Exceptional Dehn fillings on hyperbolic 3-manifolds with at least two boundary components

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Abstract

We estimate the number of exceptional slopes for hyperbolic 3-manifolds with a torus boundary component and at least one other boundary component.

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1. Introduction

Consider a compact, connected, and orientable 3-manifold $M$ with a torus boundary component $\partial_0 M$. A slope $r$ is the isotopy class of an essential simple closed curve on $\partial_0 M$. As usual, we denote by $M(r)$ the 3-manifold obtained from $M$ by $r$-Dehn filling, that is, by attaching a solid torus $J$ to $M$ along $\partial_0 M$ in such a way that $r$ bounds a disk in $J$. For two slopes $r_1, r_2$ on $\partial_0 M$, $\Delta(r_1, r_2)$ denotes the distance between the slopes, which is their minimal geometric intersection number.

We shall say that a compact 3-manifold $M$ is hyperbolic if $M$ with its boundary tori removed admits a complete finite volume hyperbolic structure with a totally geodesic boundary. It is a fundamental result of Thurston [29] that if $M$ is hyperbolic, then $M(r)$ will be hyperbolic for all but a finite number of

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exceptional slopes. A good deal of energy has been devoted to the understanding of exceptional slopes of hyperbolic manifolds over the past twenty five years or so. See [4] or [10] for more details.

One of the main goals of Dehn surgery theory is to find universal bounds on the number of exceptional slopes. Given a hyperbolic 3-manifold $M$, define $\mathcal{E}(M)$ to be the set of exceptional slopes of $M$ on a specified boundary torus. It is well known that if $M$ is the exterior of the figure eight knot, then $|\mathcal{E}(M)| = 10$, and it is conjectured that $|\mathcal{E}(M)| \leq 8$ for other hyperbolic 3-manifolds.

Recall that a closed orientable 3-manifold is hyperbolike if it is irreducible, atoroidal, and is not a small Seifert fibered space [8]. A hyperbolike 3-manifold is hyperbolic modulo Thurston’s Geometrization Conjecture. For a hyperbolic 3-manifold with $\partial M$ a torus, define $\mathcal{E}'(M) = \{ r : M(r) \text{ is not hyperbolike} \} \subset \mathcal{E}(M)$. A universal bound on $|\mathcal{E}'(M)|$ was obtained by Bleiler and Hodgson [3]; $|\mathcal{E}'(M)| \leq 24$. Agol [1] and Lackenby [21] greatly improved their estimation; $|\mathcal{E}'(M)| \leq 12$. At the moment this seems to be the best universal bound on $|\mathcal{E}'(M)|$.

For a 3-manifold with a boundary, Thurston’s Geometrization Theorem for Haken manifolds asserts that it is hyperbolic if and only if it is irreducible, boundary irreducible, atoroidal, and anannular. This means that if $M$ is a hyperbolic 3-manifold with a torus boundary component and at least one other boundary component, then $M(r)$ contains an essential surface with a non-negative Euler characteristic for $r \in \mathcal{E}(M)$. For such manifolds, 6 is the maximal observed value for $|\mathcal{E}(M)|$. For example, consider the links in Fig. 1. The first three links are the Whitehead link, the Whitehead sister (or $(-2, 3, 8)$-pretzel) link, the 2-bridge link associated to the rational number $3/10$, and the last one is the link with the property that its exterior is the unique hyperbolic 3-manifold that has 3 Dehn fillings creating a solid torus [2]. Martelli and Petronio [24] showed that their exteriors admit 6 exceptional slopes. They (together with Frigerio [6]) also showed that for any $g \geq 2$ there exist infinitely many hyperbolic 3-manifolds with 6 exceptional Dehn fillings among which three yield a handlebody of genus $g$ and the others yield an annular manifold.

The main result of the present paper is the following.

**Theorem 1.1.** Let $M$ be a hyperbolic 3-manifold with a torus boundary component and at least one other boundary component. Then $|\mathcal{E}(M)| \leq 6$, and any two exceptional slopes have a mutual distance no larger than 4, unless $M$ is the exterior of the Whitehead sister link, in which case there are two exceptional slopes at distance 5.

This follows immediately from the works of Gordon, Luecke, Oh, Qiu, Scharlemann, Wu, and Theorem 1.2. Most parts of the paper are devoted to proving the following three theorems.
**Theorem 1.2.** Let $M$ be a hyperbolic 3-manifold with a torus boundary component $\partial_0 M$ and at least one other boundary component. If $r_1$ and $r_2$ are slopes on $\partial_0 M$ with $\Delta(r_1, r_2) = 5$ such that both $M(r_1)$ and $M(r_2)$ are toroidal, then $M$ is the exterior of the Whitehead sister link.

This gives a partial answer to [10, Question 5.2]. See also [7]. Recently, Gordon and Wu [17] completely determined hyperbolic 3-manifolds having two toroidal Dehn fillings at distance 4 or 5.

**Theorem 1.3.** Let $M$ be a hyperbolic 3-manifold with a torus boundary component $\partial_0 M$ and at least one other boundary component. If $r_1$ and $r_2$ are slopes on $\partial_0 M$ such that $M(r_1)$ contains a Klein bottle and $M(r_2)$ is toroidal, then $\Delta(r_1, r_2) \neq 5$.

Let $W$ denote the exterior of the Whitehead link. Let $T$ be one of the components of $\partial W$. Under a standard meridian-longitude coordinate system, we parameterize the slopes on $T$ by $\mathbb{Q} \cup \{\infty\}$ in the usual way. It is well known that $\mathcal{E}(W) = \{\infty, 0, 1, 2, 3, 4\}$. Hence, $M = W(-4)$, which can be described as in Fig. 2(a), is hyperbolic. The shaded surface in Fig. 2(b) is a once punctured Klein bottle having boundary slope 4 on $\partial M$, so it can be extended to a Klein bottle in $M(4)$. The surgery instruction in Fig. 2(c) describes $M(-1)$ and is equivalent to that in Fig. 2(d), where the knot is the figure eight knot. It is also well known that $-4$-surgery on the figure eight knot yields a Klein bottle.

The following theorem asserts that $W(-4)$ is the unique hyperbolic 3-manifold that admits two Dehn fillings at distance 5 yielding Klein bottles. A similar result has been obtained by Matignon and Sayari [25].

**Theorem 1.4.** Let $M$ be a hyperbolic 3-manifold with a torus boundary component $\partial_0 M$. If $r_1$ and $r_2$ are slopes on $\partial_0 M$ with $\Delta(r_1, r_2) = 5$ such that both $M(r_1)$ and $M(r_2)$ contain a Klein bottle, then $M$ is homeomorphic to $W(-4)$.

**Corollary 1.5.** Let $M$ be a hyperbolic 3-manifold with a torus boundary component $\partial_0 M$. If $r_1$ and $r_2$ are slopes on $\partial_0 M$ such that both $M(r_1)$ and $M(r_2)$ contain a Klein bottle, then either

1. $\Delta(r_1, r_2) \leq 4$; or
2. $\Delta(r_1, r_2) = 5$ and $M$ is homeomorphic to $W(-4)$; or
3. $\Delta(r_1, r_2) = 6$ and $M$ is homeomorphic to $W(-2)$; or
4. $\Delta(r_1, r_2) = 8$ and $M$ is homeomorphic to either $W(-1)$ or $W(5)$.

**Proof.** This follows immediately from [9] and Theorem 1.4. See also [28, Theorem 1.2].

We remark that there are infinitely many hyperbolic 3-manifolds having two Dehn fillings at distance 4 which yield Klein bottles. See [22].
Assume that \( M \) is the exterior of the Whitehead sister link. Let \( T \) be the boundary torus of \( M \) corresponding to the unknot component of the link. Then the exceptional slopes on \( T \) are given by \( \mathcal{E}(M) = \{\infty, 2, 3, 7/2, 11/3, 4\} \) under the standard slope parametrization. (In fact, there is an automorphism of \( M \) interchanging two boundary tori, so \( |\mathcal{E}(M)| \) is independent of the choice of \( T \).) Note that \( \Delta(r_1, r_2) < 5 \) for \( r_1, r_2 \in \mathcal{E}(M) \) unless \( \{r_1, r_2\} = \{2, 11/3\} \). Hence, if \( r_1 \) and \( r_2 \) are slopes on \( T \) with \( \Delta(r_1, r_2) = 5 \) such that \( M(r_1) \) contains a Klein bottle and \( M(r_2) \) is toroidal, then \( r_1, r_2 \in \mathcal{E}(M) \) and \( \{r_1, r_2\} = \{2, 11/3\} \). However, from [15, Fig. 7.5] or [24, Table 6], one can see that \( M(2) \) and \( M(11/3) \) are homeomorphic, so neither contains a Klein bottle by Theorem 1.4. Therefore, Theorem 1.3 can be reformulated as the following weak version, which is more suitable to our method of proof.

**Theorem 1.3'**. Let \( M \) be a hyperbolic 3-manifold with a torus boundary component \( \partial_0 M \) and at least one other boundary component. If \( r_1 \) and \( r_2 \) are slopes on \( \partial_0 M \) with \( \Delta(r_1, r_2) = 5 \) such that \( M(r_1) \) contains a Klein bottle and \( M(r_2) \) is toroidal, then \( M \) is the exterior of the Whitehead sister link.

Here is an outline of the paper. We assume \( \Delta(r_1, r_2) = 5 \), and shall first prove for Theorem 1.4 that \( M \) is homeomorphic to \( W(-4) \), and then prove for Theorems 1.3' and 1.2 that \( M \) is homeomorphic to the Whitehead sister link exterior. Finally, we prove Theorem 1.1. We adapt the combinatorial techniques developed by Gordon and others to prove the theorems. Our main tool in this paper is the concept of an \( x\)-face, which was earlier exploited in [23].

As usual, the intersection of the involved surfaces gives a pair of labelled graphs. See [5]. Section 2 gives some definitions and prepares basic facts about the graph pair. The orientability of \( M \) defines signs of the edges of the graphs. In Section 3 (resp. 5), we assume one surface is a Klein bottle (resp. a torus), and estimate the number of positive (or negative) edge endpoints around each vertex of the graph on the surface, which will be used to prove Theorems 1.4, 1.3' and 1.2 in Sections 4, 6 and 7, respectively.

2. Preliminaries

Hereafter, we assume that \( M \) is a hyperbolic 3-manifold with a torus boundary component \( \partial_0 M \). For two slopes \( r_1, r_2 \) on \( \partial_0 M \), we suppose that both \( M(r_1) \) and \( M(r_2) \) contain an essential torus or a Klein bottle.

To prove Theorems 1.2, 1.3' and 1.4, we assume \( \Delta(r_1, r_2) = 5 \) throughout the paper. We shall prove the theorems in reverse order. Throughout, we shall use the indices \( \alpha \) and \( \beta \) to denote 1 and 2, with the convention that, when they are used together, \( \{\alpha, \beta\} = \{1, 2\} \).

**Lemma 2.1.** \( M(r_\alpha) \) is irreducible.

**Proof.** This follows from [26,27,30].

Let \( \hat{P}_\alpha \) be an essential torus or a Klein bottle in \( M(r_\alpha) \) with \( n_\alpha = |\hat{P}_\alpha \cap J_\alpha| \) minimal over all such surfaces (here, \( n_\alpha > 0 \) by the hyperbolicity of \( M \)). It is natural to put the following assumption throughout the paper to simplify the arguments.

**Assumption.** If \( M(r_\alpha) \) contains a Klein bottle \( \hat{P}_\alpha' \) and an essential torus \( \hat{P}_\alpha'' \), both realizing the minimal value of \( n_\alpha \) over all such surfaces, then we will assume that \( \hat{P}_\alpha \) is the Klein bottle \( \hat{P}_\alpha' \).
The surface $\hat{P}_\alpha$ meets the attached solid torus $J_\alpha$ in $n_\alpha$ meridional disks $v_1^\alpha, v_2^\alpha, \ldots, v_n^\alpha$, numbered successively along $J_\alpha$. On occasion, we will consider the indices $1, 2, \ldots, n_\alpha$ modulo $n_\alpha$, and omit the superscript $\alpha$ from $v_x^\alpha$ when it is understood in the context.

Let $P_\alpha = \hat{P}_\alpha \cap M$. By [23, Lemma 2.1], $P_\alpha$ is incompressible and boundary incompressible. We may assume that $P_1$ and $P_2$ intersect transversely and minimally. Then no circle component of $P_1 \cap P_2$ bounds a disk in $P_1$ or $P_2$, and no arc component is boundary parallel in $P_1$ or $P_2$.

Let $G_\alpha$ be the graph on $\hat{P}_\alpha$ with $v_x$ as (fat) vertices and the arc components of $P_1 \cap P_2$ as edges. There is no trivial loop in $G_\alpha$, since $P_\beta$ is boundary incompressible. For an edge of $G_\alpha$, if its endpoint lies in $\partial v_x^\alpha \cap \partial v_y^\beta$, then the point has label $y$ at $v_x^\alpha$ and label $x$ at $v_y^\beta$. An edge with label $x$ at its one endpoint is called an $x$-edge, and is called an $(x, y)$-edge if it has label $y$ at the other endpoint. An edge is said to be level if its endpoints have the same label.

We can give a sign to each edge of $G_\alpha$ as follows. Orient the boundary circles of $P_\alpha$ so that they are mutually homologous on $\partial_0 M$. Any edge $e$ in $G_\alpha$ has a rectangular neighborhood $R$ in $P_\alpha$, where two opposite edges of $R$ are contained in two components (possibly equal) of $\partial P_\alpha$. If $\partial R$ can be oriented to agree with compatible orientations of the two components of $\partial P_\alpha$, then $e$ is positive. Otherwise, it is negative. See Fig. 3. Then we have the following rule, which is a natural generalization of the usual parity rule [5].

**Parity rule.** An edge in one graph is positive if and only if it is negative in the other.

When $\hat{P}_\alpha$ is a torus, each vertex is given a sign, depending on whether the core of $J_\alpha$ passes $\hat{P}_\alpha$ from the positive side or the negative side at the vertex. A positive edge connects vertices of the same sign, while a negative one connects vertices of opposite signs.

**Lemma 2.2.** There are no two edges which are parallel in both graphs.

**Proof.** This is [9, Lemma 2.1]. □

Let $G_\alpha^+$ denote the subgraph of $G_\alpha$ consisting of all vertices and all positive edges of $G_\alpha$. Also, $G_\alpha^+(x)$ denotes the subgraph of $G_\alpha^+$ consisting of all vertices and all $x$-edges of $G_\alpha^+$ for a label $x$. A disk face of $G_\alpha^+(x)$ is called an $x$-face. That is, an $x$-face is a disk in $G_\alpha$ bounded by positive $x$-edges. If an $x$-face is a face of $G_\alpha$, then its boundary is called a Scharlemann cycle, and the face is called a Scharlemann cycle face. We remark that the edges of a Scharlemann cycle have the same label pair at their endpoints. In particular, a Scharlemann cycle of length 2 is called an $S$-cycle. A triple $\{e_1, e_2, e_3\}$ of mutually parallel positive edges in succession is called a generalized $S$-cycle if $e_2$ is level and $n_\beta \geq 3$. If a Scharlemann cycle $\sigma$ is surrounded by a cycle $\tau$, that is, each edge of $\sigma$ is immediately parallel to an edge of $\tau$, then $\tau$ is called an extended Scharlemann cycle. The reduced graph $\overline{G}_\alpha$ of $G_\alpha$ is obtained from $G_\alpha$ by amalgamating each family of parallel edges into a single edge.
Lemma 2.3. Suppose that \( \hat{P}_\alpha \) is a Klein bottle. If \( n_\alpha \geq 2 \), then \( G_\beta \) satisfies the following.

1. \( G_\beta \) cannot contain a Scharlemann cycle.
2. If \( n_\alpha \geq 3 \), then \( G_\beta \) cannot contain a generalized S-cycle.
3. Any family of parallel positive edges contains at most \( n_\alpha/2 + 1 \) edges.
4. Any family of parallel negative edges contains at most \( n_\alpha \) edges.
5. At most two labels can be labels of positive level edges.

Proof. For (1) and (2), see [20, Lemma 5.1]. (3) If \( n_\alpha \geq 3 \), see also [20, Lemma 5.1]. Assume \( n_\alpha = 2 \). Suppose that there is a family of at least 3 parallel positive edges in \( G_\beta \). Since \( G_\beta \) cannot contain a Scharlemann cycle, all the edges in the family are level. Then there would be two edges which are parallel in both graphs, contradicting Lemma 2.2. (4) Let \( e_1, e_2, \ldots, e_{n_\alpha}, e'_1 \) be mutually parallel edges in \( G_\beta \) such that \( e_i \) has label \( i \) at one vertex for \( i = 1, 2, \ldots, n_\alpha \) and \( e'_1 \) has label 1 at the same vertex. Then a permutation \( \rho \) is defined on the set of labels \( \{1, 2, \ldots, n_\alpha\} \) such that \( \rho(i) \) is the label of the endpoint of \( e_i \) at the other vertex. The permutation \( \rho \) is not the identity, otherwise \( e_1 \) and \( e'_1 \) would be parallel loops in \( G_\alpha \), contradicting Lemma 2.2. For each orbit \( \theta \) of the permutation \( \rho \), the edges \( e_i \) belonging to \( \theta \) define a cycle \( C_\theta \) in \( \hat{P}_\alpha \), which is essential by [9, Lemma 2.3]. Let \( C_{\theta_0} \) be the cycle containing the edge \( e_1 \). Then \( (C_{\theta_0} \setminus e_1) \cup e'_1 \) forms an inessential circle in \( \hat{P}_\alpha \), since \( e_1 \) and \( e'_1 \) are not parallel in \( G_\alpha \) by Lemma 2.2. This is impossible by [9, Lemma 2.3]. (5) Let \( e \) be a positive level edge in \( G_\beta \) with label \( x \). Then \( e \) is a negative loop edge at vertex \( v_x \) of \( G_\alpha \), so \( N(e \cup v_x) \) is a Möbius band in \( \hat{P}_\alpha \). Since a Klein bottle contains at most two disjoint Möbius bands, the result follows.

Lemma 2.4. Suppose that \( \hat{P}_\alpha \) is a torus. Then \( G_\beta \) satisfies the following:

1. If \( G_\beta \) contains a Scharlemann cycle, then \( \hat{P}_\alpha \) is separating.
2. The edges of a Scharlemann cycle of \( G_\beta \) do not lie in a disk in \( \hat{P}_\alpha \).
3. If \( n_\alpha \geq 3 \), then \( G_\beta \) cannot contain an extended Scharlemann cycle.
4. If \( n_\alpha \geq 3 \), then any family of parallel positive edges contains at most \( n_\alpha/2 + 1 \) edges. Moreover, if the family contains exactly \( n_\alpha/2 + 1 \) edges, then the family has an S-cycle.
5. If all vertices of \( G_\alpha \) do not have the same sign, and if \( n_\alpha \geq 3 \), then any family of parallel negative edges contains at most \( n_\alpha \) edges.
6. If \( n_\alpha \geq 4 \), then any family of parallel negative edges contains at most \( 2n_\alpha \) edges.

Proof. (1), (2) and (3) are [15, Lemma 2.2(4), (5) and (6)]. (4) Any family of parallel positive edges contains at most \( n_\alpha/2 + 2 \) edges by [30, Lemma 1.4]. Moreover, if the family contains \( n_\alpha/2 + 2 \) edges, then the family contains two S-cycles on disjoint label pairs. Then the argument as in the proof of [12, Lemma 3.10] gives a Klein bottle in \( M(r_\alpha) \), contradicting our assumption. (5) follows from [15, Lemma 2.3(1)] and Lemma 2.2. (6) is [9, Corollary 5.5].

Lemma 2.5. Assume that \( \hat{P}_\alpha \) is a Klein bottle with \( n_\alpha = 2 \). Also, assume that \( \hat{P}_\beta \) is either a Klein bottle with \( n_\beta \geq 2 \) or a separating torus with \( n_\beta \geq 4 \). Then

1. there are no three positive level x-edges at a vertex in \( G_\beta \) for each \( x = 1, 2 \); and
2. there are no two consecutive pairs of parallel positive edges at a vertex in \( G_\beta \).

Proof. The proofs of [20, Lemma 6.2(i) and 6.3] remain valid here once we note that any family of negative loops in \( G_\alpha \) cannot contain more than \( n_\beta \) edges.
Lemma 2.6. Let \( \sigma \) be a Scharlemann cycle in \( G_\alpha \) of length 2 or 3. Then the edges of \( \sigma \) lie in an essential annulus in \( \hat{P}_\beta \).

Proof. Note that \( \hat{P}_\beta \) is a torus. The result then follows from [12, Lemma 3.7]. \( \square \)

Suppose that \( \hat{P}_\alpha \) is a Klein bottle. Recall that an edge in \( G_\beta \) is level if its endpoints have the same label, say, \( x \). Then, the corresponding edge in \( G_\alpha \) is a loop at the vertex \( v_\alpha \), orientation-reversing or orientation-preserving according to whether the edge is positive or negative in \( G_\beta \). In the former case, the vertex \( v_\alpha \) is referred to as a level vertex.

Suppose that \( \hat{P}_\alpha \) is a torus. If \( G_\beta \) has a Scharlemann cycle \( \sigma \) with label \( x \), then the vertex \( v_\alpha \) of \( G_\alpha \) is called a Scharlemann vertex or especially an \( S \)-vertex if \( \sigma \) is an \( S \)-cycle.

Let \( G \) be a graph on a surface \( S \). Let \( V(G) \), \( E(G) \), and \( F(G) \) denote the number of vertices, edges, and disk faces of \( G \), respectively. Then \( F(G) \geq E(G) - V(G) + \chi(S) \).

Lemma 2.7. Any vertex of \( G_\alpha \), which is neither a level vertex nor a Scharlemann vertex, has at most \( 3n_\beta - 1 \) negative edge endpoints. Equivalently, it has at least \( 2n_\beta + 1 \) positive edge endpoints.

Proof. Let \( v_\alpha \) be a vertex of \( G_\alpha \) which is neither a level vertex nor a Scharlemann vertex. Assume that \( v_\alpha \) has at least \( 3n_\beta \) negative edge endpoints. Then we have \( E(G_\beta^+(x)) \geq 3n_\beta \) by the parity rule.

Let \( V = V(G_\beta^+(x)), E = E(G_\beta^+(x)), \) and \( F = F(G_\beta^+(x)) \). Then \( V = n_\beta, E \geq 3n_\beta, \) and \( F \geq E - V + \chi(\hat{P}_\beta) = E - V \). Any disk face in \( G_\beta^+(x) \) has at least 3 sides by Lemmas 2.3(2) and 2.4(3). So, \( 2E \geq 3F \geq 3(E - V) \), giving \( E \leq 3V = 3n_\beta \). Hence, \( E = 3n_\beta \) and the above inequalities are all equalities. In particular, all faces in \( G_\beta^+(x) \) are 3-sided disk faces, which fill up the graph \( G_\beta^+(x) \). Then \( G_\beta^+ = G_\beta \) and hence we can conclude that \( G_\beta \) contains \( 5n_\beta \) positive \( x \)-edges. This contradicts the fact that \( E = 3n_\beta \). \( \square \)

Lemma 2.8. Suppose that \( \hat{P}_\beta \) is either a Klein bottle with \( n_\beta \geq 2 \) or a separating torus with \( n_\beta \geq 4 \). Then any two edges in \( G_\alpha \) bounding a bigon disk in \( \hat{P}_\alpha \) are parallel.

Proof. Suppose that two edges in \( G_\alpha \) bound a bigon disk \( D \subset \hat{P}_\alpha \). Assume that \( D \) has vertices of \( G_\alpha \) in its interior. Otherwise we are done.

Claim. Any vertex in \( \text{Int} D \) has valency at least 6 as a vertex of \( \tilde{G}_\alpha \).

Proof. Assume for contradiction that a vertex \( v_\alpha \) in \( \text{Int} D \) has valency at most 5. Notice that \( v_\alpha \) is neither a level vertex nor a Scharlemann vertex, since the edges incident to it lie in a disk.

Assume that \( \hat{P}_\beta \) is a Klein bottle with \( n_\beta = 2 \). Then, by Lemma 2.3(3), (4), \( v_\alpha \) is incident to exactly 5 pairs of parallel edges, among which at least 3 pairs are pairs of positive edges by Lemma 2.7. Then there are two consecutive pairs of parallel positive edges at \( v_\alpha \), contradicting Lemma 2.5(2).

Assume that \( \hat{P}_\beta \) is either a Klein bottle with \( n_\beta \geq 3 \) or a separating torus with \( n_\beta \geq 4 \). Let \( k \) be the number of families of parallel positive edges incident to \( v_\alpha \). Since any family of parallel positive edges contains at most \( n_\beta/2 + 1 \) by Lemmas 2.3(3) and 2.4(4), and since any family of parallel negative edges contains at most \( n_\beta \) edges by Lemmas 2.3(4) and 2.4(5), by counting edge endpoints around \( v_\alpha \), we have

\[
5n_\beta \leq k(n_\beta/2 + 1) + (5 - k)n_\beta.
\]

Equivalently, \((n_\beta/2 - 1)k \leq 0\), so \( k = 0 \). This contradicts Lemma 2.7. \( \square \)
Let $\overline{G}_D$ be the restriction of $\overline{G}_\alpha$ to the disk $D$. Then any vertex of $\overline{G}_D$ in $\text{Int} \ D$ has valency at least 6 by the above claim, and $\overline{G}_D$ has two vertices on $\partial D$. Taking two copies of $(D, \overline{G}_D)$ and gluing them along their boundaries, we get a graph $\Gamma$ on a 2-sphere.

Let $V = V(\Gamma)$, $E = E(\Gamma)$, and $F = F(\Gamma)$. Then $F \geq E - V + \chi(\text{sphere}) = E - V + 2$. Since all but two vertices on $\partial D$ have valency at least 6, $2E \geq 6(V - 2) + 2 \cdot 2$, giving $E \geq 3V - 4$. On the other hand, any disk face has at least 3 sides, so $2E \geq 3F \geq 3(E - V + 2)$, giving $E \leq 3V - 6$. The two inequalities conflict. □

Note that if $f$ is a disk face of $G_\alpha$, then $\partial f$ consists of an alternating sequence of corners (subarcs of $\partial P_\alpha$) and edges of $G_\alpha$. To each corner, we associate an abstract interval $(x, x + 1)$ if its endpoints are labeled $x$ and $x + 1$. By abuse of terminology we shall refer to the abstract interval as a corner. Let $C(f)$ denote the set of corners appeared in $\partial f$. A disk face $f$ in $G_\alpha$ is two-cornered if $|C(f)| = 2$ and its edges are positive. For example, the 3-gon in the configuration illustrated in Fig. 4(a) is a two-cornered face with corners $\{(1, 2), (n_\alpha, 1)\}$.

3. Positive edge endpoints of the graphs on Klein bottles

The goal of this section is to prove Proposition 3.4; assuming one surface is a Klein bottle, we estimate the number of positive edge endpoints around each vertex of the graph on this surface. To do this, we assume that $\hat{P}_\alpha$ is a Klein bottle throughout this section.

We prepare the following three lemmas, in which we investigate $x$-faces in $G_\beta$.

**Lemma 3.1.** Assume $n_\alpha \geq 3$. For any label $x$ of $G_\beta$, any $x$-face in $G_\beta$ contains at least one level edge in its interior. Moreover, the labels of such level edges differ from $x$.

**Proof.** Although label $x$ may be the label of a positive level edge in $G_\beta$, the proof of [23, Proposition 3.1] works in our context to show that any $x$-face contains at least one level edge in its interior. The label of such a level edge is definitely different from $x$. □

Let $X$ be a thin regular neighborhood of $\hat{P}_\alpha$ in $M(r_\alpha)$, and set $Y = M(r_\alpha) - \text{Int} \ X$. Then $\hat{P}'_\alpha = \partial X$ is a torus that meets the solid torus $J_\alpha$ in $2n_\alpha$ meridian disks $v'_1, v'_2, \ldots, v'_{2n_\alpha}$, where two disks $v'_{2i-1}, v'_{2i}$ are assumed to cut off a 1-handle from $J_\alpha$ containing $v'_i$. (Recall that the vertices of $G_\alpha$ are $v_1, v_2, \ldots, v_{n_\alpha}$.) As done in Section 2, we obtain a labeled graph pair $G'_\alpha, G'_\beta$ from the surfaces $P'_\alpha = M \cap \hat{P}'_\alpha$ and $P_\beta'$.
by taking arc components of their intersection as edges. Here, $G'_\alpha$ double-covers $G_\alpha$ and $G'_\beta$ is obtained from $G_\beta$ by thickening each edge of $G_\beta$ and then taking boundary edges of the resulting bigon.

**Lemma 3.2.** Assume $n_\alpha \geq 3$. Let $x$ be a label of $G_\beta$ which is not a label of a positive level edge. Then any $x$-face in $G_\beta$ has at least 4 sides.

**Proof.** Assume for contradiction that there is an $x$-face $D$ in $G_\beta$ with at most 3 sides. By Lemmas 2.3(2) and 3.1, $D$ must be a 3-sided face with a level edge in its interior. Consider the graph $\Gamma$ obtained by restricting $G_\beta$ to $D$, that is, $\Gamma = G_\beta \cap D$. Notice that the faces of $\Gamma$ consist of a single 3-gon and bigons. Since $\Gamma$ cannot contain a generalized $S$-cycle, any level edge appears in the boundary of the 3-gon. Thus $\Gamma$ contains one or two level edges. We distinguish two cases.

**Case I.** $\Gamma$ contains a single level edge.

We may assume that the level edge has label 1 at its endpoints. Also, we may assume that $\Gamma$ contains a configuration as shown in Fig. 4(a). By slightly abusing our notation, we denote that configuration by $\Gamma$.

**Claim.** There is an orientation-reversing circle $c$ in $\hat{P}_\alpha$ which does not meet any vertex of $G_\alpha$ and any edge of $G_\alpha$ corresponding to an edge of $\Gamma'$.

**Proof.** The graph $\Gamma$ has two $(2, n_\alpha)$-edges, which form an orientation-preserving cycle of length 2 in $G_\alpha$. Thus, we see one of the configurations in Fig. 4(b), where the level 1-edge of $\Gamma$ forms an orientation-reversing loop at vertex $v_1$. Here, the sides of each rectangle are identified as indicated in the figure to form a Klein bottle. Let $c$ be the circle defined by the bottom edge of the rectangle. We may assume that $c$ is disjoint from the vertices of $G_\alpha$. Then $c$ is a desired circle. $\square$

Consider the graph $\Gamma' = G'_\beta \cap D$. In $\Gamma'$, the level edge of $\Gamma'$ goes to an $S$-cycle face $f$ on label pair $\{1, 2\}$, as shown in Fig. 5(a). Let $g_1, g_2$ be the bigon and the 3-gon next to $f$, respectively.

By the above claim, there is an orientation-reversing circle $c$ in $P_\alpha$ which is disjoint from the edges coming from $\Gamma'$. Collecting $I$-fibers over $c$ in $X$, we can obtain a properly embedded Möbius band $B_1$ in $X$ such that $\partial B_1 = c$. Notice that $B_1 \cap K_\alpha = \emptyset$, where $K_\alpha$ is the core of the attached solid torus $J_\alpha$. The edges of $f$ along with $\partial B_1$ partition $\hat{P}'_\alpha = \text{Int}(v'_1 \cup v'_2)$ into two annuli $A, A'$. See Fig. 6(a). There are two $(3, 2n_\alpha)$-edges in $\Gamma''$, so the vertices $v'_3, v'_{2n_\alpha}$ lie in the same annulus, say, $A$. By the construction of $\hat{P}'_\alpha$, each annulus contains $n_\alpha - 1$ vertices of $G'_\alpha$ in its interior, i.e., $|A \cap K_\alpha| = |A' \cap K_\alpha| = n_\alpha - 1$. 
Let $H_j$ be the annulus in $\partial J_\alpha$ cobounded by $\partial v'_j$ and $\partial v'_{j+1}$ modulo $2n_\alpha$. Let $R, R'$ be the two components of $H_1$ cut along the two corners of $f$. We may assume that $R \cap A$ is an arc in $\partial v'_1$. Then $B_2 = f \cup R \cup v'_1$ becomes a properly embedded Möbius band in $X$ after its interior is pushed into $X$. Let $F = (A - \text{Int}(v'_3 \cup v'_{2n_\alpha})) \cup (H_2 \cup H_{2n_\alpha})$, isotope $g_1, g_2$ in $F$ so that the edges of $g_1, g_2$ lie in the interior of $A$, and compress $F$ using the disks $g_1, g_2$. It is not hard to see that the resulting surface is an annulus $B \subset Y$ with $\partial B = \partial B_1 \cup \partial B_2$. Then $\hat{P}_\alpha = B \cup B_1 \cup B_2$ is a new Klein bottle in $M(r_\alpha)$ that intersects $K_\alpha$ fewer than $n_\alpha$ times because $|\hat{P}_\alpha \cap K_\alpha| = |B \cap K_\alpha| + |B_1 \cap K_\alpha| + |B_2 \cap K_\alpha| = (n_\alpha - 3) + 0 + 1 = n_\alpha - 2$. This contradicts the choice of $\hat{P}_\alpha$.

Case II. $\Gamma$ contains two level edges.

We may assume that the labels at their endpoints are 1 and 2, respectively. Consider $\Gamma'' = G'_\beta \cap D$ again. In $\Gamma''$, these two level edges go to two $S$-cycle faces $f_1$ and $f_2$ on label pairs $\{1, 2\}$ and $\{3, 4\}$, respectively. Let $g_1, g_2, g_3$ be the three disk faces next to $f_1$ and $f_2$ as shown in Fig. 5(b). The edges of $f_1$ and $f_2$ partition $\hat{P}'_\alpha - \text{Int}(v'_1 \cup v'_2 \cup v'_3 \cup v'_4)$ into two annuli $A, A'$, where $A$ is chosen to contain the vertices $v'_5$ and $v'_{2n_\alpha}$. See Fig. 6(b). Notice that $|A \cap K_\alpha| = |A' \cap K_\alpha| = n_\alpha - 2$.

Let $H_j$ be as above. Let $F = (A - \text{Int}(v'_5 \cup v'_{2n_\alpha})) \cup (H_2 \cup H_3 \cup H_{2n_\alpha})$. The circles $\partial g_1, \partial g_2, \partial g_3$ can be isotoped in $F$ so that the edges of $g_1, g_2, g_3$ lie in the interior of $A$. Cutting $F$ using the disks $g_1, g_2, g_3$ clearly gives an annulus $B \subset Y$. As before, we obtain two properly embedded Möbius bands $B_1, B_2 \subset X$ from $f_1$, $f_2$ and $H_1, H_2$ such that $\partial B = \partial B_1 \cup \partial B_2$ and $|B_1 \cap K_\alpha| = |B_2 \cap K_\alpha| = 1$. Then $\hat{P}_\alpha = B \cup B_1 \cup B_2$ is a new Klein bottle that intersects $K_\alpha$ fewer than $n_\alpha$ times because $|\hat{P}_\alpha \cap K_\alpha| = |B \cap K_\alpha| + |B_1 \cap K_\alpha| + |B_2 \cap K_\alpha| = (n_\alpha - 4) + 1 + 1 = n_\alpha - 2$. This also contradicts the choice of $\hat{P}_\alpha$. 

Lemma 3.3. Suppose that $n_\alpha \geq 3$ and that $G_\beta$ contains a positive level $x$-edge. Then $G_\beta$ cannot contain an $x$-face.

Proof. Assume for contradiction that $G_\beta$ contains an $x$-face $D$. By Lemma 3.1, $D$ contains level edges in its interior, and the labels at their endpoints, which differ from $x$, are the same by Lemma 2.3(5). Without loss of generality, we may assume that the label of the level edges in $\text{Int}D$ is 2.

By [19, Lemma 3.6], $D$ contains a pair of two-cornered faces $f_1, f_2$ with corners $(1, 2)$ and $(2, 3)$, sharing a level 2-edge, $e$, on their boundaries, such that at least one of them contains only one level 2-edge. Note that neither $f_1$ nor $f_2$ has a level $x$-edge; otherwise $x = 1$ or 3 and then $f_1$ or $f_2$ would contain
an \((n_\alpha, 1)\)-corner or \((3, 4)\)-corner (or, a \((3, 1)\)-corner when \(n_\alpha = 3\)). Each edge in \(G_\alpha\) corresponding to an edge of \(f_1\) or \(f_2\) runs between two (possibly equal) of the vertices \(v_1, v_2, v_3\) of \(G_\alpha\).

In \(G'_\beta\), the level edge \(e\) goes to an \(S\)-cycle face \(f'\) on label pair \(\{3, 4\}\), while the two-cornered faces \(f_1, f_2\) go to two-cornered faces \(f'_1, f'_2\) with corners \((2, 3)\) and \((4, 5)\). For example, see Fig. 7. By our hypothesis, there is a level \(x\)-edge in \(G'_\beta\), which goes to an \(S\)-cycle face \(g'\) on label pair \(\{2x - 1, 2x\}\).

The edges of \(f', g'\) partition \(\hat{\mathcal{P}}_\alpha - \text{Int}(v'_3 \cup v'_4 \cup v'_{2x-1} \cup v'_{2x})\) into two annuli \(A, A'\). By the construction of \(\hat{\mathcal{P}}_\alpha\), each annulus contains \(n_\alpha - 2\) vertices of \(G'_\alpha\), i.e. |\(A \cap K_\alpha\)| = |\(A' \cap K_\alpha\)| = \(n_\alpha - 2\), where \(K_\alpha\) is the core of the attached solid torus \(J_\alpha\). We may assume that \(A\) contains the vertex \(v'_1\). Then we see one of configurations in Fig. 8 (by the existence of \((2, 5)\)-edges in \(G'_\beta\)) according to whether \(x = 1\) or not. The edges in \(G'_\alpha\) corresponding to the edges of \(f'_1, f'_2\) are contained in \(A\).

Let \(H_j\) be the annulus in \(\partial J_\alpha\) cobounded by \(\partial v'_j\) and \(\partial v'_{j+1}\) modulo \(2n_\alpha\). Let \(R, R'\) (resp. \(S, S'\)) be two components of \(H_3\) (resp. \(H_{2x-1}\)) cut along the two corners of \(f'\) (resp. \(g'\)). We may assume that \(R \cap A\) (resp. \(S \cap A\)) is an arc in \(\partial v'_3\) (resp. \(\partial v'_{2x-1}\)). See Fig. 8.

Let \(B_1 = f' \cup R \cup v'_3\), and let \(B_2 = g' \cup S\) if \(x = 1\) or \(B_2 = g' \cup S \cup v'_{2x-1}\) if \(x \neq 1\). Then \(B_1\) and \(B_2\) become properly embedded Möbius bands in \(X\) after their interiors are slightly pushed into \(X\). Here, |\(B_1 \cap K_\alpha\)| = 1, and |\(B_2 \cap K_\alpha\)| = 0 if \(x = 1\) or |\(B_2 \cap K_\alpha\)| = 1 if \(x \neq 1\).

The surface \(F = (A - \text{Int}(v'_3 \cup v'_4 \cup v'_5)) \cup (H_2 \cup H_3)\) is a twice-punctured surface of genus 2, and |\(F \cap K_\alpha\)| = \(n_\alpha - 3\) or \(n_\alpha - 4\) according to whether \(x = 1\) or not. Isotope \(\partial f'_1, \partial f'_2\) in \(F\) so that the edges of \(f'_1, f'_2\) are contained in the interior of \(A\). The two circles \(\partial f'_1, \partial f'_2\) are non-separating in \(F\), since the vertices of \(f'_1, f'_2\) have the same sign. The second paragraph in the proof of [19, Lemma 3.9] applies here to show that \(\partial f'_1, \partial f'_2\) are not parallel in \(F\).

Compress \(F\) using the disks \(f'_1, f'_2\). The resulting surface is an annulus or a disjoint union of an annulus and a torus according to whether \(F - \partial f'_1 \cup \partial f'_2\) is connected or not. In any case, the surgery gives an annulus \(B \subset Y\) such that |\(B \cap K_\alpha\)| \(\leq n_\alpha - 3\) if \(x = 1\) or |\(B \cap K_\alpha\)| \(\leq n_\alpha - 4\) if \(x \neq 1\), and \(\partial B = \partial B_1 \cup \partial B_2\). Then \(\tilde{\mathcal{P}}_\alpha = B \cup B_1 \cup B_2\) is a new Klein bottle in \(M(r_\alpha)\) that intersects \(K_\alpha\) fewer than \(n_\alpha\) times because |\(\tilde{\mathcal{P}}_\alpha \cap K_\alpha\)| = |\(B \cap K_\alpha\)| + |\(B_1 \cap K_\alpha\)| + |\(B_2 \cap K_\alpha\)| \(\leq n_\alpha - 2\). This contradicts the choice of \(\mathcal{P}_\alpha\). \(\square\)
Proposition 3.4. Suppose that \( \hat{\mathcal{P}}_\alpha \) is a Klein bottle with \( n_\alpha \geq 3 \). Then the following holds.

1. Any non-level vertex of \( G_\alpha \) has at most \( 2n_\beta - 1 \) negative edge endpoints. Equivalently, it has at least \( 3n_\beta + 1 \) positive edge endpoints.
2. Any level vertex of \( G_\alpha \) has at most \( 2n_\beta \) negative edge endpoints. Equivalently, it has at least \( 3n_\beta \) positive edge endpoints.

Proof. (1) Assume for contradiction that a non-level vertex \( v_x \) of \( G_\alpha \) has at least \( 2n_\beta \) negative edge endpoints. By the parity rule, \( E(G_\alpha^+(x)) \geq 2n_\beta \). Let \( V = V(G_\alpha^+(x)), \ E = E(G_\alpha^+(x)), \ and \ F = F(G_\alpha^+(x)) \). Then \( V = n_\beta, \ E \geq 2n_\beta = 2V \) and \( F \geq E - V \). By Lemma 3.2, any disk face of \( G_\alpha^+(x) \) has at least 4 sides, so we have \( 2E \geq 4F \geq 4(E - V) \), giving \( E \leq 2V \). Therefore, \( E = 2V \) and all inequalities above are equalities. In particular, all the faces of \( G_\alpha^+(x) \) are disk faces with 4 sides and they fill up the graph \( G_\alpha^+(x) \), so \( G_\alpha^+ = G_\beta \). Then all x-edges in \( G_\beta \) are positive, contradicting the fact that \( E = 2n_\beta \).

(2) Assume for contradiction that a level vertex \( v_x \) of \( G_\alpha \) has more than \( 2n_\beta \) negative edge endpoints. By the parity rule, the label \( x \) appears more than \( 2n_\beta \) times in \( G_\beta \) as positive edge endpoints. We remark that among the positive x-edges there may exist level x-edges. However, one sees that there are more than \( n_\beta \) positive x-edges in \( G_\beta \).

Let \( V, \ E, \ F \) be as above. Then \( V = n_\beta, \ E > n_\beta \) and \( F \geq E - V > 0 \). Hence \( G_\beta \) contains an x-face. This contradicts Lemma 3.3. □

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Assume that both \( \hat{\mathcal{P}}_1 \) and \( \hat{\mathcal{P}}_2 \) are Klein bottles.

4.1. Generic case

The following lemma shows that \( G_1 \) or \( G_2 \) has at most two vertices.

Lemma 4.1. \( n_1 \leq 2 \) or \( n_2 \leq 2 \).

Proof. Assume for contradiction that \( n_1, n_2 \geq 3 \). Let \( l_\alpha (\leq 2) \) be the number of level vertices of \( G_\alpha \). Let \( N \) be the number of positive edge endpoints of \( G_1 \). Then by Proposition 3.4, we have

\[
(3n_2 + 1)(n_1 - l_1) + 3n_2l_1 \leq N \leq (2n_2 - 1)(n_2 - l_2) + 2n_1l_2.
\]

Hence, \( n_1n_2 + n_1 + n_2 \leq l_1 + l_2 \leq 4 \), giving

\[
(n_1 + 1)(n_2 + 1) \leq 5.
\]

This is impossible. □

4.2. The case where \( n_\alpha = 1 \)

The reduced graph \( \mathcal{G}_\alpha \) is a subgraph of one of the graphs illustrated in Fig. 9, where each graph has one positive edge \( e_0 \) of multiplicity \( p_0 \) and two negative edges \( e_1 \) and \( e_2 \) of multiplicities \( p_1 \) and \( p_2 \), respectively. We write \( G_\alpha = H_1(p_0, p_1, p_2) \) or \( H_2(p_0, p_1, p_2) \) according to whether \( \mathcal{G}_\alpha \) looks like Fig. 9(a) or (b). Up to homeomorphism of \( \hat{\mathcal{P}}_\alpha, \ H_i(p_0, p_1, p_2) \cong H_i(p_0, p_2, p_1) \) for \( i = 1, 2 \).
Lemma 4.2. If \( n_\alpha = 1 \), then \( n_\beta = 2 \).

**Proof.** Assume \( n_\alpha = 1 \) and \( n_\beta \neq 2 \). Then \( n_\beta \) is even and \( n_\beta \geq 4 \). By Lemma 2.3(3) and (4), we have \( p_0 \leq n_\beta / 2 + 1 \) and \( p_1, p_2 \leq n_\beta \). The total valency of the (unique) vertex \( v \) of \( G_\alpha \) is \( 5n_\beta \); so \( p_0 = n_\beta / 2 \) or \( n_\beta / 2 + 1 \).

If \( p_0 = n_\beta / 2 + 1 \), then the first and last edges in the family of edges represented by \( \tilde{e}_0 \) are level. For any such level edge, there are an odd number of edge endpoints along \( \partial v \) from one endpoint of the level edge to the other endpoint, since \( n_\beta \) is even. From this, one easily sees that \( p_0 = n_\beta / 2 + 1 \) and \( G_\alpha = H_2(p_0, p_1, p_2) \) cannot happen simultaneously. Therefore, there are three possibilities for \( G_\alpha: G_\alpha = H_1(n_\beta / 2, n_\beta, n_\beta), H_1(n_\beta / 2 + 1, n_\beta, n_\beta - 1) \) or \( H_2(n_\beta / 2, n_\beta, n_\beta) \). For each case, we fix the labeling around \( v \) as in Fig. 10.

**Case I.** Assume that \( G_\alpha = H_1(n_\beta / 2, n_\beta, n_\beta) \).

Let \( e_0(i) \) be the edge of \( G_\alpha \) represented by \( \tilde{e}_0 \) with label \( i \) at one end of \( \tilde{e}_0 \) for \( i = 1, 2, \ldots, n_\beta / 2 \). Similarly, let \( e_k(j) \) be the edge of \( G_\alpha \) represented by \( \tilde{e}_k \) with label \( j \) at the end of \( \tilde{e}_k \) on the right side of \( \tilde{e}_0 \) \((k = 1, 2, j = 1, 2, \ldots, n_\beta) \). Note that \( e_0(i) \) is a positive \((i, n_\beta - i + 1)\)-edge, while \( e_k(j) \) is a negative \((j, n_\beta / 2 + j)\)-edge. Hence, \( e_k(j) \) and \( e_k(n_\beta / 2 + j) \) have the same label pair at their endpoints, so they form an orientation-preserving cycle \( \sigma_k(j) \) of length 2 in \( \hat{P}_\beta \). The cycles \( \sigma_k(j) \) are essential in \( \hat{P}_\beta \); otherwise some cycle \( \sigma_k(j_0) \) would bound a bigon with no vertex of \( G_\beta \) in its interior, and then the edges \( e_k(j_0), e_k(n_\beta / 2 + j_0) \) would be parallel in both graphs, contradicting Lemma 2.2. In particular, the cycles \( \sigma_1(j) \) and \( \sigma_2(j) \) are parallel for all \( j \).
The graph $G_\alpha$ contains two 3-gons, which are two-cornered faces with corners $\{(n_\beta, 1), (n_\beta/2, n_\beta/2 + 1)\}$. Two cycles $\sigma_1(1), \sigma_1(n_\beta/2)$ cut off an annulus $A$ from $\hat{P}_\beta$. Here, $e_0(1), e_0(n_\beta/2)$ are spanning arcs of $A$. See Fig. 11. Hence, there are no vertices in the interior of $A$. After slightly enlarging $A$, we may assume that $A$ contains the edges of the 3-gons and only four vertices $v_{n_\beta}, v_1, v_{n_\beta/2}, v_{n_\beta/2+1}$ in its interior. Let $H_1$ (resp. $H_2$) be the annulus in $\partial J_\beta$ cobounded by $\partial v_{n_\beta}$ and $\partial v_1$ (resp. $\partial v_{n_\beta/2}$ and $\partial v_{n_\beta/2+1}$). Let $F = (A - \text{Int}(v_{n_\beta} \cup v_1 \cup v_{n_\beta/2} \cup v_{n_\beta/2+1})) \cup (H_1 \cup H_2)$. Then $F$ is a twice-punctured surface of genus 2.

One 3-gon has two $(n_\beta, 1)$-corners, while the other has one such corner, so their boundary circles are not mutually parallel in $F$. The circles are non-separating in $F$. Hence, compressing $F$ along the 3-gons gives an annulus or a disjoint union of an annulus and a torus, depending on whether $F$ cut along the circles is connected or not. Let $A'$ be the annulus obtained by the surgery. Then $A'$ does not intersect the core of $J_\beta$ and $\partial A' = \partial A$. Replacing $A$ with $A'$ gives a new Klein bottle intersecting the core of $J_\beta$ fewer than $n_\beta$ times. This contradicts the choice of $\hat{P}_\beta$.

Case II. Assume that $G_\alpha = H_1(n_\beta/2 + 1, n_\beta, n_\beta - 1)$.

As above, let $e_0(i)$ be the edge of $G_\alpha$ represented by $\tilde{e}_0$ with label $i$ at one end of $\tilde{e}_0$, and let $e_2(j)$ be the edge of $G_\alpha$ represented by $\tilde{e}_2$ with label $j + 1$ at the end of $\tilde{e}_2$ on the right side of $\tilde{e}_0$, $(1 \leq i \leq n_\beta/2 + 1, 1 \leq j \leq n_\beta - 1)$. Then, $e_0(1)$ and $e_0(n_\beta/2 + 1)$ are positive level edges; so they form disjoint orientation-reversing loops in $\hat{P}_\beta$. For each $j$, $e_2(j)$ is a negative $(j + 1, n_\beta/2 + j + 1)$-edge. In particular, $e_2(n_\beta/2)$ has label pair $[1, n_\beta/2 + 1]$ at its endpoints; so the three edges $e_0(1), e_0(n_\beta/2 + 1), e_2(n_\beta/2)$ together with vertices $v_1, v_{n_\beta/2+1}$ of $G_\beta$ cut $\hat{P}_\beta$ into a disk. Other edges $e_2(j), j \neq n_\beta/2$, must lie in this disk. Observe that two edges $e_2(j)$ and $e_2(n_\beta/2 + j)$ have the same label pair at their endpoints, so they form a cycle in the disk. It follows that for some $j_0$, two edges $e_2(j_0)$ and $e_2(n_\beta/2 + j_0)$ are parallel in both graphs $G_1, G_2$, contradicting Lemma 2.2.

Case III. Assume that $G_\alpha = H_2(n_\beta/2, n_\beta, n_\beta)$.

Consider the two 3-gons in $G_\alpha$. For each 3-gon, its corners have the same pair of labels at their endpoints. Let $f$ be the 3-gon with corner $(n_\beta, 1)$ and $H$ be the annulus in $\partial J_\beta$ cobounded by $\partial v_1$ and $\partial v_{n_\beta}$. Then compressing $(\hat{P}_\beta - \text{Int}(v_{n_\beta} \cup v_1)) \cup \partial v_{n_\beta}$ along $f$ gives a new Klein bottle that intersects the core of $J_\beta$ fewer than $n_\beta$ times. This contradicts the choice of $\hat{P}_\beta$. □

4.3. The case where $n_\alpha = 2$

Note that $E(G_\alpha) = E(G_\beta) = 5n_\beta$. We begin with the following observation.
Lemma 4.3. Assume $n_\beta \geq 2$. Then both vertices of $G_\alpha$ are bases of negative loops.

Proof. Notice that any disk face in $G_\beta^+$ has at least one positive level $x$-edge for $x = 1, 2$ by Lemma 2.3(1). Hence, it is enough to show $F(G_\beta^+) > 0$.

If $n_\beta \geq 3$, then each vertex of $G_\beta$ has at least $3n_\alpha(=6)$ positive edge endpoints by Proposition 3.4. Thus, $E(G_\beta^+) \geq 3n_\beta$ and hence, $F(G_\beta^+) \geq E(G_\beta^+) - V(G_\beta^+) \geq 2n_\beta > 0$.

Assume $n_\beta = 2$. Also, assume $E(G_\beta^+) \leq n_\beta$ (otherwise, $F(G_\beta^+) > 0$). By the parity rule, $E(G_\alpha^+) \geq 4n_\beta = 8$. Any family of parallel positive edges in $G_\alpha$ contains at most 2 edges by Lemma 2.3(3).

Using an Euler characteristic argument, one can easily see that $E(G_\alpha) \leq 6$. In particular, $E(G_\alpha^+) \leq 6$. If $E(G_\alpha^+) = 6$, then $G_\alpha^+ = G_\alpha$ and $E(G_\alpha^+) = 5n_\beta = 10$; so there are 4 pairs of parallel positive edges in $G_\alpha$. If $E(G_\alpha^+) \leq 5$, then there are at least 3 pairs of parallel positive edges in $G_\alpha$, since $E(G_\alpha^+) \geq 8$.

In any case, there are 3 pairs of parallel positive edges incident to some vertex of $G_\alpha$. This contradicts Lemma 2.5(1). □

Assume $n_\beta \geq 2$. Then there are negative loops $e_1$ and $e_2$ of $G_\alpha$ at the vertices $v_1$ and $v_2$, respectively. These edges $e_1$ and $e_2$ cut $P_\alpha$ into an annulus $A$. One boundary circle of $A$ consists of two copies of $e_1$, denoted by $e_1^\pm$, and two arcs on $\partial v_1$, denoted by $v_1^\pm$. The situation is similar for the other boundary circle. See Fig. 12.

Positive edges of $G_\alpha$ lie in the interior of $A$. If we consider $G_\alpha^+$ as a graph on $A$ and $v_1^\pm$, $v_2^\pm$ as its vertices, then we may assume that each edge of $G_\alpha^+$ runs either between $v_1^+$ and $v_2^+$ or between $v_1^-$ and $v_2^-$.

Lemma 4.4. Assume $n_\beta \geq 2$. Then, $E(G_\alpha^+) \geq 2n_\beta$.

Proof. Assume $E(G_\alpha^+) < 2n_\beta$. By the parity rule, $E(G_\beta^+) > 3n_\beta$, and hence $F(G_\beta^+) > 2n_\beta$. As noted before, each disk face of $G_\beta^+$ has at least one positive level 1-edge (or 2-edge). Since any positive level 1-edge can be shared by at most two disk faces of $G_\beta^+$, the number of positive level 1-edges are bigger than $n_\beta$. This contradicts Lemma 2.3(4). □

Lemma 4.5. $n_\beta \leq 2$.

Proof. Assume $n_\beta \geq 3$. By Lemmas 2.3(3) and 4.4, we have $E(G_\alpha^+) \geq 3$. Thus $G_\alpha$ is one of the graphs in Fig. 13, where the solid edges are positive and the dashed edges are negative.
For the first graph in the figure, one vertex of $G_\alpha$ has valency 5. Then each family of parallel edges incident to the vertex contains exactly $n_\beta$ edges, contradicting Lemma 2.3(3).

For the last two graphs, there are only two families of negative edges in $G_\alpha$, each containing at most $n_\beta$ edges by Lemma 2.3(4). Thus, $E(G_\alpha^+) \geq 3n_\beta$. Since each family of parallel positive edges in $G_\alpha$ contains at most $n_\beta/2 + 1$ edges, $E(G_\alpha^+) \leq 3(n_\beta/2 + 1)$ or $E(G_\alpha^+) \leq 4(n_\beta/2 + 1)$ according to whether $G_\alpha$ is the second or third graph in Fig. 13. Thus, we have $3n_\beta \leq 3(n_\beta/2 + 1)$ or $3n_\beta \leq 4(n_\beta/2 + 1)$. The first case is impossible and the second case can happen only if $n_\beta \leq 4$. If $n_\beta = 4$, then each family of parallel positive edges in $G_\alpha$ contains exactly 3 edges and then $G_\alpha$ contains $S$-cycles, contradicting Lemma 2.3(1). If $n_\beta = 3$, then each family of parallel positive edges contains at most 2 edges by Lemma 2.3(3), so $E(G_\alpha^+) \leq 8$, contradicting the fact that $E(G_\alpha^+) \geq 3n_\beta = 9$. □

**Lemma 4.6.** $n_\beta = 1$.

**Proof.** Assume $n_\beta = 2$. Then $E(G_1^+) = E(G_2^+) = 10$. We may assume $E(G_1^+) \geq 5$. Then $E(G_1^+) \geq 3$, since any family of parallel positive edges in $G_1$ contains at most 2 edges. Thus, $G_1$ is one of the graphs in Fig. 13.

For the first graph, there are two consecutive pairs of parallel positive edges at a vertex of $G_1$, since $E(G_1^+) \geq 5$. This contradicts Lemma 2.5(2).

For the second graph, if $E(G_1^+) \geq 6$ then a similar situation as above would occur, contradicting Lemma 2.5(2) again. Thus, $E(G_1^+) = 5$ and $G_1^+$ consists of two pairs of parallel loop edges and one non-loop edge. Then, each vertex of $G_1$ has exactly 5 negative edge endpoints, which must be paired up by negative loops at the vertex. This is absurd.

For the last graph, $E(G_1^+) \leq 6$ by Lemma 2.5(2). First, assume $E(G_1^+) = 5$. Since each vertex of $G_1$ has an even number of negative edge endpoints, $G_1^+$ consists of a pair of parallel loop edges at one vertex and a loop edge at the other vertex and two non-parallel non-loop edges. Then, the pair of parallel loop edges in $G_1^+$ forms an $S$-cycle, contradicting Lemma 2.3(1). Next, assume $E(G_1^+) = 6$. Then, $G_1^+$ consists of two pairs of parallel loop edges and two non-parallel non-loop edges. As above, $G_1$ contains $S$-cycles, a contradiction. □

**4.4. The case where $n_\alpha = 2$ and $n_\beta = 1$**

There are four possibilities for $G_\beta$, as shown in Fig. 14. We claim that $G_\beta$ cannot be any of the last three graphs in the figure. Assume that $G_\beta$ is one of the last three graphs. Then, $G_\beta$ contains a bigon $f$ bounded by negative level edges. See the shaded bigons in Fig. 14. Each of the edges of $f$ forms an
orientation-preserving loop at a vertex of $G_\alpha$, and the edges cut off an annulus $A$ from $P_\alpha$. See Fig. 15. Let $H$ be the annulus in $\partial J_\alpha$ which connects the two vertices $v_1, v_2$ of $G_\alpha$ and intersects $\partial f$. Then, there are four possibilities for the pair $(A \cup H, \partial f)$ as shown in Fig. 15. The annulus $H$ is cut into two disks $D, D'$ by $\partial f$, where we choose $D$ so that $\partial D \cap A \subset \partial D$. For Fig. 15(a), $A \cup D \cup f \cup v_1$ is a Klein bottle in $M(r_\alpha)$ intersecting the core of $J_\alpha$ in one point. For Fig. 15(b), $A \cup D \cup f$ is a Klein bottle in $M(r_\alpha)$ disjoint from the core of $J_\alpha$. For Fig. 15(c), $(P_\alpha - A) \cup D' \cup f$ is a Klein bottle in $M(r_\alpha)$ disjoint from the core of $J_\alpha$. For Fig. 15(d), $(P_\alpha - A) \cup D' \cup f \cup v_1$ is a Klein bottle in $M(r_\alpha)$ intersecting the core of $J_\alpha$ in one point. Any of these cases contradicts the choice of $\hat{P}_\alpha$.

Assume that $G_\beta$ is the first graph in Fig. 14. There are two possibilities for $G_\alpha$, as illustrated in Fig. 16. Here, any two parallel edges of $G_\beta$ form an essential cycle in $G_\alpha$ by Lemma 2.2. Orient the negative edges of $G_\beta$ as shown in Fig. 14. Then the edges of the two bigons of $G_\alpha$ are correspondingly oriented. For any bigon in $G_\alpha$, its boundary circle cannot be oriented to agree with the orientations of its two edges; otherwise, using the argument below [12, Claim 7.5], one could find a properly embedded Möbius band in $M$. Thus, if we label the edges of $G_\beta$ as $a, b, c, d, e$, then up to homeomorphism of $\hat{P}_\alpha$, there are four possibilities for labeling the edges of $G_\alpha$ as shown in Fig. 17. Let $p, q, r, s, t$ be the edge endpoints of $G_\beta$ labeled 1 as in the figure. These points must appear at vertex $v_1$ of $G_\alpha$ in the order of either $p, q, r, s, t$ or $p, r, t, q, s$ along $\partial v_1$, since $\partial v_1$ and $\partial P_\beta$ are circles at distance 5 in the torus $\partial_0 M$. (Consider $\partial_0 M$ as a standardly embedded torus in $S^3$, in which $\partial v_1$ is a meridian curve and $\partial P_\beta$ is either a $(5,1)$-curve or $(5,2)$-curve.) Therefore, the first three labelings of the edges of $G_\alpha$ are not
appropriate, and the edge-correspondence between the graphs $G_\alpha$ and $G_\beta$ is uniquely determined. Then one can verify that the frontier of $N(P_1 \cup P_2 \cup \partial_0 M)$ in $M$ is a sphere. Hence, $M$ is uniquely determined to be $W(-4)$, where $W$ is the exterior of the Whitehead link and the surgery coefficient is parameterized in the usual way.

This completes the proof of Theorem 1.4.

5. Positive edge endpoints of the graphs on tori

Hereafter, we assume that $M$ has at least one boundary component other than $\partial_0 M$. To prove Theorems 1.2 and 1.3', we also assume that $M$ is not the exterior of the Whitehead sister link. The purpose of this section is to prove Propositions 5.2 and 5.12.

Lemma 5.1. $M(r_\alpha)$ is an annular for $\alpha = 1, 2$. 

We shall prove only (2). (The proof of (1) is similar.)

Assume that \( \hat{P}_a \) is a Klein bottle. Then a thin regular neighborhood \( N(\hat{P}_a) \) of \( \hat{P}_a \) in \( M(r_\beta) \) has a torus as its boundary. By the above argument, the torus \( \partial N(\hat{P}_a) \) must be inessential in \( M(r_\beta) \). It follows from the irreducibility of \( M(r_\beta) \) that \( \partial N(\hat{P}_a) \) is boundary parallel in \( M(r_\beta) \). Hence, \( M(r_\beta) \) is annular and \( M \) is the exterior of the Whitehead sister link by [16, Theorem 1.1(3)], contradicting our assumption. \( \square \)

**Proposition 5.2.** Suppose that \( \hat{P}_a \) is a non-separating torus in \( M(r_\alpha) \) with \( n_\alpha \geq 2 \). Then any vertex of \( G_\alpha \) has at most \( n_\beta \) negative edge endpoints. Equivalently, it has at least \( 4n_\beta \) positive edge endpoints.

**Proof.** Assume that a vertex \( v_x \) of \( G_\alpha \) has more than \( n_\beta \) negative edge endpoints. An Euler characteristic argument shows that \( G_\beta \) contains an \( x \)-face, and then it contains a Scharlemann cycle by [18, Proposition 5.1]. This contradicts Lemma 2.4(1). \( \square \)

In the rest of this section, we assume that \( \hat{P}_a \) is a separating torus in \( M(r_\alpha) \). Recall that \( M(r_\alpha) \) does not contain a Klein bottle by our assumption at the beginning of Section 2.

**Lemma 5.3.** Suppose that \( \hat{P}_a \) divides \( M(r_\alpha) \) into two pieces \( X \) and \( Y \), where \( Y \) is chosen to contain at least one boundary component of \( M(r_\alpha) \). Then the following holds.

1. Let \( f \) be a Scharlemann cycle face in \( G_\beta \) such that the edges of \( f \) lie in an annulus in \( \hat{P}_a \). Then \( f \) is contained in \( X \), and \( \partial X = \hat{P}_a \).
2. Let \( f, g \) be the two-cornered faces of \( G_\beta \) with corners \( C(f) = C(g) = \{(r-1, r), (r+1, r+2)\} \) such that \( f \) is a bigon and \( g \) is a 3-gon. (For example, see Fig. 18(a).) Then \( f, g \) are contained in \( X \), and \( \partial X = \hat{P}_a \).

**Proof.** We shall prove only (2). (The proof of (1) is similar.)

Notice that either \( f, g \subset X \) or \( f, g \subset Y \) necessarily, due to the condition on the corners of \( f, g \), and since \( \hat{P}_a \) is separating. Assume for contradiction that \( f, g \) are contained in \( Y \). Up to a homeomorphism, the edges of \( f \) and \( g \) appear on \( G_\alpha \) as one of the configurations in Fig. 18(b). Thus, we can take an essential annulus \( A \) in \( \hat{P}_a \) disjoint from the edges.

Let \( H_1 \) (resp. \( H_2 \)) be the annulus in \( \partial J_a \) cobounded by \( \partial v_{r-1} \) and \( \partial v_r \) (resp. \( \partial v_{r+1} \) and \( \partial v_{r+2} \)). Let \( F = (\hat{P}_a - \text{Int}(v_{r-1} \cup v_r \cup v_{r+1} \cup v_{r+2})) \cup (H_1 \cup H_2) \). Then \( F \) is a surface of genus 3 with \( A \subset F \). The circles \( \partial f \) and \( \partial g \) are non-separating and mutually non-parallel in \( F \). There is a circle in \( \hat{P}_a \) that intersects \( \partial f \cup \partial g \) transversely in a single point (see Fig. 18(b)), so \( F - (\partial f \cup \partial g) \) is connected. Hence, compressing \( F \) using the disks \( f \) and \( g \) gives a torus \( T \) in \( Y \). During the surgery, \( A \) remains intact and hence is contained in \( T \). The torus \( T \) intersects the core of \( J_a \) fewer than \( n_\alpha \) times. By the choice of \( \hat{P}_a \), \( T \) is an inessential torus in \( M(r_\alpha) \). Since \( Y \) contains at least one boundary component of \( M(r_\alpha) \), it follows from the irreducibility of \( M(r_\alpha) \) that \( T \) is boundary parallel into a component of \( Y \cap \partial M(r_\alpha) \).

Therefore, the region realizing the boundary parallelism contains an annulus \( A' \) such that one boundary circle of \( A' \) lies in \( \partial M(r_\alpha) \) and the other is a core of \( A \). The boundary circle of \( A' \) in \( \partial M(r_\alpha) \) is essential, otherwise \( \hat{P}_a \) would be compressible. Surgering \( \hat{P}_a \) along \( A' \) gives rises to a properly embedded annulus in \( M(r_\alpha) \), which is essential, since \( \hat{P}_a \) is essential. This contradicts Lemma 5.1. Thus, \( f \) and \( g \) are contained in \( X \).

If \( X \) had contained at least one component of \( \partial M(r_\alpha) \), then \( f, g \) could not lie in \( X \) by the same argument as above. Therefore, \( X \) is bounded only by \( \hat{P}_a \). \( \square \)
Lemma 5.4. Assume \( n_\alpha \geq 4 \). Let \( x \) be a label of \( G_\beta \) which is not a label of a Scharlemann cycle. Then any \( x \)-face in \( G_\beta \) has at least 4 sides.

**Proof.** Recall that any \( x \)-face contains a Scharlemann cycle. By Lemma 2.4(3) \( G_\beta \) cannot contain a 2-sided \( x \)-face. Assume that \( G_\beta \) contains a 3-sided \( x \)-face. Applying Lemma 2.4(3) and [12, Lemma 5.1], one can see that the \( x \)-face contains an \( S \)-cycle, and a bigon \( f \) and a 3-gon \( g \) adjacent to the \( S \)-cycle, where \( f \) and \( g \) are two-cornered faces with the same pair of corners. This contradicts Lemma 5.3, since the \( S \)-cycle face and the faces \( f \) and \( g \) lie on the opposite sides of \( \hat{P}_\alpha \).

Lemma 5.5. Assume \( n_\alpha \geq 4 \). Then any 4-sided \( x \)-face in \( G_\beta \) contains a Scharlemann cycle of length 2 or 4.

**Proof.** Assume otherwise. Then \( G_\beta \) has a 4-sided \( x \)-face \( D \) that contains only Scharlemann cycles of length 3. Since \( D \) has 4 sides, it contains at most two Scharlemann cycles of length 3.

First, assume that \( D \) contains two Scharlemann cycles of length 3, one with label pair \( \{p, p + 1\} \) and the other with label pair \( \{q, q + 1\} \). An edge of one Scharlemann cycle must be parallel to an edge of the other, and there are \( n_\alpha/2 - 2 \) edges between them. See Fig. 19. At one end of the family of these parallel edges, label \( x \) must appear, contradicting the fact that \( D \) is an \( x \)-face.

Next, assume that \( D \) contains only one Scharlemann cycle \( \sigma \) of length 3. Without loss of generality, we may assume that the labels appear anticlockwise around each vertex of \( D \), and that \( x = 1 \). Let \( \{r, r + 1\} \) be the label pair of \( \sigma \). Since \( G_\beta \) cannot contain an extended Scharlemann cycle, two bigons and one 3-gon are adjacent to \( \sigma \) within \( D \). Let \( f \) be any such bigon. Then it is easy to see that \( f \) has corners \( \{(r - 1, r), (r + 1, r + 2)\} \). The 3-gon, \( g \), also has two such corners near an edge of \( \sigma \). Let \( \{(s, s + 1) \) be the remaining corner of \( g \). Within \( D \), we see one of the configurations of Fig. 20. We shall consider only Fig. 20(a); the cases of Fig. 20(b) and (c) are similar.

Let \( a, b, c \) be the vertices of \( D \) containing the corners \( (s, s + 1), (r - 1, r), (r + 1, r + 2) \) of \( g \), respectively. Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be the edge families in \( D \) connecting the vertices \( a, b \) and \( a, c \), respectively.
The set of labels of $F_1$ at vertex $b$ is $\{1, 2, \ldots, r - 1\}$, while the set of labels of $F_2$ at vertex $a$ is $\{1, 2, \ldots, s\}$. Since each edge family $F_i$ does not contain an $S$-cycle, the sets of labels at the two ends of the family are disjoint by [5, Lemma 2.6.6]. Thus, we have $r - 1 < s + 1$ and $s < r + 2$, giving $r - 2 < s < r + 2$. Hence, $s = r - 1$, $r$ or $r + 1$. The parity rule guarantees that each edge of $F_i$ connects vertices with one label even and the other label odd. This means that $s = r - 1$ or $r + 1$ and hence, $C(g) = \{(r - 1, r), (r + 1, r + 2)\}$. Then using the disks $f$ and $g$, we get a contradiction as in the proof of Lemma 5.4. □

**Lemma 5.6.** At most two labels of $G_\beta$ can be labels of $S$-cycles.

**Proof.** Assume that there are three such labels. Then $G_\beta$ contains two $S$-cycles $\sigma_1, \sigma_2$ on distinct label pairs. By Lemma 2.6, the edges of $\sigma_i$ ($i = 1, 2$) are contained in an essential annulus in $\hat{P}_g$. Let $f_i$ be the face of $G_\beta$ bounded by $\sigma_i$. Lemma 5.3 implies that $f_1$ and $f_2$ lie on the same side of $\hat{P}_\alpha$. Then $\sigma_1$ and $\sigma_2$ have disjoint label pairs. The construction in the proof of [12, Lemma 3.10] gives a Klein bottle in $M(r_\alpha)$. This contradicts our assumption. □

**Lemma 5.7.** At most four labels of $G_\beta$ can be labels of Scharlemann cycles.

**Proof.** Assume that there are three Scharlemann cycles $\sigma_1, \sigma_2, \sigma_3$ on distinct label pairs. Let $\{x_i, x_i + 1\}$ be the label pair of $\sigma_i$ for $i = 1, 2, 3$. Assume for contradiction that $\bigcup \{x_i, x_i + 1\}$ has at least 5 elements. Let $f_i$ be the face of $G_\beta$ bounded by $\sigma_i$. We may assume that $f_1$ and $f_2$ lie on the same side of $\hat{P}_\alpha$, say, $X$. Then the edges of $\sigma_1$ and $\sigma_2$ lie on disjoint essential annuli $A_1$ and $A_2$ in $\hat{P}_\alpha$. By Lemma 5.3, $X$ is bounded by $\hat{P}_\alpha$ and the other side, $Y$, of $\hat{P}_\alpha$ contains $\partial M(r_\alpha)$.

First, assume that $f_3$ lies in $Y$. If $\{x_3, x_3 + 1\} \cap (\{x_1, x_1 + 1\} \cup \{x_2, x_2 + 1\})$ is empty, then the edges of $\sigma_3$ must lie in an essential annulus in $\hat{P}_\alpha$, disjoint from $A_1$ and $A_2$. This is impossible by Lemma 5.3. Hence, $\{x_3, x_3 + 1\} \cap (\{x_1, x_1 + 1\} \cup \{x_2, x_2 + 1\})$ has a single element, and then the edges of $\sigma_3$ lie in
a region in $\hat{P}_\alpha$ between the cores of $A_1$ and $A_2$. The region must be an essential annulus in $\hat{P}_\alpha$, which is also impossible by Lemma 5.3.

Next, assume that $f_3$ lies in $X$. Then the edges of $\sigma_3$ lie in an essential annulus $A_3$ in $\hat{P}_\alpha$, disjoint from $A_1$ and $A_2$. Define $M_i = N(A_i \cup V_i \cup f_i; X)$, where $V_i$ is the part of $J_\alpha$ between consecutive vertices $v_x$ and $v_{x+1}$. Let $B_i = \text{cl}(\partial M_i - A_i)$. Then a new torus $T_i = (\hat{P}_\alpha - A_i) \cup B_i$ meets the core of $J_\alpha$ fewer than $n_\alpha$ times, and hence it is boundary parallel or compressible. If any $T_i$ is boundary parallel, then a similar construction as in the proof of Lemma 5.3 shows that $M(r_\alpha)$ is annular, contradicting Lemma 5.1. Thus, any $T_i$ is compressible. Then the argument in the proof of [12, Theorem 3.5] (without any change) gives a contradiction. □

**Lemma 5.8.** Let $\sigma_1, \sigma_2$ be Scharlemann cycles of $G_\beta$ of length $n$ on the same label pair, where $n = 2$ or 3. Then there is an annulus in $\hat{P}_\alpha$ that contains the edges of $\sigma_1, \sigma_2$.

**Proof.** By Lemma 2.6, there is an annulus $A_i$ in $\hat{P}_\alpha$ that contains the edges of $\sigma_i$ for each $i = 1, 2$. Let $X$, $Y$ be two sides of $\hat{P}_\alpha$, where $Y$ is chosen to contain at least one component of $\partial M(r_\alpha)$. Let $f_i$ be the face of $G_\beta$ bounded by $\sigma_i$. By Lemma 5.3, $f_1$ and $f_2$ lie in $X$, and $\partial X = \hat{P}_\alpha$.

If the cores of $A_1$ and $A_2$ are isotopic in $\hat{P}_\alpha$, then we are done. Assume not. Let $\{x, x + 1\}$ be the label pair of the Scharlemann cycles, and let $V$ be the part of $J_\alpha$ between vertices $v_x$ and $v_{x+1}$ of $G_\alpha$. Define $M_i = N(A_i \cup V \cup f_i; X)$. Then $M_i$ is a solid torus, and $B_i = \partial M_i - \text{Int} A_i$ is a properly embedded annulus in $X$. Let $T_i = (\hat{P}_\alpha - A_i) \cup B_i$. Then $T_i$ is a torus in $X$ which the core of $J_\alpha$ intersects fewer than $n_\alpha$ times. By the choice of $\hat{P}_\alpha$, the torus $T_i$ bounds a solid torus $M'_i$ in $X$. Hence, $X = M_i \cup B_i$, $M'_i$ is a Seifert fibered space over the disk with two exceptional fibers such that the core of $A_i$ is a Seifert fiber. For such a manifold, the Seifert fiberation is unique unless it is a twisted $I$-bundle over a Klein bottle. As $M(r_\alpha)$ does not contain a Klein bottle, $X$ has a unique Seifert fiberation. However, $X$ admits two different Seifert fibrations, since the cores of $A_1$ and $A_2$ are not isotopic in $\hat{P}_\alpha$. This is a contradiction. □

**Lemma 5.9.** Assume $n_\alpha \geq 4$. Let $\sigma_1$ and $\sigma_2$ be Scharlemann cycles of $G_\beta$ of lengths 2 and 3, respectively. Then their label pairs are disjoint.

**Proof.** Assume otherwise. Let $f_i$ be the disk face in $G_\beta$ bounded by $\sigma_i$ for $i = 1, 2$. Since the edges of $f_i$ lie in an essential annulus $A_i$ in $\hat{P}_\alpha$, the disks $f_1$ and $f_2$ lie on the same side, say, $X$ of $\hat{P}_\alpha$ by Lemma 5.3 (so, $\sigma_1, \sigma_2$ have the same label pair, say, $\{x, x + 1\}$), and $X$ is bounded by $\hat{P}_\alpha$. The proof of Lemma 5.8 shows that $X$ is a Seifert fibered space with the core of $A_i$ a fiber and has a unique Seifert fiberation. Thus the cores of $A_1$ and $A_2$ are isotopic, and so we may assume that the edges of $f_1, f_2$ lie in an essential annulus $A \subset \hat{P}_\alpha$.

Let $H$ be an annulus in $\partial J_\alpha$ cobounded by $\partial v_x$ and $\partial v_{x+1}$. Using the disks $f_1, f_2$, compress $(A - \text{Int}(v_x \cup v_{x+1}) \cup H$. The resulting surface is a disjoint union of two disks, any of which becomes a compressing disk of $\hat{P}_\alpha$. This is a contradiction. □

**Lemma 5.10.** Suppose that $\hat{P}_\beta$ is either a Klein bottle with $n_\beta \geq 2$ or a separating torus with $n_\beta \geq 4$. Suppose that $G_\beta$ contains Scharlemann cycles of length 3 on label pair $\{x, x + 1\}$. Then there are families $F_1$ and $F_2$ of parallel edges in $G_\alpha$ connecting vertices $v_x$ and $v_{x+1}$ such that any such Scharlemann cycle has exactly one edge in family $F_1$ and two edges in family $F_2$.

**Proof.** Lemma 5.8 guarantees that there is an essential annulus $A$ in $\hat{P}_\alpha$ such that the edges of any such Scharlemann cycle lie in $A$. By Lemma 2.1 $M(r_\alpha)$ is irreducible. Then the proof of [14, Theorem 5.8]
Lemma 5.11. Suppose that $\hat{P}_\beta$ is either a Klein bottle with $n_\beta \geq 2$, or a separating torus with $n_\beta \geq 4$. Then:

1. There can be at most $n_\beta$ $S$-cycles in $G_\beta$ on the same label pair.
2. There can be at most $n_\beta/2$ Scharlemann cycles in $G_\beta$ of length 3 on the same label pair.

Proof. We prove only (2). The proof of (1) is very similar. Assume that there are more than $n_\beta/2$ Scharlemann cycles in $G_\beta$ of length 3 on label pair $\{x, x + 1\}$. By Lemma 5.10, there is a family $F$ in $G_\alpha$ of more than $n_\beta$ parallel edges connecting vertices $v_x$ and $v_{x+1}$. This is impossible by Lemma 2.3(4) and 2.4(5). □

Proposition 5.12. Suppose that $\hat{P}_\alpha$ is a separating torus in $M(r_\alpha)$ with $n_\alpha \geq 4$. Also, suppose that $\hat{P}_\beta$ is either a Klein bottle with $n_\beta \geq 2$, or a separating torus with $n_\beta \geq 4$. Then the following hold:

1. Any vertex of $G_\alpha$, except Scharlemann vertices, has at most $2n_\beta - 1$ negative edge endpoints. Equivalently, it has at least $3n_\beta + 1$ positive edge endpoints.
2. Any Scharlemann vertex of $G_\alpha$, which is not an $S$-vertex, has at most $18n_\beta/7$ negative edge endpoints. Equivalently, it has at least $17n_\beta/7$ positive edge endpoints.
3. Any $S$-vertex of $G_\alpha$ has at most $30n_\beta/11$ negative edge endpoints. Equivalently, it has at least $25n_\beta/11$ positive edge endpoints.

Proof. (1) Assume that a vertex $v_x$ of $G_\alpha$, which is not a Scharlemann vertex, has at least $2n_\beta$ negative edge endpoints. By the parity rule, $E(G_\beta^+(x)) \geq 2n_\beta$. Let $V = V(G_\beta^+(x))$, $E = E(G_\beta^+(x))$, and $F = F(G_\beta^+(x))$. Then $V = n_\beta, E \geq 2n_\beta$ and $F \geq E - V$. Since any $x$-face has at least 4 sides by Lemma 5.4, we have $2E \geq 4F \geq 4(E - V)$, giving $E \leq 2V = 2n_\beta$. Thus, $E = 2n_\beta$ and the above inequalities are all equalities. In particular, $G_\beta^+(x)$ is filled up with 4-sided disk faces. Thus, $G_\beta^+ = G_\beta$ and then $E = 5n_\beta$, a contradiction.

(2) Let $v_x$ be a Scharlemann vertex of $G_\alpha$ which is not an $S$-vertex. By Lemma 2.4(3) $G_\beta^+(x)$ does not contain a bigon. Let $V = V(G_\beta^+(x))$, $E = E(G_\beta^+(x))$, and $F = F(G_\beta^+(x))$. (This means that $v_x$ has $E$ negative edge endpoints.) Then $V = n_\beta$ and $F \geq E - V$. Let $F_k$ be the number of disk faces of $G_\beta^+(x)$ with $k$ sides. Then, we have

$$2E \geq \sum_{k \geq 3} kF_k \geq 5F - 2F_3 - F_4.$$

This, along with $F \geq E - V$, gives

$$3E \leq 5V + 2F_3 + F_4.$$

Let $S$ be the number of $S$-cycles, and let $R$ be the number of Scharlemann cycles of length 4 with label $x$. Then, Lemma 5.5 gives $S + R \geq F_4$. By Lemma 5.6 any two $S$-cycles have the same label pair, so Lemma 5.11 gives $S \leq n_\beta$.

For any 3-sided $x$-face, its boundary is a Scharlemann cycle with label $x$, since there is no extended Scharlemann cycle in $G_\beta$ and since the $x$-face cannot contain an $S$-cycle (see the proof of Lemma 5.4).

applies here without change, once we note that if two edges connecting the vertices $v_x$ and $v_{x+1}$ bound a bigon disk in $\hat{P}_\alpha$, then they are parallel in $G_\alpha$ by Lemma 2.8. □
Any two Scharlemann cycles with label \( x \) cannot share an edge because \( n_\alpha \geq 4 \). Thus \( 3F_3 + 4R \leq E \), giving \( R \leq E/4 - 3F_3/4 \), and also giving \( F_3 \leq E/3 \). Therefore, we have

\[
3E \leq 5V + 2F_3 + F_4 \leq 5V + 2F_3 + S + R \leq 5n_\beta + 2F_3 + n_\beta + \frac{E}{4} - \frac{3F_3}{4} \leq 6n_\beta + \frac{2E}{3}.
\]

Hence \( E \leq 18n_\beta/7 \) and the result follows.

(3) Let \( v_i \) be an \( S \)-vertex of \( G_\alpha \). Then there is an \( S \)-cycle in \( G_\beta \) with label \( x \). First, note that \( G_\beta^+(x) \) does not contain a 3-sided face; otherwise either a 3-sided \( x \)-face would contain an \( S \)-cycle with labels different from \( x \), contradicting Lemma 5.6, or would itself be a Scharlemann cycle face, contradicting Lemma 5.9. For any 4-gon in \( G_\beta^+(x) \), its boundary is a Scharlemann cycle with label \( x \) by Lemmas 5.5 and 5.6. Any bigon in \( G_\beta^+(x) \) is an \( S \)-cycle face.

Hence, any two disk faces of \( G_\beta^+(x) \) with 2 or 4 sides cannot share an edge, since \( n_\alpha \geq 4 \). This leads to \( 2F_2 + 4F_4 \leq E \), giving \( F_4 \leq E/4 - F_2/2 \). (We use the same notation as above.) Since

\[
2E \geq 2F_2 + 4F_4 + \sum_{k \geq 3} kF_k \geq 5F - 3F_2 - F_4,
\]

combining with \( F \geq E - V \), we obtain

\[
3E \leq 5V + 3F_2 + F_4 \leq 5V + 3F_2 + \frac{E}{4} - \frac{F_2}{2} = 5n_\beta + \frac{5F_2}{2} + \frac{E}{4}.
\]

By Lemma 5.11, \( F_2 \leq n_\beta \). So, we have \( E \leq 30n_\beta/11 \). \( \square \)

6. Proof of Theorem 1.3’

Recall that \( M \) has at least two boundary components. In this section, we assume that \( \hat{P}_1 \) is a Klein bottle and \( \hat{P}_2 \) is a torus. Then \( M(r_2) \) does not contain a Klein bottle by our assumption. See also Theorem 1.4.

Lemma 6.1. If \( \hat{P}_2 \) is non-separating in \( M(r_2) \), then \( n_1 \leq 2 \) or \( n_2 = 1 \).

Proof. Assume \( n_1 \geq 3 \) and \( n_2 \geq 2 \). Let \( l (\leq 2) \) be the number of level vertices of \( G_1 \). Let \( N \) be the number of positive edge endpoints of \( G_1 \). Then Propositions 3.4 and 5.2 give

\[
(3n_2 + 1)(n_1 - l) + 3n_2l \leq N \leq n_1n_2.
\]

Since \( l \leq 2 \), we have \((3n_2 + 1)(n_1 - 2) + 6n_2 \leq (3n_2 + 1)(n_1 - l) + 3n_2l\), so

\[
(3n_2 + 1)(n_1 - 2) + 6n_2 \leq n_1n_2.
\]

Equivalently, \( n_1(2n_2 + 1) \leq 2 \). This is impossible. \( \square \)

Lemma 6.2. \( n_1 \leq 2 \) or \( n_2 \leq 2 \).

Proof. Assume for contradiction that \( n_1, n_2 \geq 3 \). Then \( \hat{P}_2 \) is a separating torus by Lemma 6.1, so \( n_2 \) is an even integer no less than 4.

Let \( l \) be the number of level vertices of \( G_1 \), and let \( s \) and \( s' \) be the number of \( S \)-vertices and Scharlemann vertices of \( G_2 \), respectively. Then \( l, s \leq 2 \) and \( s' \leq 4 \). Let \( N \) be the number of positive edge endpoints of \( G_1 \). Then Propositions 3.4 and 5.12 give

\[
(3n_2 + 1)(n_1 - l) + 3n_2l \leq N \leq (2n_1 - 1)(n_2 - s') + \frac{18n_1}{7}(s' - s) + \frac{30n_1}{11} \cdot s
\]
Since \( l, s \leq 2 \) and \( s' \leq 4 \), we have
\[
(3n_2 + 1)(n_1 - 2) + 6n_2 \leq (3n_2 + 1)(n_1 - l) + 3n_2l
\]
and
\[
(2n_1 - 1)(n_2 - s') + \frac{18n_1}{7}(s' - s) + \frac{30n_1}{11} \cdot s \leq (2n_1 - 1)(n_2 - 4) + \frac{36n_1}{7} + \frac{60n_1}{11}.
\]
Hence,
\[
(3n_2 + 1)(n_1 - 2) + 6n_2 \leq (2n_1 - 1)(n_2 - 4) + \frac{36n_1}{7} + \frac{60n_1}{11}.
\]
Equivalently, \((n_1 + 1)(n_2 - 123/77) \leq 339/77\). This is impossible because \( n_1 \geq 3 \) and \( n_2 \geq 4 \).

**Lemma 6.3.** If \( n_1 = 2 \), then \( G_2^+ \) cannot contain a bigon and a 3-gon simultaneously.

**Proof.** Let \( X \) be a thin regular neighborhood of \( \hat{P}_1 \) in \( M(r_1) \). Then \( X \) is a twisted \( I \)-bundle over the Klein bottle \( \hat{P}_1 \), and its boundary is an essential torus in \( M(r_1) \) by Lemmas 2.1 and 5.1, which intersects the core of \( J_1 \) four times. Let \( P'_1 = \partial X \cap M \). As in Section 3, we obtain a graph pair \( G_1' \), \( G_2' \) from the surfaces \( P'_1 \) and \( P_2 \). If \( G_2^+ \) contains a bigon and a 3-gon, then \( G_2^{1+} \) also contains a bigon and a 3-gon which lie on the outside of \( X \). By Lemma 5.3(2), \( M(r_1) - \text{Int} X \) is bounded by \( \partial X \), so \( \partial M \) is a single torus. This is a contradiction. \( \square \)

**Lemma 6.4.** \( n_2 \neq 1 \).

**Proof.** Assume that \( n_2 = 1 \). There are \( 5n_1/2 \) edges (so, \( n_1 \) is even) in \( G_2 \), which are divided into at most three families of mutually parallel edges. See [9, Lemma 5.1]. Since each family contains at most \( n_1/2 + 1 \) edges by Lemma 2.3(3), we have \( 5n_1 \leq 6(n_1/2 + 1) \) and hence \( n_1 = 2 \). Then the labels around the vertex of \( G_2 \) appear as shown in Fig. 21. This contradicts Lemma 6.3. \( \square \)

**Lemma 6.5.** \( n_1 \neq 1 \).

**Proof.** Assume that \( n_1 = 1 \). Recall that \( \overline{G}_1 \) is a subgraph of one of the graphs illustrated in Fig. 9. An Euler characteristic calculation shows that \( G_2 \) contains a disk face \( f \) with at most 3 sides. By Lemma 2.2, \( f \) cannot be a bigon bounded by negative edges, since any two positive edges in \( G_1 \) are parallel. Hence if \( f \) has a negative edge, then \( f \) is 3-sided.

Let \( X \) be a thin regular neighborhood of \( \hat{P}_1 \) in \( M(r_1) \), and let \( Y = M(r_1) - \text{Int} X \). Then \( \hat{P}_1' = \partial X \) is an essential torus in \( M(r_1) \). Let \( P'_1 = \hat{P}_1' \cap M \). As before, the arc components of \( P'_1 \cap P_2 \) define two labeled graphs \( G_1' \) and \( G_2' \), whose edges have signs.
Lemma 6.6. If $n_2 = 2$, then $n_1 = 2$.

Proof. Assume that $n_2 = 2$. By Lemma 6.5, $n_1 \geq 2$. Assume for contradiction that $n_1 \geq 3$. Claim. The two vertices of $G_2$ have opposite signs.

Proof. Assume not. Then $G_2^+=G_2$. Let $x$ be a label of $G_2$ which is not a label of a level edge. Applying an Euler characteristic argument to the graph $G_2^+(x)$, one can see that there is a 2-sided $x$-face in $G_2$. This contradicts Lemma 2.3(1) or (2).

The reduced graph $G_2$ is a subgraph of the graph shown in Fig. 22, which has two loop edges and four non-loop edges. Let $F_1$ and $F_2$ be the families of parallel loop edges at vertices $v_1$ and $v_2$ of $G_2$, respectively, and let $F_3, F_4, F_5, F_6$ be the four families of parallel non-loop edges. Here, loop edges are positive and non-loop edges are negative. Let $|F_i|$ denote the number of edges in the family $F_i$. Then $|F_1| = |F_2| \leq n_1/2 + 1$ and $|F_j| \leq n_1 (j = 3, 4, 5, 6)$ by Lemma 2.3(3) and (4).

Since $|F_1| = |F_2| \leq n_1/2 + 1$, $G_1$ contains at least $4n_1 - 2$ positive edges and hence contains Scharlemann cycles. Thus, $P_2$ is separating in $M(r_2)$. Any two $S$-cycle faces of $G_1$ lie on the same side of $P_2$ by Lemma 5.3, and their two edges belong to the same pair of the edge families, say, $F_3$ and $F_4$; otherwise $M(r_2)$ would contain a Klein bottle by the argument in the proof of [13, Lemma 5.2].

If some label appears twice at ends of $F_1$ (or $F_2$), then it is a label of a positive level edge. From this fact, one easily sees that any non-level vertex of $G_1$ has at least 8 positive edge endpoints, and any level vertex has at least 6 positive edge endpoints.

If $G_1$ contains 3 parallel edges, then either two of the 3 edges are loop edges at the same vertex of $G_2$ (if the 3 edges are negative), or $G_1$ contains two $S$-cycle faces lying on the opposite sides of $P_2$ (if the
3-edges are positive). Both are impossible by Lemmas 2.2 and 5.3. Thus, any family of parallel edges in $G_1$ contains at most 2 edges.

Suppose that a vertex $v_x$ of $G_1$ has valency at most 5. Then $v_x$ has valency exactly 5, and 5 pairs of parallel edges are incident to $v_x$. Since $v_x$ has at least 6 positive edge endpoints, at least 3 pairs of parallel edges at $v_x$ are pairs of positive edges. So, at least 3 edges belonging to $\mathcal{F}_3$ are incident to $v_x$. The label $x$ must appear at least twice at one end of $\mathcal{F}_3$. Then $|\mathcal{F}_3| \geq n_1 + 1$, giving a contradiction.

Therefore, any vertex of $G_1$ has valency 6, and 4 pairs of parallel edges are incident to it. Consider a non-level vertex of $G_1$ (it exists, since $n_1 \geq 3$). Since the vertex has at least 8 positive edge endpoints, at least 3 pairs of parallel edges around the vertex are pairs of positive edges. A similar argument as above gives a contradiction. □

Lemma 6.1 through Lemma 6.6 show that $n_1 = 2$, $n_2 \geq 2$. In the rest of this section we shall consider this case.

Lemma 6.7. $\hat{P}_2$ is a separating torus in $M(r_2)$.

Proof. Assume not. Then each vertex of $G_2$ has at least 8 positive edge endpoints by Proposition 5.2. An Euler characteristic calculation shows that $G_2^+$ contains bigons. By Lemma 6.3, $G_2^+$ cannot contain a 3-gon.

Claim. Any family of parallel negative edges in $G_1$ contains at most $2n_2$ edges.

Proof. This is Lemma 2.4(6) if $n_2 \geq 4$.

Suppose that $G_2^+$ has $V(= n_2)$ vertices, $E$ edges, and $F$ disk faces. Since each vertex of $G_2$ has at least 8 positive edge endpoints, we have $E \geq 4V$. Let $F_k$ be the number of disk faces of $G_2^+$ with $k$ sides, with $k \geq 2$. Then $2E \geq 2F_2 + \sum_{k \geq 4} kF_k \geq 4F - 2F_2$, giving $F_2 \geq 2F - E \geq E - 2V$. Any bigon in $G_2^+$ is bounded by a level 1-edge and a level 2-edge. Such level 1-edges (or 2-edges) are parallel orientation-reversing loops at one vertex in $G_1$. So $F_2 \leq 2n_2 = 2V$ by the above claim, and combining $F_2 \geq E - 2V$, we have $E \leq 4V$. Since $E \geq 4V$, we have $E = 4V$ and the above inequalities are all
equalities. In particular, any disk face with more than 2 sides has exactly 4 sides, and $G_2^+$ is filled up with disk faces. Then $G_2^+ = G_2$ and hence $E = 5V$. This is a contradiction.  \[\square\]

**Lemma 6.8.** $n_2 = 2$ or $4$.

**Proof.** Since $\hat{P}_2$ is separating, $n_2$ is even. Assume that $n_2 \geq 6$. Any family of parallel edges in $G_2$ contains at most 2 edges by Lemma 2.3(3) and (4).

By Proposition 5.12, any vertex of $G_2$ has at least 5 positive edge endpoints. If some vertex of $\overline{G}_2$ has valency 5, then there are at least three pairs of parallel positive edges at this vertex, contradicting Lemma 2.5(1). Thus, each vertex of $\overline{G}_2$ has valency 6.

There is a non-Scharlemann vertex in $G_2$, since $n_2 \geq 6$. This vertex has at least 7 positive edge endpoints by Proposition 5.12(1). Then, there are at least three pairs of parallel positive edges at this vertex. This contradicts Lemma 2.5(1) again.  \[\square\]

**Lemma 6.9.** $n_2 \neq 4$.

**Proof.** Assume $n_2 = 4$.

By Proposition 5.12, any vertex of $G_2$ has at least 5 positive edge endpoints. Hence, each vertex of $\overline{G}_2$ has valency 6 by the argument in the proof of Lemma 6.8. Then, there are at least two pairs of parallel positive edges at each vertex of $G_2$. Thus, each vertex of $G_1$ is a base of negative loops.

Note that $E(G_2) = 20$. Hence $E(G_2^+) \geq 10$, since there are at least 5 positive edge endpoints at any vertex of $G_2$. Assume $E(G_2^+) = 10$. Then $E(G_1^+) = 10$ by the parity rule. Each family of parallel positive edges in $G_1$ contains at most 3 edges by Lemma 2.4(4), so $E(G_1^+) = 4$ and $\overline{G}_1$ is the last graph in Fig. 13. Then at least 5 negative loop edges are incident to a vertex of $G_1$, contradicting Lemma 2.4(5). Thus $E(G_2^+) \geq 11$.

Recall that any family of parallel edges in $G_2$ contains at most 2 edges. Thus any vertex of $\overline{G}_2^+$ has valency at least 3, since it has at least 5 positive edge endpoints. Since $\hat{P}_2$ is separating, $\overline{G}_2^+$ has at least two components, each having at most two vertices. It is easy to see that any component cannot be contained in a disk in $\hat{P}_2$, since each vertex of $\overline{G}_2^+$ has valency at least 3. Hence, each component is contained in an annulus and is one of the graphs shown in Fig. 24.

Since $E(G_2^+) \geq 11$, one component of $G_2^+$ has at least 6 edges. By Lemma 2.5(2), the reduced graph corresponding to that component is the first graph in Fig. 24, and that component has exactly 6 edges, where 2 edges are loop edges at one vertex, another 2 edges are loop edges at the other vertex, and the
remaining 2 edges are mutually non-parallel non-loop edges. By examining the labels around the vertices of the component, one can find $S$-cycles in the loop edges. This contradicts Lemma 2.3(1).

\[\square\]

**Lemma 6.10.** $n_2 \neq 2$.

**Proof.** Assume $n_2 = 2$. Then $G_2$ is a subgraph of the graph in Fig. 22 such that the two vertices have the opposite signs. Each edge family in $G_2$ contains at most 2 edges.

If there are two parallel negative level edges in $G_2$, then by applying the argument in the first paragraph of Section 4.4, one can show that $M(r_1)$ contains a Klein bottle intersecting the core of $J_1$ in at most one point. This contradicts the choice of $P_1$. Thus, $G_2$ does not contain two parallel negative level edges, and up to a homeomorphism of $P_2$, we have only three possibilities for $G_2$ as shown in Fig. 25.

For the first configuration, we can decide $G_1$ as in Fig. 16 with edges there doubled. Then $G_1$ has two $S$-cycle faces on opposite sides of $P_2$, contradicting Lemma 5.3.

For the last two configurations, we can decide $G_1^+$ as in Fig. 26. Then $G_1$ has two $S$-cycle faces and two Scharlemann cycle faces of length 3 lying on the opposite sides of $P_2$. This also contradicts Lemma 5.3.

This completes the proof of Theorem 1.3’.

\[\square\]

### 7. Proofs of Theorems 1.1 and 1.2

Throughout this section, we assume that both $P_1$ and $P_2$ are tori. By our assumption, neither $M(r_1)$ nor $M(r_2)$ contains a Klein bottle. See also Theorems 1.3 and 1.4. We first prove Theorem 1.2.

**Lemma 7.1.** If $n_1, n_2 \geq 3$, then both $P_1$ and $P_2$ are separating tori.
Proof. Let $N$ be the number of positive edge endpoints of $G_1$.

If both $\hat{P}_1$ and $\hat{P}_2$ are non-separating, then $4n_1n_2 \leq N \leq n_1n_2$ by Proposition 5.2. This is impossible.

Suppose that $\hat{P}_1$ is non-separating, while $\hat{P}_2$ is separating. Then $n_2$ is an even integer no less than 4. As noted before, an Euler characteristic argument shows that there is a vertex of $\overline{G}_1$ having valency at most 6. Let $k$ be the number of the families of parallel positive edges incident to the vertex. Counting the edge endpoints around the vertex, we have $5n_2 \leq k(n_2/2 + 1) + (6 - k)n_2$ by Lemma 2.4(4) and (5). Equivalently, we have $(n_2 - 2)(k - 2) \leq 4$, so $k \leq 4$, since $n_2 \geq 4$. Then the vertex has at least $3n_2 - 4(> n_2)$ negative edge endpoints, contradicting Proposition 5.2. □

Lemma 7.2. $n_1 \leq 2$ or $n_2 \leq 2$.

Proof. Assume that $n_1, n_2 \geq 3$. Then both $\hat{P}_1$ and $\hat{P}_2$ are separating, so $n_1, n_2$ are even and $n_1, n_2 \geq 4$. Let $N$ be the number of positive edge endpoints of $G_1$.

By Proposition 5.12,

$$(3n_2 + 1)(n_1 - 4) + \frac{34n_2}{7} + \frac{50n_2}{11} \leq N \leq (2n_1 - 1)(n_2 - 4) + \frac{36n_1}{7} + \frac{60n_1}{11}.$$ 

Equivalently, $(n_1 - 123/77)(n_2 - 123/77) \leq 8 + (123/77)^2$. This implies $n_1 = n_2 = 4$.

Consider the case where $n_1 = n_2 = 4$. Notice that $E(G_1) = E(G_2) = 40$. There is a vertex $v_x$ of $\overline{G}_\alpha$ having valency at most 6. Let $k$ be the number of families of parallel positive edges incident to $v_x$. Then we have $20 = 5n_\beta \leq k(n_\beta/2 + 1) + (6 - k)n_\beta = 24 - k$, giving $k \leq 4$. Since $v_x$ has at least 10 positive edge endpoints by Proposition 5.12, $k = 4$ and $v_x$ has exactly 12 positive edge endpoints. Any other vertex also has at least 10 positive edge endpoints, so $E(G_\alpha^+) \geq 21$. This inequality holds for both $\alpha = 1, 2$. This contradicts the parity rule. □

Lemma 7.3. $n_\alpha \neq 1$ for $\alpha = 1, 2$.

Proof. Assume that $n_\alpha = 1$. Then $n_\beta$ is even. The graph $G_\alpha$ has at most three families of parallel edges. By Lemma 2.4(4), we have $5n_\beta \leq 6(n_\beta/2 + 1)$ and equivalently, $n_\beta \leq 3$. Hence, $n_\beta = 2$.

Assume $n_\beta = 2$. Notice that $G_\alpha$ contains no level edge by the parity rule. If $G_\alpha$ has three parallel edges, then it has two $S$-cycle faces on opposite sides of $\hat{P}_\beta$, contradicting Lemma 5.3. Thus, $G_\alpha$ has exactly three families of parallel edges, and then it contains level edges as shown in Fig. 21. This is a contradiction. □

Lemma 7.4. If $n_\alpha = 2$, then $n_\beta = 2$.

Proof. Assume that $n_\beta \geq 3$. Since $n_\alpha = 2$, $E(\overline{G}_\alpha) \leq 6$. See Fig. 22. If the vertices of $G_\alpha$ have the same sign, then Lemma 2.4(4) leads to $5n_\beta \leq 6(n_\beta/2 + 1)$, giving $n_\beta \leq 3$. But if $n_\beta = 3$, then the right side of the above inequality is $6 \cdot (n_\beta + 1)/2$, and we get a contradiction. Thus the vertices of $G_\alpha$ have opposite signs. Since only loop edges of $G_\alpha$ are positive, we have $E(G_\alpha^+) \leq n_\beta + 2$, and by the parity rule $E(G_\beta^+) \geq 4n_\beta - 2$.

Note that any disk face of $G_\beta^+$ is a Scharlemann cycle face. Let $V = V(G_\beta^+), E = E(G_\beta^+)$, and $F = F(G_\beta^+)$. Then $V = n_\beta, E \geq 4V - 2$ and $F \geq E - V \geq 3V - 2 > 0$, so $\hat{P}_\alpha$ is separating in $M(r_\alpha)$ by Lemma 2.4(1).

Let $F_k$ be the number of disk faces of $G_\beta^+$ with $k$ sides. Then $2E \geq \sum_{k \geq 2} kF_k \geq 4F - 2F_2 - F_3 \geq 4E - 4V - 2F_2 - F_3$, giving

$$2F_2 + F_3 \geq 2E - 4V.$$
By Lemmas 2.6 and 5.3, all the disk faces of $G^+_\beta$ with at most 3 sides must lie on the same side of $\hat{P}_\alpha$. Thus,

$$E \geq 2F_2 + 3F_3.$$ 

Combining these two inequalities, we obtain

$$E \geq 2F_2 + 3F_3 \geq 2F_2 + F_3 \geq 2E - 4V. \quad (1)$$

Hence, $4V - 2 \leq E \leq 4V$. Since the number of loop edges of $G_\alpha$ is even, so is $E$. Thus $E = 4V - 2$ or $4V$. If $E = 4V$, then the inequality (1) gives $F_3 = 0$ and $F_2 = 2V$. If $E = 4V - 2$, then $4V - 2 \geq 2F_2 + 3F_3 \geq 2F_2 + F_3 \geq 4V - 4$, so $F_3 \leq 1$ and $F_2 \geq 2V - 5/2$. In any case, $F_2 \geq V = n_\beta$.

The edges of all bigons of $G^+_\beta$ belong to the same pair of families of mutually parallel negative edges in $G_\alpha$ by [13, Lemma 5.2]. (Otherwise, $M(r_\alpha)$ would contain a Klein bottle.) Since $F_2 \geq n_\beta$, some family of parallel negative edges in $G_\alpha$ contains more than $n_\beta$ edges. Lemma 2.4(5) implies that all the vertices of $G_\beta$ have the same sign. Then any edge in $G_\beta$ is positive, implying $E = 5V$. This is a contradiction. □

**Lemma 7.5.** $n_1 = n_2 = 2$ is impossible.

**Proof.** Assume the vertices of $G_\beta$ have the same sign. If we keep the notation in the proof of Lemma 7.4, then we have $E = 5V$. On the other hand, the proof of Lemma 7.4 shows the inequality (1) holds. This is a contradiction.

Thus the vertices of $G_\beta$ have opposite signs for $\beta = 1, 2$. Hence, the loop edges of $G_\beta$ are positive, and the others are negative. As observed in the proof of Lemma 7.3, $G_\beta$ cannot contain three parallel positive edges. This implies that $G_\beta$ contains at most 4 positive edges; equivalently, it contains at least 6 negative edges ($\beta = 1, 2$). This is impossible by the parity rule. □

This completes the proof of Theorem 1.2.

**Proof of Theorem 1.1.** Let $M$ be a hyperbolic 3-manifold with a torus boundary component $\partial_0 M$ and at least one other boundary component. Assume for contradiction that there are at least 7 exceptional slopes on $\partial_0 M$. Then there are two such slopes $r_1, r_2$ with $\Delta(r_1, r_2) \geq 5$ by [11, Corollary 2.2]. Thurston’s Geometrization Theorem for Haken manifolds guarantees that both $M(r_1)$ and $M(r_2)$ contain essential surfaces of nonnegative Euler characteristic.

Since $\Delta(r_1, r_2) \geq 5$, $M(r_\alpha)$ contains an essential annulus or essential torus ($\alpha = 1, 2$) by the works of Gordon, Luecke, Oh, Qiu, Scharlemann, and Wu. See [10, Section 2]. Gordon [9] showed that if $\Delta(r_1, r_2) \geq 6$, then both $M(r_\alpha)$ are toroidal and $M$ is bounded by a single torus. This is impossible in our situation. If $\Delta(r_1, r_2) = 5$, then $M$ is the exterior of the Whitehead sister link by [15, Theorem 1.1], [16, Theorem 1.1] and Theorem 1.2. As mentioned before, the Whitehead sister link exterior admits exactly 6 exceptional slopes. This completes the proof. □

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