1. Let Γ denote a Q-polynomial kite-free distance-regular graph with diameter $d \geqslant 3$ and intersection number $a_1 \neq 0$, which is not a near polygon (i.e., a distance regular graph with $a_i = c_i a_1$ $(1 \leqslant i \leqslant d - 1)$ and no induced $K_{1,2,1}$). By Theorem 4.1, Γ has classical parameters (d, b, α, β) with $b < -1$. We fix adjacent vertices x, y of Γ and let D_j^i, $D_i^i(0)$, $D_i^i(1)$ denote the corresponding sets as in Definitions 3.1 and 4.3.

Our goal for the next two sections is to establish points (i)–(iii) from the Introduction. We begin with a lemma.

Lemma 5.2. *With reference to Definition* 5.1 *the following* (i)–(iii) *hold*:

(i) *There are no edges between* $D_i^{i-1} \cup D_{i-1}^i$ *and* $D_{i-1}^{i-1}(0) \cup D_{i-1}^{i-1}(1)$ $(2 \leqslant i \leqslant d)$.

(ii) *There are no edges between* $D_i^i(1)$ *and* $D_{i+1}^{i+1}(0)$ $(1 \leqslant i \leqslant d-1)$.

(iii) *There are no edges between* $D_i^i(1)$ *and* $D_{i-1}^{i-1}(0)$ $(3 \leqslant i \leqslant d)$.

Proof. (i) Let $v \in D_i^{i-1}$ (resp., $v \in D_{i-1}^i$) and $w \in D_{i-1}^{i-1}(0) \cup D_{i-1}^{i-1}(1)$. Suppose that v and w are adjacent. Then the 4-tuple $yxwv$ (resp., $xywv$) is an *i*-parallelogram, contradicting Theorem 4.2.

(ii) There are no edges between $D_i^i(1)$ and $D_{i+1}^{i+1}(0)$ by the definition of the set $D_{i+1}^{i+1}(0)$.

(iii) Assume $v \in D_i^i(1)$ is adjacent to $w \in D_{i-1}^{i-1}(0)$. Let z be the unique vertex in D_1^1 such that $\partial(z, v) = i-1$. By Lemma 4.4(ii), $\partial(z, w) = i-1$. But now the 4-tuple $xzwv$ is an *i*-parallelogram, contradicting Theorem 4.2. $\square$

Lemma 5.3. *Pick an integer* i $(2 \leqslant i \leqslant d)$. *Then with reference to Definition* 5.1 *the following* (i)–(iii) *hold*.

(i) *Each* $v \in D_{i-1}^i$ (*resp.*, D_i^{i-1}) *is adjacent to*

(a) *precisely*	0	*vertices in* $D_{i-1}^{i-1}(0)$, $D_{i-1}^{i-1}(1)$,
(b) *precisely*	c_{i-1}	*vertices in* D_{i-2}^{i-1} (*resp.*, D_{i-1}^{i-2}),
(c) *precisely*	$c_i - c_{i-1}$	*vertices in* D_i^{i-1} (*resp.*, D_{i-1}^i),
(d) *precisely*	a_{i-1}	*vertices in* D_{i-1}^i (*resp.*, D_i^{i-1}),
(e) *precisely*	b_i	*vertices in* D_i^{i+1} (*resp.*, D_{i+1}^i),
(f) *precisely*	$a_i - a_{i-1} - \lvert\Gamma(v) \cap D_i^i(1)\rvert$	*vertices in* $D_i^i(0)$.

(ii) *Each* $v \in D_i^i(0)$ *is adjacent to*

(a) *precisely*	0	*vertices in* $D_{i-1}^{i-1}(1)$, $D_{i+1}^{i+1}(1)$, D_{i+1}^i, D_i^{i+1},
(b) *precisely*	$c_i - \lvert\Gamma(v) \cap D_{i-1}^{i-1}(0)\rvert$	*vertices in* D_{i-1}^i,
(c) *precisely*	$c_i - \lvert\Gamma(v) \cap D_{i-1}^{i-1}(0)\rvert$	*vertices in* D_i^{i-1},
(d) *precisely*	b_i	*vertices in* $D_{i+1}^{i+1}(0)$,
(e) *precisely*	$a_i + \lvert\Gamma(v) \cap D_{i-1}^{i-1}(0)\rvert - c_i - \lvert\Gamma(v) \cap D_i^i(1)\rvert$	*verticesin* $D_i^i(0)$.

(iii) *Each* $v \in D_i^i(1)$ *is adjacent to*

(a) *precisely*	0	*vertices in* $D_{i-1}^{i-1}(0)$, $D_{i+1}^{i+1}(0)$, D_{i+1}^i, D_i^{i+1},
(b) *precisely*	$c_i - \lvert\Gamma(v) \cap D_{i-1}^{i-1}(1)\rvert$	*vertices in* D_{i-1}^i,
(c) *precisely*	$c_i - \lvert\Gamma(v) \cap D_{i-1}^{i-1}(1)\rvert$	*vertices in* D_i^{i-1},
(d) *precisely*	b_i	*vertices in* $D_{i+1}^{i+1}(1)$,
(e) *precisely*	$a_i + \lvert\Gamma(v) \cap D_{i-1}^{i-1}(1)\rvert - c_i - \lvert\Gamma(v) \cap D_i^i(1)\rvert$	*vertices in* $D_i^i(0)$.

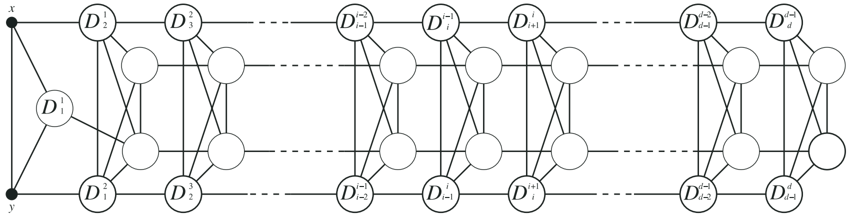

Fig. 1. The partition corresponding to a pair of adjacent vertices x and y. The circles in the middle of the figure represent the sets $D_i^i(0)$ $(2 \leqslant i \leqslant d)$ (upper line) and the sets $D_i^i(1)$ $(2 \leqslant i \leqslant d)$ (lower line). Observe that $D_{i-1}^i \cup D_i^i(0) \cup D_i^i(1) \cup D_{i+1}^i = \Gamma_i(x)$ and $D_i^{i-1} \cup D_i^i(0) \cup D_i^i(1) \cup D_i^{i+1} = \Gamma_i(y)$.

Proof. Routine using Lemma 5.2 and the fact that $D_{i-1}^i \cup D_i^i(0) \cup D_i^i(1) \cup D_{i+1}^i = \Gamma_i(x)$ and $D_i^{i-1} \cup D_i^i(0) \cup D_i^i(1) \cup D_i^{i+1} = \Gamma_i(y)$. $\square$

With reference to Definition 5.1 and Lemma 5.3 we visualize $D_{i-1}^i,\ D_i^{i-1},\ D_i^i(0),\ D_i^i(1)$ in Fig. 1.

6. The main result II

In this section we continue to establish points (i)–(iii) from the Introduction.

Lemma 6.1. *With reference to Definition* 5.1 *the following holds*: *For each integer* i $(2 \leqslant i \leqslant d)$ *and for all* $v \in D_i^{i-1} \cup D_{i-1}^i$,

$$|\Gamma(v) \cap D_i^i(1)| = a_1(c_i - c_{i-1}).$$

Proof. Without loss we may assume $v \in D_i^{i-1}$. Then v is at distance i from every vertex in D_1^1, else there exists an i-kite. Hence there are exactly $a_1 c_i c_{i-1} \cdots c_1$ paths of length i from v to D_1^1. Since v has c_{i-1} neighbours in D_{i-1}^{i-2}, exactly $c_{i-1} a_1 c_{i-1} c_{i-2} \cdots c_1$ of these paths pass through D_{i-1}^{i-2}. The remaining $a_1(c_i - c_{i-1}) c_{i-1} c_{i-2} \cdots c_1$ paths must pass through $D_i^i(1)$. Let $w \in D_i^i(1)$. Since there is a unique vertex $z \in D_1^1$ such that $\partial(z, w) = i - 1$, there are exactly $c_{i-1} c_{i-2} \cdots c_1$ paths of length $i - 1$ between w and D_1^1. Hence v has exactly $a_1(c_i - c_{i-1})$ neighbours in $D_i^i(1)$. $\square$

Lemma 6.2. *With reference to Definition* 5.1 *the following* (i), (ii) *hold*:

(i) *For each integer* i $(2 \leqslant i \leqslant d)$ *and for all* $v \in D_i^i(0)$,

$$|\Gamma(v) \cap D_{i-1}^{i-1}(0)| = c_i \frac{b^{i-2} - 1}{b^i - 1}.$$

(ii) *For each integer* i $(2 \leqslant i \leqslant d)$ *and for all* $v \in D_i^i(0)$,

$$|\Gamma(v) \cap D_i^i(1)| = a_1 c_i \frac{b^i - b^{i-2}}{b^i - 1}.$$

Proof. (i) We abbreviate $\tau = |\Gamma(v) \cap D_{i-1}^{i-1}(0)|$ and $\eta = |\Gamma(v) \cap D_i^{i-1}|$. We observe $\tau + \eta = c_i$. Recall by Definition 5.1 that Γ has classical parameters (d, b, α, β). Let the eigenvalue θ and the dual eigenvalue sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ be as in Lemma 3.5. Let E denote the associated primitive idempotent of Γ. By Lemma 3.4 and since Γ is Q-polynomial with respect to E we find

$$\sum_{\substack{\partial(x,z)=i-1\\ \partial(v,z)=1}} Ez - \sum_{\substack{\partial(x,z)=1\\ \partial(v,z)=i-1}} Ez = c_i \frac{\theta_{i-1}^* - \theta_1^*}{\theta_0^* - \theta_i^*}(Ex - Ev). \tag{19}$$

Observe that by the definition of $D_i^i(0)$, we have $\{z \in V\Gamma | \partial(x,z) = 1,\ \partial(v,z) = i-1\} \subseteq D_2^1$. Taking the inner product of (19) with Ey using Lemma 2.4(ii) we get (after multiplying by $|V\Gamma|$)

$$\tau\theta_{i-1}^* + \eta\theta_i^* - c_i\theta_2^* = c_i \frac{\theta_{i-1}^* - \theta_1^*}{\theta_0^* - \theta_i^*}(\theta_1^* - \theta_i^*).$$

Evaluating the above line using $\eta = c_i - \tau$ we obtain

$$\tau = c_i \frac{(\theta_{i-1}^* - \theta_1^*)(\theta_1^* - \theta_i^*) + (\theta_0^* - \theta_i^*)(\theta_2^* - \theta_i^*)}{(\theta_0^* - \theta_i^*)(\theta_{i-1}^* - \theta_i^*)}.$$

Simplifying the above line using Lemma 3.5(iii) we get

$$\tau = c_i \frac{b^{i-2} - 1}{b^i - 1}.$$

(ii) By Lemma 4.4(ii) the vertex v is at distance i from every vertex in D_1^1. Hence there are exactly $a_1 c_i c_{i-1} \cdots c_1$ paths of length i from v to D_1^1. By Lemma 6.2(i), exactly $a_1 \frac{b^{i-2}-1}{b^i-1} c_i c_{i-1} \cdots c_1$ of these paths pass through $D_{i-1}^{i-1}(0)$. The remaining $a_1 \frac{b^i - b^{i-2}}{b^i-1} c_i \cdots c_1$ paths must pass through $D_i^i(1)$. Let $w \in D_i^i(1)$. Since there is a unique vertex $z \in D_1^1$ such that $\partial(z, w) = i-1$, there are exactly $c_{i-1} \cdots c_1$ paths of length $i-1$ between w and D_1^1. Hence v has exactly $a_1 c_i \frac{b^i - b^{i-2}}{b^i - 1}$ neighbours in $D_i^i(1)$. □

Lemma 6.3. *With reference to Definition* 5.1 *the following* (i), (ii) *hold*:

(i) *For each integer* i $(2 \leqslant i \leqslant d)$ *and for all* $v \in D_i^i(1)$,

$$|\Gamma(v) \cap D_{i-1}^{i-1}(1)| = c_{i-1}.$$

(ii) *For each integer* i $(1 \leqslant i \leqslant d)$ *and for all* $v \in D_i^i(1)$,

$$|\Gamma(v) \cap D_i^i(1)| = (a_1 - 1)(c_i - c_{i-1}) + a_{i-1}.$$

Proof. (i) Let z be the unique vertex in D_1^1 such that $\partial(z, v) = i - 1$. Then there are exactly $c_{i-1} \cdots c_1$ paths of length $i-1$ joining z and v, and all of these paths pass through $D_{i-1}^{i-1}(1)$. Since every neighbour of v in $D_{i-1}^{i-1}(1)$ is joined to z via $c_{i-2} \cdots c_1$ paths of length $i - 2$, we find v has exactly c_{i-1} neighbours in $D_{i-1}^{i-1}(1)$.

(ii) First assume $i = 1$. Then the result holds by Lemma 4.4(i). Next assume $i \geqslant 2$. Let z denote the unique vertex in D_1^1 such that $\partial(z, v) = i - 1$. If z' is another vertex from D_1^1, then $\partial(z', v) = i$. Therefore there are exactly $(a_1 - 1)c_i c_{i-1} \cdots c_1 + (a_{i-1} + \cdots + a_1)c_{i-1} \cdots c_1$ paths of length i from v to D_1^1. Since v has c_{i-1} neighbours in $D_{i-1}^{i-1}(1)$, there are exactly $c_{i-1}((a_1 - 1)c_{i-1} \cdots c_1 + (a_{i-2} + \cdots + a_1)c_{i-2} \cdots c_1)$ of these paths, for which v is the only vertex in $D_i^i(1)$. Recall there are exactly $c_{i-1} \cdots c_1$ paths of length $i - 1$ from each vertex in $D_i^i(1)$ to D_1^1. From these comments we find v has exactly $(a_1 - 1)(c_i - c_{i-1}) + a_{i-1}$ neighbours in $D_i^i(1)$. □

Lemma 6.4. *With reference to Definition* 5.1 *the following* (i), (ii) *hold*:

(i) *For each integer* i $(1 \leqslant i \leqslant d)$,

$$|D_i^i(1)| = a_1 \frac{b_1 \cdots b_{i-1}}{c_1 \cdots c_{i-1}}.$$

(ii) *For each integer* i $(1 \leqslant i \leqslant d)$,

$$|D_i^i(0)| = \frac{b_1 \cdots b_{i-1}}{c_1 \cdots c_i}(a_i - a_1 c_i).$$

Proof. (i) Assume $i \geqslant 2$; otherwise the result is clear. By Lemma 5.3 every vertex in $D_{i-1}^{i-1}(1)$ has b_{i-1} neighbours in $D_i^i(1)$. Moreover, by Lemma 6.3(i) every vertex in $D_i^i(1)$ is adjacent to exactly c_{i-1} vertices in $D_{i-1}^{i-1}(1)$. Counting the edges between $D_{i-1}^{i-1}(1)$ and $D_i^i(1)$ in two different ways we obtain

$$|D_{i-1}^{i-1}(1)|\, b_{i-1} = |D_i^i(1)|\, c_{i-1}.$$

Evaluating this equation using induction we obtain $|D_i^i(1)| = |D_1^1(1)| \frac{b_1 \cdots b_{i-1}}{c_1 \cdots c_{i-1}}$. Recall $D_1^1(1) = D_1^1$ and $|D_1^1| = a_1$, so $|D_i^i(1)| = a_1 \frac{b_1 \cdots b_{i-1}}{c_1 \cdots c_{i-1}}$.

(ii) Observe $|D_i^i(0)| = |D_i^i| - |D_i^i(1)|$ and that $|D_i^i| = p_{ii}^1$. To finish the proof, evaluate p_{ii}^1 using Lemma 2.2 and evaluate $|D_i^i(1)|$ using (i) above. □

Lemma 6.5. *With reference to Definition* 5.1 *the following* (i), (ii) *hold*:

(i) $D_i^i(1) \neq \emptyset$ $(1 \leqslant i \leqslant d)$.
(ii) $D_i^i(0) \neq \emptyset$ $(2 \leqslant i \leqslant d)$.

Proof. (i) This follows from Lemma 6.4(i).

(ii) Since Γ is not a near polygon, we obtain $b \neq -a_1 - 1$ from [1, Theorems 6.2.1 and 6.4.1]. The assertion now follows from Lemmas 6.4(ii) and 2.3. □

Theorem 6.6. *Let Γ denote a Q-polynomial kite-free distance-regular graph with diameter $d \geqslant 3$. Assume the intersection number $a_1 \neq 0$, and that Γ is not a near polygon. Let x, y denote adjacent vertices of Γ. Then with reference to Definitions* 3.1 *and* 4.3, *the partition of $V\Gamma$ into the sets D_i^{i-1}, D_{i-1}^i, $D_i^i(1)$ $(1 \leqslant i \leqslant d)$ and $D_i^i(0)$ $(2 \leqslant i \leqslant d)$ is equitable. Moreover, the corresponding parameters are independent of the choice of x, y.*

Proof. Immediate from Lemmas 5.3, 6.1–6.3 and 6.5. □

We end the paper with some comments on Theorem 6.6.

Corollary 6.7. *Let Γ denote a Q-polynomial kite-free distance-regular graph with diameter $d \geqslant 3$. Assume the intersection number $a_1 \neq 0$. Then the following* (i), (ii) *are equivalent.*

(i) Γ *is* 1*-homogeneous,*
(ii) Γ *is a near polygon.*

Proof. Since Γ is a Q-polynomial kite-free distance-regular graph, it has classical parameters (d, b, α, β) by Theorem 4.1. If Γ is a near polygon, then we have $a_2 = a_1 c_2$ by Brouwer et al. [1, Theorem 6.4.1]. So, by Lemma 2.3(ii), we obtain $1 + b + a_1 = 0$. But then also $a_d = a_1 c_d$ by Lemma 2.3(ii). Therefore, Γ is a near $2d$-gon by Brouwer et al. [1, Theorem 6.4.1] and it is 1-homogeneous by Nomura [4, Theorem 1]. On the other hand, if Γ is not a near polygon then, by Lemma 6.5, we have $D_2^2(0) \neq \emptyset$ and $D_2^2(1) \neq \emptyset$. Using this we find Γ is not 1-homogeneous. □

Let Γ be as in Theorem 6.6. By Theorem 4.1, Γ has classical parameters (d, b, α, β) with $b < -1$. Now by a result of Weng [9, Theorem 10.3], at least one of the following (i)–(iv) holds:

(i) $d = 3$,
(ii) $c_2 = 1$,
(iii) Γ is the Hermitean form graph $\mathrm{Her}_{-b}(d)$,
(iv) $\alpha = (b-1)/2$, $\beta = -(1 + b^d)/2$, and $-b$ is a power of an odd prime.

See [1, Section 9.5] for the definition of the Hermitean form graph.

We suspect that Theorem 6.6 implies new feasibility conditions which will enable us to classify cases (i), (ii) and (iv) above. We will pursue this in a future paper.

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