Theoretical Computer Science 65 (1989) 197-212
North-Holland

STATISTIC AND ERGODIC PROPERTIES OF MINKOWSKI'S DIAGONAL CONTINUED FRACTION

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Received March 1988
Revised June 1988

Abstract. Recently the author introduced a new class of continued fraction expansions, the S-expansions. Here it is shown that Minkowski's diagonal continued fraction (DCF) is an S-expansion. Due to this, statistic and ergodic properties of the DCF can be given.

Contents

1. Introduction

Let x be an irrational number between 0 and 1, let

$$\text{(1.1)}\qquad \cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{\ddots}}}$$

be its expansion as a regular continued fraction, denoted by RCF, and let $(p_n/q_n)_{n=-1}^{\infty}$ be the corresponding sequence of convergents. Here $q_{-1}=0$.

1.2. Definitions. The operator $T:[0,1]\to[0,1]$ is defined by

$$Tx := x^{-1}-[x^{-1}],\quad x\neq 0,$$

$$T0 := 0.$$

Put

$$T_n := T^n(x)\text{: the } n\text{th-iterate of } T \text{ on } x, \text{ where } n \geq 1 \text{ and } T_0 := x$$

and

$$V_n := q_{n-1}/q_n, \quad n \geq 0.$$

Notice that $(T_n, V_n)_{n\geq 0}$ is a sequence in Ω, where $\Omega := [0, 1] \times [0, 1]$. Let the sequence of regular approximation constants $\theta_n = \theta_n(x)$, $n \geq -1$, be given by

$$\theta_n := q_n|q_n x - p_n|, \quad n \geq -1.$$

Then we have

$$0 < \theta_n < 1, \quad n \geq 0,$$

and

$$\textbf{(1.3)} \quad \theta_n = \frac{T_n}{1 + T_n V_n}, \quad n \geq 0;$$

see e.g. [7, p. 29, Eq. (11)].

Furthermore we have the following classical theorems of Legendre and Vahlen.

1.4. Theorem (Legendre). *Let x be an irrational number and let $P, Q \in \mathbb{Z}$ such that $(P, Q) = 1$, $Q > 0$ and $\theta = Q|Qx - P| < \frac{1}{2}$. Then P/Q is a regular convergent of x.*

1.5. Theorem (Vahlen). *For all $k \in \mathbb{N}$ and $x \notin \mathbb{Q}$ we have* $\min(\theta_k, \theta_{k+1}) < \frac{1}{2}$.

1.6. Definitions. Here and in the following, $[a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots]$ is the abbreviation of

$$a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cfrac{\varepsilon_3}{a_3 + \ddots}}}$$

where $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}$, $i \geq 1$ and $\varepsilon_i \in \{\pm 1\}$, $i \in \mathbb{N}$. We call $[a_0; \varepsilon_1 a_1, \ldots]$ a *semi-regular continued fraction expansion* (SRCF) in case $\varepsilon_i + a_i \geq 1$, $\varepsilon_{i+1} + a_i \geq 1$ for $i \geq 1$ and $\varepsilon_{i+1} + a_i \geq 2$ infinitely often.

Let x be an irrational number. Consider the sequence σ of all irreducible rational fractions P/Q, with $Q > 0$, satisfying

$$\left| x - \frac{P}{Q} \right| < \frac{1}{2} \frac{1}{Q^2},$$

ordered in such a way that the denominators form an increasing sequence.

From Legendre's Theorem 1.4 it follows that σ consists exactly of those regular convergents p_k/q_k for which $\theta_k < \frac{1}{2}$. Due to this and Vahlen's Theorem 1.5 we see that σ is an infinite subsequence of the sequence of regular convergents of x. In [13, Section 41] it is shown that there exists a unique SRCF-expansion of x such that σ is the sequence of convergents of this expansion of x. By definition, this is Minkowski's diagonal continued fraction expansion (DCF) of x; see also [10].

1.7. Definition. Let the irrational number x have the continued fraction expansion $[a_0;\ \varepsilon_1 a_1, \dots]$ and suppose that for a certain $k \geqslant 0$ one has $a_{k+1} = 1$ and $\varepsilon_{k+1} = \varepsilon_{k+2} = 1$. The operation by which this continued fraction is replaced by

$$[a_0;\ \varepsilon_1 a_1, \varepsilon_2 a_2, \dots, \varepsilon_k(a_k+1), -(a_{k+2}+1), \varepsilon_{k+3}a_{k+3}, \dots],$$

which is again a continued fraction expansion of x, is called the *singularization of the partial quotient* a_{k+1} *equal to* 1, see also [3, Section 1].

1.8. Remarks. (i) The singularization of a partial quotient equal to 1 is based upon the equality

$$A + \cfrac{1}{1+\cfrac{1}{B+\xi}} = A + 1 - \frac{1}{B+1+\xi},$$

where B, $\xi > 0$.

(ii) Notice that if we repeat this singularization operation, we can never singularize two consecutive partial quotients.

(iii) Let $(A_n/B_n)_{n\geqslant -1}$ be the sequence of convergents of expansion (1.6) and $(C_n/D_n)_{n\geqslant -1}$ that of the expansion obtained by singularizing in (1.6) an a_{k+1} equal to 1. Then the sequence $(C_n/D_n)_{n\geqslant -1}$ is obtained from the sequence $(A_n/B_n)_{n\geqslant -1}$ by skipping the term A_k/B_k.

1.9. Theorem. *Minkowski's DCF-expansion of an irrational number x is obtained from the RCF-expansion of x by singularizing all those regular partial quotients b_{k+1} for which $\theta_k > \frac{1}{2}$.*

Proof. The theorem is an immediate consequence of the observation that $\theta_k > \frac{1}{2}$ implies $b_{k+1} = 1$ and of Vahlen's Theorem 1.5. □

1.10. Definition. Let x be an irrational number and let $[a_0;\ \varepsilon_1 a_1, \varepsilon_2 a_2, \dots]$ be its DCF-expansion. We denote the sequence of DCF-convergents of x by

$$\frac{r_n(x)}{s_n(x)}, \quad n \geqslant -1 \qquad \text{or shortly by} \qquad \frac{r_n}{s_n}, \quad n \geqslant -1.$$

The DCF-approximation constants $\Theta_n = \Theta_n(x)$, $n \geqslant -1$, are defined by

$$\Theta_n := s_n |r_n x - s_n|, \quad n \geqslant -1.$$

In this paper we show that the one-sided shift-operator connected with the DCF-expansion comes from a certain dynamical system. Due to this, the distribution of some sequences connected with the DCF can be given. In fact, the DCF is an example of a wider class of continued fraction expansions, so called S-expansions; see also [9]. For S-expansions the underlying dynamical system can be given; a short description of these expansions will be given in Section 2. Finally, it is shown that the DCF-expansion of a quadratic surd is periodic.

We conclude this section with an example.

1.11. Example. Let $x=(-39+\sqrt{3029})/58=0.2764\ldots$. One has $\mathrm{RCF}(x)=[0;\overline{3,1,1,1,1,1,1}]$. Hence

n	-1	0	1	2	3	4	5	6	7	8	9	...
b_n	-	0	3	1	1	1	1	1	1	3	1	...
p_n	1	0	1	1	2	3	5	8	13	47	60	...
q_n	0	1	3	4	7	11	18	29	47	170	217	...
θ_n	0	x	0.51..	0.42..	0.45..	0.45..	0.41..	0.52..	0.23..	0.52..	0.41..	...

Thus,

n	-1	0	1	2	3	4	5	6	7	...
ε_n	-	-	1	-1	1	1	1	-1	-1	...
a_n	-	0	4	2	1	1	2	5	2	...
r_n	1	0	1	2	3	5	13	60	107	...
s_n	0	1	4	7	11	18	47	217	387	...
Θ_n	0	x	0.42..	0.45..	0.45..	0.41..	0.23..	0.41..	0.45..	...

One finds that $\mathrm{DCF}(x)=[0;4,\overline{-2,1,1,2,-5}]$. The periodicity of this expansion does not follow from the above short calculation but from the algorithm given in Section 5.

2. S-expansions

Fundamental in the theory of S-expansions is the following theorem; see also [11, 12].

2.1. Theorem. *Let $\boldsymbol{B}$ be the collection of Borel-subsets of Ω and μ the probability measure on $(\Omega, \boldsymbol{B})$ with density $(\log 2)^{-1}(1+xy)^{-2}$. Define the operator $\mathcal{T}:\Omega\to\Omega$ by*

$$\mathcal{T}(x,y):=(Tx,(y+[x^{-1}])^{-1}),\quad (x,y)\in\Omega.$$

Then $(\Omega, \boldsymbol{B}, \mu, \mathcal{T})$ forms an ergodic system.

A simple way to derive a strategy for singularization is given by a singularization area S.

2.2. Definition. A subset S from Ω is called *a singularization area* when it satisfies

(i) $S\in\boldsymbol{B}$ is μ-continuous;
(ii) $S\subseteq[\frac{1}{2},1]\times[0,1]$;
(iii) $(\mathcal{T}S)\cap S=\emptyset$.

2.3. Theorem. *Let S be a singularization area. Then*

$$0\leq\mu(S)\leq\frac{\log 2g}{\log 2}=1-\frac{\log G}{\log 2}=0.30575\ldots.$$

For a proof of this, see [9].

Here and in the sequel we put

$$g := \frac{\sqrt{5}-1}{2}, \qquad G := \frac{\sqrt{5}+1}{2} = g+1 = g^{-1}.$$

2.4. Definition. Let S be a singularization area and x an irrational number. The S-expansion of x is obtained from the regular expansion of x by singularizing b_{n+1} if and only if $(T_n, V_n) \in S$. Here T_n and V_n are defined as in Definition 1.2.

2.5. Definition. Let S be a singularization area, x an irrational number and let $[a_0; \varepsilon_1 a_1 . \varepsilon_2 a_2, \ldots]$ be the S-expansion of x. The shift t which acts on $x - a_0$ is defined by

$$t([0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots]) := [0; \varepsilon_2 a_2, \varepsilon_3 a_3, \ldots].$$

Moreover, let t_k be the kth iterate of t on $x - a_0$ and let $v_k := s_{k-1}/s_k$, $k \geq 0$, where s_k is the denominator of the kth S-convergent of x.

2.6. Remark. By definition, $t_k = [0; \varepsilon_{k+1} a_{k+1}, \varepsilon_{k+2} a_{k+2}, \ldots]$. One easily sees that $v_k = [0; a_k, \varepsilon_k a_{k-1}, \ldots, \varepsilon_2 a_1]$ and that the numerators and denominators of the sequence $(r_k/s_k)_{k \geq 1}$, the sequence of S-convergents, satisfy the following recurrence relations:

$$r_{-1} := 1, \qquad r_0 := a_0, \qquad r_n := a_n r_{n-1} + \varepsilon_n r_{n-2}, \qquad n \geq 1,$$

$$s_{-1} := 0, \qquad s_0 := 1, \qquad s_n := a_n s_{n-1} + \varepsilon_n s_{n-2}, \qquad n \geq 1.$$

2.7. Lemma. *With S and x as in Definition 2.5, $(p_n/q_n)_{n \geq -1}$ the sequence of regular convergents and $(r_k/s_k)_{k \geq -1}$ the sequence of S-convergents of x, we have*

$$x = \frac{p_{n-1}(b_n + T_n) + p_{n-2}}{q_{n-1}(b_n + T_n) + q_{n-2}} = \frac{p_n + T_n p_{n-1}}{q_n + T_n q_{n-1}}, \qquad n \geq 1$$

and

$$x = \frac{r_{k-1}(a_k + t_k) + \varepsilon_k r_{k-2}}{s_{k-1}(a_k + t_k) + \varepsilon_k s_{k-2}} = \frac{r_k + t_k r_{k-1}}{s_k + t_k s_{k-1}}, \qquad k \geq 1.$$

From Remark 1.8(iii), Definition 2.4 and Lemma 2.7 one easily derives the following theorem.

2.8. Theorem. *Using the same notation as in Definition 2.5 and putting $\Delta := \Omega \backslash S$, $\Delta^- := \mathcal{T}S$ and $\Delta^+ := \Delta \backslash \Delta^-$, we have*

(i) $(T_n, V_n) \in S \Leftrightarrow p_n/q_n$ *is not an S-convergent*;

(ii) p_n/q_n *is not an S-convergent* $\Rightarrow$ *both p_{n-1}/q_{n-1} and p_{n+1}/q_{n+1} are S-convergents*;

(iii) $(T_n, V_n) \in \Delta^+ \Leftrightarrow \exists k$: $\begin{cases} r_{k-1} = p_{n-1},\ r_k = p_n \\ s_{k-1} = q_{n-1},\ s_k = q_n \end{cases}$ *and*

$$\begin{cases} t_k = T_n & (\textit{hence } \varepsilon_{k+1} := \operatorname{sgn}(t_k) = +1) \\ v_k = V_n; \end{cases}$$

(iv) $(T_n, V_n) \in \Delta^- \Leftrightarrow \exists k$: $\begin{cases} r_{k-1} = p_{n-2},\ r_k = p_n \\ s_{k-1} = q_{n-2},\ s_k = q_n \end{cases}$ *and*

$$\begin{cases} t_k = -T_n/(1+T_n) & (\textit{hence } \varepsilon_{k+1} = -1) \\ v_k = 1 - V_n. \end{cases}$$

2.9. Remarks. Define the transformation $\mathcal{S}: \Delta \to \Delta$ by

$$\mathcal{S}(x, y) := \begin{cases} \mathcal{T}(x, y), & \mathcal{T}(x, y) \in \Delta, \\ \mathcal{T}^2(x, y), & \mathcal{T}(x, y) \in S, \end{cases}$$

where Δ is defined as in Theorem 2.8. Due to the fact that $\mathcal{S}$ is an induced transformation, we now have that $(\Delta, \boldsymbol{B}, \rho, \mathcal{S})$ forms an ergodic system. Here ρ is the probability measure on $(\Delta, \boldsymbol{B})$ with density

$$\frac{1}{\mu(\Delta)\log 2} \frac{1}{(1+tv)^2},$$

see e.g. [14]. Since $h(T)$, the entropy of the RCF, equals

$$h(T) = \frac{\pi^2}{6\log 2},$$

see [11], we have, due to a formula of Abramov, $h(\mathcal{S}) = h(T)/\mu(\Delta)$, see [1]. It is now natural to consider the following definition.

2.10. Definition. Let the map $M: \Delta \to \mathbb{R}^2$ be defined by

$$M(T, V) := \begin{cases} (T, V), & (T, V) \in \Delta^+, \\ (-T/(1+T), 1-V), & (T, V) \in \Delta^-. \end{cases}$$

2.11. Theorem. *Let S be a singularization area and put $\Omega_S := M(\Delta) = \Delta^+ \cup M(\Delta^-)$. Let again $\boldsymbol{B}$ be the collection of Borel-subsets of Ω_S and let ρ be the probability measure on $(\Omega_S, \boldsymbol{B})$ with density $(\mu(\Delta)\log 2)^{-1}(1+tv)^{-2}$. Define the map $\tau: \Omega_S \to \Omega_S$ by $\tau(t, v) := M(\mathcal{S}(M^{-1}(t, v)))$. Then τ is conjugate to $\mathcal{S}$ by M and we have*

(i) $(t_k, v_k) \in \Omega_S$, $\forall k \geq 0$;

(ii) $(\Omega_S, \boldsymbol{B}, \rho, \tau)$ *forms an ergodic system*;

(iii) $h(\tau) = h(\mathcal{S})$.

Moreover we have the following theorem.

2.12. Theorem. *Let the map* $f: \Omega_S \to \mathbb{R}$ *be defined by*

$$f(t, v) := |t^{-1}| - \tau_1(t, v), \quad (t, v) \in \Omega_S,$$

where τ_1 *is the first coordinate function of* τ. *Let* $b(t) := [t^{-1}]$, $\forall t \in \mathbb{R}$, $t \neq 0$. *Using the same notations as in Theorem* 2.11 *we now have*

$$\text{(i)} \quad f(t, v) = \begin{cases} b(t), & \textit{when } \operatorname{sgn}(t) = 1,\ \mathcal{T}(t, v) \notin S, \\ b(t)+1, & \textit{when } \operatorname{sgn}(t) = 1,\ \mathcal{T}(t, v) \in S, \\ b(-t(1+t))+1, & \textit{when } \operatorname{sgn}(t) = -1,\ \mathcal{T}(M^{-1}(t, v)) \notin S, \\ b(-t/(1+t))+2, & \textit{when } \operatorname{sgn}(t) = -1,\ \mathcal{T}(M^{-1}(t, v)) \in S; \end{cases}$$

$$\text{(ii)} \quad \tau(t, v) = (|t^{-1}| - f(t, v), (\operatorname{sgn}(t)v + f(t, v))^{-1}), \quad \forall (t, v) \in \Omega_S.$$

A consequence of this is the following corollary.

2.13. Corollary

(i) $f(t, v) \in \mathbb{N}$, $\forall (t, v) \in \Omega_S$.

(ii) $a_{k+1} = f(t_k, v_k)$, $\forall k \geq 0$, *where* $(t_0, v_0) = (x - a_0, 0)$.

For proofs and more results on S-expansions, see [9].

3. Minkowski's diagonal expansion as an S-expansion

From the definition of Minkowski's diagonal continued fraction (DCF) and formula (1.3) it follows at once that the DCF is an S-expansion with

$$S = S_{\text{DCF}} := \left\{ (T, V) \in \Omega; \frac{T}{1+TV} > \frac{1}{2} \right\}$$

(see Fig. 1). Notice that we now have

$$\Delta^+ = \left\{ (T, V) \in \mathbb{R}^2; \frac{T}{1+TV} < \frac{1}{2}, \frac{V}{1+TV} < \frac{1}{2}, T > 0, V > 0 \right\}$$

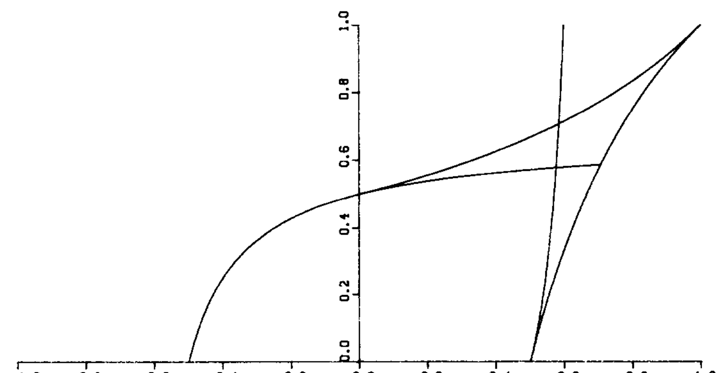

Fig. 1.

and

$$M(\Delta^-)=\left\{(T, V)\in\mathbb{R}^2; \frac{(1+T)(1-V)}{1+TV}<\frac{1}{2}, -\tfrac{1}{2}<T<0, V>0\right\}$$

(see also Fig. 1) where Δ^+, Δ^- are defined as in Theorem 2.8 and M is defined as in Definition 2.10. Since $\mu(S_{\mathrm{DCF}})=1-1/(2\log 2)$ (see [4, p. 286]) we find the following theorem (see also Definition 2.5).

3.1. Theorem. *The two-dimensional ergodic system for the DCF is* $(\Omega_{S_{\mathrm{DCF}}}, \boldsymbol{B}, \rho, \tau)$ *where* ρ *is the probability measure on* $\Omega_{S_{\mathrm{DCF}}}$ *with density* $2/(1+tv)^2$.

In some cases, e.g. Nakada's α-expansions, Bosma's OCF, which are all examples of S-expansions, it is possible to obtain an explicit expression for $f(t, v)$. See also [9, 5]. In these cases, one no longer depends on the RCF to obtain the S-expansion of x. Since S_{DCF} has relatively smooth boundaries, it is possible to obtain an explicit expression for $f=f_{\mathrm{DCF}}$, using Remark 2.6 and Lemma 2.7. Indeed we have the following theorem.

3.2. Theorem. *For all* $(t, v)\in\Omega_{S_{\mathrm{DCF}}}$,

$$f(t, v)=\left[|t^{-1}|+\frac{[|t^{-1}|]+\operatorname{sgn}(t)\cdot v-1}{2([|t^{-1}|]+\operatorname{sgn}(t)\cdot v)-1}\right].$$

Proof. Let $n\in\mathbb{N}$. Put

$$A_n^+:=\left\{(t, v)\in\Delta^+; \frac{1}{n+1}<t<\frac{1}{n}, \mathcal{T}(t, v)\in S_{\mathrm{DCF}}\right\},$$

$$B_n^+:=\left\{(t, v)\in\Delta^+; \frac{1}{n+1}<t<\frac{1}{n}, \mathcal{T}(t, v)\notin S_{\mathrm{DCF}}\right\},$$

$$A_n^-:=\left\{(t, v)\in M(\Delta^-); \frac{1}{n+1}<\frac{-t}{1+t}<\frac{1}{n}, \mathcal{T}(M^{-1}(t, v))\in S_{\mathrm{DCF}}\right\}$$

and

$$B_n^-:=\left\{(t, v)\in M(\Delta^-); \frac{1}{n+1}<\frac{-t}{1+t}<\frac{1}{n}, \mathcal{T}(M^{-1}(t, v))\notin S_{\mathrm{DCF}}\right\}.$$

We will only prove the theorem for $(t, v)\in B:=B_1^+$; the other cases are proved in the same way.

A simple calculation yields, using Theorem 3.1 and the definition of $\mathcal{T}$,

$$\textbf{(3.3)}\quad B:=\left\{(t, v)\in\Omega_{S_{\mathrm{DCF}}}; \frac{(1-t)(1+v)}{1+tv}>\frac{1}{2}, \frac{t}{1+tv}<\frac{1}{2}, \frac{v}{1+tv}<\frac{1}{2}\right\}$$

$$=\left\{(t, v)\in\Omega_{S_{\mathrm{DCF}}}; v<\frac{1-2t}{3t-2}, v>\frac{2t-1}{t}, v<\frac{1}{2-t}\right\}.$$

From Theorem 2.13 we have $f(t, v) = 1$; hence we must show that

$$\left[|t^{-1}| + \frac{[|t^{-1}|] + \operatorname{sgn}(t) \cdot v - 1}{2([|t^{-1}|] + \operatorname{sgn}(t) \cdot v) - 1}\right] = 1.$$

Now

$$\left[|t^{-1}| + \frac{[|t^{-1}|] + \operatorname{sgn}(t) \cdot v - 1}{2([|t^{-1}|] + \operatorname{sgn}(t) \cdot v) - 1}\right] = \left[\frac{1}{t} + \frac{v}{2v+1}\right]$$

and we have, due to $(t, v) \in B$,

$$\frac{1+2v}{3v+2} < t < \frac{1}{2-v}.$$

Since

$$2 - v + \frac{v}{2v+1} > 1 \quad \text{for } (t, v) \in B,$$

we thus find

$$\left[\frac{1}{t} + \frac{v}{2v+1}\right] = 1. \qquad \square$$

3.4. Remark. Notice that we also have that $h(\tau) = \frac{1}{3}\pi^2$.

4. The distribution of some sequences connected with Minkowski's diagonal expansion

Only a few metrical results are known for the DCF; they are to be found in [4]. These results are

(i) for almost all x the sequence $(\Theta_k(x))_{k \geq 0}$ is uniformly distributed over the interval $[0, \frac{1}{2}]$;

(ii) let x be an irrational number and let the monotonic function $k: \mathbb{N} \to \mathbb{N}$ be such that

$$\frac{r_n}{s_n} = \frac{p_{k(n)}}{q_{k(n)}}, \quad n = 1, 2, \ldots;$$

then one has, for almost all x,

$$\lim_{n \to \infty} \frac{k(n)}{n} = 2 \log 2 = 1.3862 \ldots.$$

Using the theory of S-expansions we are able to extend this considerably. For instance, we will obtain for almost all x the distribution of the sequences $(\Theta_{k-1}, \Theta_k)_{k \geq 1}$, $(\Theta_{k-1} + \Theta_k)_{k \geq 1}$ and the relative frequency of the partial quotient 1.

Let $[a_0; \varepsilon_1 a_1, \varepsilon_2 a_2, \ldots]$ be the DCF-expansion of the irrational number x and $(r_k/s_k)_{k \geq 1}$ be its sequence of DCF-convergents. From Theorem 3.1 we derive, using techniques analogous to those used in [6, 8], the following theorem.

4.1. Theorem. *For almost all irrational numbers x, the two-dimensional sequence* $(t_k, v_k)_{k\geq 0}$ *is distributed over* $\Omega_{S_{\mathrm{DCF}}} =: \Omega_S$ *according to the density function h, given by* $h(x, y) = 2/(1+xy)^2$.

Since

$$\Theta_k = \Theta_k(x) := s_k|s_k x - r_k| = \frac{|t_k|}{1+t_k v_k}, \quad \forall k \geq 0,$$

$$\Theta_{k-1} = \frac{v_k}{1+t_k v_k}, \quad \forall k \geq 1,$$

it is natural to consider the map $\psi : \Omega_S \to \mathbb{R}^2$, defined by

$$\psi(t, v) := \left(\frac{v}{1+tv}, \frac{|t|}{1+tv}\right), \quad \forall (t, v) \in \Omega_S.$$

Let $\mathcal{A}_1 := \psi(\{(t, v) \in \Omega_S;\ \mathrm{sgn}(t) = +1\})$, $\mathcal{A}_2 := \psi(\{(t, v) \in \Omega_S;\ \mathrm{sgn}(t) = -1\})$. A simple calculation shows that

$$\mathcal{A}_1 = \{(x, y) \in \mathbb{R}^2;\ 0 \leq x, y \leq \tfrac{1}{2}\},$$

$$\mathcal{A}_2 = \{(x, y) \in \mathbb{R}^2;\ 0 \leq x \leq \tfrac{1}{2},\ 0 \leq y \leq \tfrac{1}{2},\ 0 \leq (x-y)^2 + (x+y) \leq \tfrac{3}{4}\}$$

(see also Fig. 2).

Moreover, the absolute value of the Jacobian J of ψ on Ω equals

$$\frac{1-tv}{(1+tv)^3}$$

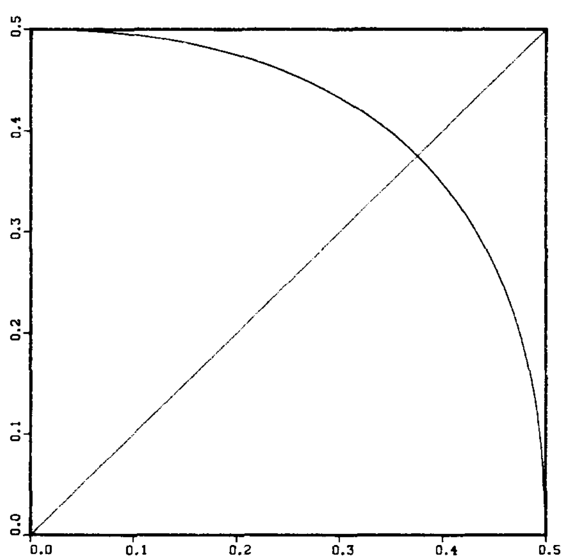

Fig. 2.

and

$$|J|^{-1}\frac{2}{(1+tv)^2}=\frac{2}{\sqrt{1-4tv/(1+tv)^2}}.$$

Hence we have proved the following theorem.

4.2. Theorem. *For all irrational numbers x the sequence* $(\Theta_{k-1}, \Theta_k)_{k\geq 1}$ *is a sequence in* $\mathcal{A} := \mathcal{A}_1 \cup \mathcal{A}_2$ *and for almost all x this sequence is distributed over* $\mathcal{A}$ *according to the density function d, where* $d := d_1 + d_2$ *and*

$$d_1(x,y) := \begin{cases} \dfrac{2}{\sqrt{(1-4xy)}}, & (x,y)\in\mathcal{A}_1,\\ 0, & (x,y)\notin\mathcal{A}_1;\end{cases}$$

$$d_2(x,y) := \begin{cases} \dfrac{2}{\sqrt{(1+4xy)}}, & (x,y)\in\mathcal{A}_2,\\ 0, & (x,y)\notin\mathcal{A}_2.\end{cases}$$

Using techniques analogous to the ones used in [8, Section 4], we find the following corollary.

4.3. Corollary. *For almost all x the sequence* $(\Theta_{k-1} + \Theta_k)_{k\geq 0}$ *is distributed over the interval* $[0, 1]$ *according to the density function m, where* $m := m_1 + m_2$ *and*

$$m_1(a) = \begin{cases} \log\dfrac{1+a}{1-a}, & 0<a<\frac{1}{2},\\ 2\log\dfrac{\sqrt{2}+\sqrt{(1-a)}}{\sqrt{(1+a)}}, & \frac{1}{2}<a<1;\end{cases}$$

$$m_2(a) = \begin{cases} 2\,\mathrm{arctg}\, a, & 0<a<\frac{1}{2},\\ 2\,\mathrm{arctg}\sqrt{\left(\dfrac{3-4a}{2+4a}\right)}, & \frac{1}{2}<a<\frac{3}{4}.\end{cases}$$

For a picture of m_1 and m_2, see Fig. 3.

In a similar way one could determine the distribution of the sequence $(\Theta_{k-1} - \Theta_k)_{k\geq 0}$ over the interval $[-\frac{1}{2}, \frac{1}{2}]$.

A classical result (see [2]) states that for the RCF one has, for almost all x,

$$\lim_{N\to\infty} N^{-1}\#\{j\leq N;\, b_j = 1\} = \frac{1}{\log 2}\log\tfrac{4}{3} = 0.41503\ldots.$$

In Example 1.11 we saw that not every regular partial quotient b_{k+1} equal to 1 disappears in the diagonal expansion of x, as is e.g. the case in the nearest integer continued fraction expansion of x. One may ask how many partial quotients equal

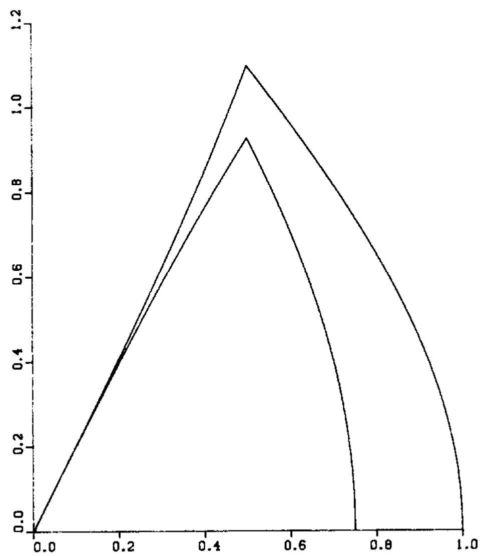

Fig. 3.

to 1 "survive" in the DCF. Note that a regular partial quotient b_{k+1} equal to 1 does not disappear in the diagonal expansion of x if and only if $(T_k, V_k) \in \Delta^+$ and $\mathcal{T}(T_k, V_k) \notin S_{\mathrm{DCF}}$. Hence we find the following theorem.

4.4. Theorem. *A regular partial quotient b_{k+1} equal to* 1 *does not disappear in the diagonal expansion of x if and only if $(T_k, V_k) \in B$, with B as in* (3.3).

For a picture of B, see Fig. 1.

Note that the hyperbolae

$$\frac{(1-t)(1+v)}{1+tv} = \frac{1}{2} \quad \text{and} \quad \frac{t}{1+tv} = \frac{1}{2}$$

are tangent to each other in $(\frac{1}{2}, 0)$ and that

$$\mu(B) = \frac{1}{\log 2}\left(\int_{1/2}^{1} \left(\int_{(2t-1)/t}^{1/(2-t)} \frac{\mathrm{d}v}{(1+tv)^2} \right) \mathrm{d}t - \int_{1/2}^{2-\sqrt{2}} \left(\int_{(2t-1)/(2-3t)}^{1/(2-t)} \frac{\mathrm{d}v}{(1+tv)^2} \right) \mathrm{d}t \right)$$

$$= \frac{1}{\log 2}\left(\log(\sqrt{2}-1) + \sqrt{2} - \tfrac{1}{2}\right) = 0.0473\ldots .$$

Thus we see that in the singularization process leading to the DCF, only 4.7% of the original 41% of partial quotients equal to 1 is saved (for almost all x). After a normalization we therefore find this result.

4.5. Theorem. *Let x be an irrational number and let $(a_k)_{k\geq 0}$ be the sequence of DCF-partial quotients of x. Then,*

$$\lim_{N\to\infty} N^{-1}\#\{j\leq N;\ a_j=1\}=2(\log(\sqrt{2}-1)+\sqrt{2}-\tfrac{1}{2})=0.0656\ldots \quad a.e.$$

5. The DCF-algorithm revisited

In this section we study the values of θ_n corresponding to a block of m consecutive partial quotients equal to 1 in the regular expansion of an irrational number x. The result, a simplification of a method described in [13, p. 183–184], enables us to obtain the diagonal expansion of x from the sequence $(b_n)_{n\geq 0}$ of regular partial quotients of x. We show that the DCF-expansion of a quadratic surd is periodical.

Before stating the main result Theorem 5.3 of this section, we mention two useful tools. A simple consequence of Definition 1.2 of the RCF-operator T is the following lemma.

5.1. Lemma. *Let* $\mathrm{RCF}(x)=[b_0;\, b_1,\ldots,b_k,\ldots]$, $\mathrm{RCF}(x')=[b_0;\, b_1,\ldots,b'_k,\ldots]$, *where* $b_k\neq b'_k$.

- *If k is even, then*

$$b_k<b'_k \Leftrightarrow x<x';$$

- *if k is odd, then*

$$b_k<b'_k \Leftrightarrow x>x'.$$

From Definition 1.7 and Remark 1.8(i) the next lemma follows at once.

5.2. Lemma. *Let* $\xi\in(0,1)$ *and* $\mathrm{RCF}(\xi)=[0;\, B_1, B_2,\ldots]$, *where* $B_1\neq 1$. *Then* $1-\xi=[0;\, 1, B_1-1, B_2,\ldots]$.

In the following we denote a block of m consecutive 1's by 1^m.

5.3. Theorem. *Let* $0<x<1$ *be an irrational number,* $\mathrm{RCF}(x)=[0;\, b_1,\ldots,b_n, 1^m, b_{n+m+1},\ldots]$, *where* $b_n, b_{n+m+1}\neq 1$ *for* $n\geq 1$, *and* $b_{m+1}\neq 1$ *for* $n=0$.

- *If $m=1$ or $n=0$, then*

$$\theta_n>\tfrac{1}{2}.$$

- *If $m>1$ and $n\geq 1$, then*

$$\theta_n>\tfrac{1}{2} \text{ if and only if } [0;\, 1^{m-1}, b_{n+m+1},\ldots]<[0;\, 1, b_n-1,\ldots,b_1],$$

$$\theta_{n+m-1}>\tfrac{1}{2} \text{ if and only if } [0;\, 1^{m-1}, b_n,\ldots,b_1]<[0;\, 1, b_{n+m+1}-1,\ldots],$$

and

$$\theta_{n+e}<\tfrac{1}{2} \quad \text{for } 0<e<m-1.$$

Proof. From the definition of $\mathcal{T}$ one easily derives the following (see also Theorem 2.1).

- In case $m=1$, $(T_n, V_n)\in[\frac{2}{3},1]\backslash\mathbb{Q}\times[0,\frac{1}{2}]\subset S_{\mathrm{DCF}}$. Hence,

(5.4) $\theta_n > \frac{1}{2}$.

- In case $m>1$, we find for $0<e<m-1$, $(T_{n+e}, V_{n+e})\in[\frac{1}{2},\frac{2}{3}]\backslash\mathbb{Q}\times[\frac{1}{2},1]$. Since $[\frac{1}{2},\frac{2}{3}]\backslash\mathbb{Q}\times[\frac{1}{2},1]\cap S_{\mathrm{DCF}}=\emptyset$, we find

(5.5) $\theta_n < \frac{1}{2}$ for $0<e<m-1$.

In general we have, for $0\le e\le m-1$, due to Definition 1.2 and formula (1.3),

(5.6)
$$\theta_{n+e}=\left(\frac{1}{T_{n+e}}+V_{n+e}\right)^{-1}=(1+T_{n+e+1}+V_{n+e})^{-1}$$
$$=(1+[0;1^{m-e-1},b_{n+m+1},\dots]+[0;1^{e},b_n,\dots,b_1])^{-1}.$$

Notice that (5.4) and (5.5) are immediate consequences of (5.6). We now have, due to Lemma 5.2, eq. (5.6) and $b_n, b_{n+m+1}\ne 1$,

$$\theta_n>\tfrac{1}{2} \text{ iff } [0;1^{m-1},b_{n+m+1},\dots]<[0;1,b_n-1,b_{n-1},\dots,b_1],$$

and

$$\theta_{n+m-1}>\tfrac{1}{2} \text{ iff } [0;1^{m-1},b_n,\dots,b_1]<[0;1,b_{n+m+1}-1,b_{n+m+2},\dots],$$

which proves the theorem. □

From Theorem 1.9 and Theorem 5.3 we have a corollary.

5.7. Corollary. *Minkowski's DCF-expansion of an irrational number* $0<x<1$ *is obtained from* RCF(x) *by singularizing each regular partial quotient* b_{k+1} *equal to* 1 *which satisfies one of the four following conditions*:

- $b_{k+1}=b_1$; *that is, the case* $k=0$,
- $b_k, b_{k+2}\ne 1$, $k>0$,
- $b_k\ne 1$, $b_{k+2}=1$ *and* $[0;b_{k+3},\dots]>[0;b_k-1,b_{k-1},\dots,b_1]$, $k>0$,
- $b_k=1$, $b_{k+2}\ne 1$ *and* $[0;b_{k-1},\dots,b_1]>[0;b_{k+2}-1,b_{k+3},\dots]$, $k>0$.

5.8. Remark. Since the conditions

$$[0;b_{k+3},\dots]>[0;b_k-1,\dots,b_1] \quad \text{and}$$
$$[0;b_{k-1},\dots,b_1]>[0;b_{k+2}-1,b_{k+3},\dots]$$

are easily checked, due to Lemma 5.1, it is relatively easy to obtain the diagonal expansion of x from its sequence of regular convergents.

Let $0<x<1$ be a quadratic irrational number. Then, by a classical theorem of Lagrange, the regular expansion of x is ultimately periodic, i.e.

$$\mathrm{RCF}(x)=[0;\, b_1, \ldots, b_{n_0}, \overline{b_{n_0+1}, \ldots, b_{n_0+L}}].$$

We take n_0 and the period length $L \geqslant 1$ both minimal. Put $b_{j,k}=b_{n_0+j+kL}$, where $k \in \mathbb{N}$, $j \geqslant 0$. Suppose that for some $i \in \{1, \ldots, L\}$ we have: $b_{n_0+i}=1$. Now if we want to obtain the DCF(x) from the RCF(x) we have to distinguish the following cases, due to Corollary 5.7.

(i) If $b_{i-1,k}$, $b_{i+1,k} \neq 1$, then $b_{i,k}$ must be singularized, for all $k \in \mathbb{N}$.

(ii) If $b_{i-1,k}=b_{i+1,k}=1$, then $b_{i,k}$ must not be singularized, for all $k \in \mathbb{N}$.

(iii) If $b_{i-1,k} \neq 1$, $b_{i+1,k}=1$, then $b_{i,k}$ must be singularized if and only if

(5.9) $[0;\, c_1, c_2, \ldots]>[0;\, d_1, \ldots, d_{n_0+i+kL}]$

where

$$[0;\, c_1, \ldots]=[0;\, \overline{b_{n_0+i+2}, \ldots, b_{n_0+L}, b_{n_0+1}, \ldots, b_{n_0+i+1}}]$$

and

$$[0;\, d_1, \ldots, d_{n_0+i+kL}]=[0;\, b_{i-1,k}-1, \ldots, b_{i,k-1}, \ldots, b_1].$$

In particular we have, in case $L=3$, $c_1=d_1+1$ and in case $L>3$,

$$c_1=b_{n_0+i+2}, \qquad c_{L-2}=b_{n_0+i-1}, \qquad d_1=b_{n_0+i-1}-1 \quad \text{and} \quad d_{L-2}=b_{n_0+i+2}.$$

Therefore it follows from Lemma 5.1 that (5.9) is equivalent to

(5.10) $$[0;\, b_{n_0+i+2}, \ldots, b_{n_0+L}, b_{n_0+1}, \ldots, b_{n_0+i-1}]$$
$$>[0;\, b_{n_0+i-1}-1, \ldots, b_{n_0+L}, b_{n_0+1}, \ldots, b_{n_0+i+2}],$$

which is independent of k.

Thus we see that $b_{i,k}$ must be singularized, for all $k \in \mathbb{N}$, if and only if (5.10) holds.

(iv) If $b_{i-1,k}=1$, $b_{i+1,k} \neq 1$, then we find, in the same way as in (iii), that $b_{i,k}$ must be singularized for all $k \in \mathbb{N}$ if and only if it holds that

$$[0;\, b_{n_0+i-2}, \ldots, b_{n_0+1}, b_{n_0+L}, \ldots, b_{n_0+i+1}]$$
$$>[0;\, b_{n_0+i+1}-1, b_{n_0+i+2}, \ldots, b_{n_0+i-2}].$$

A direct consequence of (i)-(iv) and Corollary 5.7 is the following theorem.

5.11 Theorem. *The Minkowski DCF-expansion of a quadratic irrational number is periodic.*

5.12. Remark. A different proof of this theorem can be found in [13, Section 41].

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