Neumann problem for generalized $n$-Poisson equation

Ü. Aksoy $^{a,*}$, A.O. Çelebi $^{b}$

$^a$ Department of Mathematics, Atılım University, 06836 İncek, Ankara, Turkey
$^b$ Department of Mathematics, Yeditepe University, 34755 Kaysıdağı, İstanbul, Turkey

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**A B S T R A C T**

Using a hierarchy of integral operators having higher-order Neumann functions and their derivatives as kernels, the Neumann problem for a $2n$th order linear partial complex differential equation is discussed. The solvability of the problem is obtained.

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**1. Introduction**

The Neumann problem is discussed for the homogeneous and inhomogeneous Cauchy–Riemann equations, the Poisson and higher-order Poisson ($n$-Poisson) equations extensively, see e.g. [5–10]. In this article, we discuss the Neumann problem for the “generalized $n$-Poisson equations”. By generalized $n$-Poisson equation we mean a $2n$th order linear complex partial differential equation with leading term as the polyharmonic operator of $n$th order which is a generalization of the generalized Beltrami equation.

The study of boundary value problems and in particular Neumann problems has many applications in applied sciences, among which we may mention hydrodynamics, elasticity theory, crack theory, potential theory, kinematics, medical imaging, etc. [3,11,12,18,19,17,15], besides its theoretical significance. During the derivations of mathematical models for the problems arising in applied sciences, the differential equations we get may be of higher order in a natural way together with some boundary conditions (see [4,13]). But we should note that, the higher-order Neumann functions are not easily expressed explicitly if $n \geq 3$. Thus Neumann problem for generalized $n$-Poisson equation cannot be handled employing the techniques used to derive the solutions of generalized Poisson and bi-Poisson equations given in [2].

One of the two remarkable facts on this article is the use of the functional analytic techniques and particularly Fredholm theory to determine the solvability of the Neumann problem for a higher-order linear complex partial differential equation in the unit disc of the complex plane. We employ the idea of transforming the complex boundary value problems into singular integral equations which has been initiated by Vekua [20] during the investigations of the generalized Beltrami equation. The other important result is related with a class of singular integral operators $S_{n,k,l}$. These operators have the higher-order Neumann functions as kernels. Since they are important in converting the given complex partial differential equation into singular integral equation, we have obtained some important properties of them. The technique we have developed here may be applied to other boundary value problems.

* Corresponding author.
E-mail addresses: uaksoy@atilim.edu.tr (Ü. Aksoy), acelebi@yeditepe.edu.tr (A.O. Çelebi).

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This article will be organized as follows: In Section 2, a review of Neumann functions for \( n \)-Poisson equations and their properties will be given. Section 3 is devoted to the hierarchy of the singular integral operators \( S_{n,k,l} \). In Section 4, we consider the Neumann problem for generalized \( n \)-Poisson equation. The solvability of the problem is obtained using the hierarchy of the operators \( S_{n,k,l} \) and Fredholm theory.

2. Preliminaries

The Neumann function for the unit disc \( \mathbb{D} \) is given by

\[
N_1(z, \zeta) = \log |(\zeta - z)(1 - \bar{z}\zeta)|^2
\]  

(2.1)

for \( z, \zeta \in \mathbb{D} \) [7], which is slightly different from the one given previously [14,21,20]. Eq. (2.1) satisfies

\[
\partial_\nu N_1(z, \zeta) = (z \partial_z + \bar{z} \partial_{\bar{z}}) N_1(z, \zeta) = 2
\]  

(2.2)

for \( z \in \partial \mathbb{D}, \zeta \in \mathbb{D} \). Second-order Neumann function for \( z \) and \( \zeta \) in \( \mathbb{D} \) with \( z \neq \zeta \) is given in [7] by

\[
N_2(z, \zeta) = |z - \zeta|^2 \left[ \log |(\zeta - z)(1 - \bar{z}\zeta)|^2 - 4 \right] - (1 - |z|^2)(1 - |\zeta|^2) + 2(2z - \bar{z}\zeta - \bar{z}\zeta) - \tilde{N}_2(z, \zeta)
\]

where

\[
\tilde{N}_2(z, \zeta) = -\frac{2}{2\pi i} \int_{\mathbb{D}} \left( (z - \bar{z}\zeta - \bar{z}\zeta) \log |\zeta - z|^2 - (1 + |z|^2) \right) \log |\zeta - \zeta|^2 \frac{d\zeta}{\zeta}
\]

\[
- \frac{1 - |z|^2}{\pi} \int_{\mathbb{D}} \left( \frac{1}{1 - z\zeta} + \frac{1}{1 - \bar{z}\zeta} - 1 \right) \log |\zeta - \zeta|(1 - \zeta\bar{\zeta})^2 d\zeta d\bar{\zeta}.
\]

Nevertheless, the higher-order Neumann functions are not easy to derive in their explicit forms but they may be defined iteratively for \( n \in \mathbb{N} \) where \( n \geq 2 \), as

\[
N_n(z, \zeta) = \frac{1}{\pi} \int_{\mathbb{D}} N_1(z, \zeta) N_{n-1}(\zeta, \zeta) d\zeta d\bar{\zeta}.
\]

(2.3)

These functions satisfy

\[
\partial_\nu \partial_{\bar{\nu}} N_n(z, \zeta) = N_{n-1}(z, \zeta)
\]  

(2.4)

in \( \mathbb{D} \),

\[
\partial_\nu N_n(z, \zeta) = \frac{2}{(n-1)!} ((|\zeta|^2 - 1)^n - \sum_{\mu=1}^{n-2} \frac{\mu!^2}{(n-1)!((n-1)!-(n-1)!)} \partial_\nu \partial_{\bar{\nu}} N_{n+1}(z, \zeta)
\]

on \( \partial \mathbb{D} \) and normalization condition

\[
\frac{1}{2\pi i} \int_{\partial \mathbb{D}} N_n(z, \zeta) \frac{dz}{z} = 0,
\]

see [7, Theorem 4.5]. Using the higher-order Neumann functions and higher-order Cauchy–Pompeiu representations, Neumann problems for Poisson and \( n \)-Poisson equations are solved uniquely under some normalization and solvability conditions [7,5,8,9,6].

3. A hierarchy of operators related to Neumann problem for generalized higher-order Poisson equations

In this section, we introduce a hierarchy of integral operators on \( L^p(\mathbb{D}) \) spaces. The simple forms of these operators are given previously in [2] to solve the generalized Poisson and bi-Poisson equations.

Definition 3.1. For \( n \in \mathbb{N}, k, l \in \mathbb{N}_0 \) with \( (k, l) \neq (n, n) \) and \( k + l \leq 2n \), we define

\[
(S_{n,k,l}F)(z) := S_{n,k,l}F(z) = \frac{1}{\pi} \int_{\mathbb{D}} \partial_\nu \partial_{\bar{\nu}} N_n(z, \zeta) F(\zeta) d\zeta d\bar{\zeta}
\]

for a suitable complex-valued function \( F \) given in \( \mathbb{D} \).
The operators $S_{n,k,l}$ are weakly singular for $k + l < 2n$ and strongly singular for $k + l = 2n$. Using the above definition we can obtain the following operators by some particular choices of $n, k$ and $l$:

\[ S_{1,0,0}F(z) = \frac{1}{\pi} \int_D N_1(z, \zeta)F(\zeta) \, d\zeta \, dn = \frac{1}{\pi} \int_D \log|z - \zeta|(1 - z\bar{\zeta})^2 F(\zeta) \, d\zeta \, dn, \]
\[ S_{1,1,0}F(z) = \frac{1}{\pi} \int_D \partial_2 N_1(z, \zeta)F(\zeta) \, d\zeta \, dn = -\frac{1}{\pi} \int_D \left( \frac{1}{(z - \zeta)} + \frac{\bar{\zeta}}{(1 - z\bar{\zeta})} \right) F(\zeta) \, d\zeta \, dn, \]
\[ S_{1,2,0}F(z) = \frac{1}{\pi} \int_D \partial_2^2 N_1(z, \zeta)F(\zeta) \, d\zeta \, dn = -\frac{1}{\pi} \int_D \left( \frac{1}{(z - \zeta)^2} + \frac{\bar{\zeta}^2}{(1 - z\bar{\zeta})^2} \right) F(\zeta) \, d\zeta \, dn. \]

Thus, $S_{1,0,0}$ and $S_{1,1,0}$ are modified forms of the operators $\tilde{P}_0$, $\tilde{T}_1$ and $S_{1,2,0}$ is the operator $\tilde{T}_2$ and all three operators are given by Vinogradov [21] and Vekua [20]. Besides, the properties of $S_{2,k,l} = P_{k,l}$ are investigated in [2]. It is known that $P_{k,l}$ are Calderon–Zygmund type singular integral operators when $k + l = 4$. It can be shown that these operators satisfy

\[ \partial_2 S_{1,0,0}F = S_{1,1,0}F \quad \text{and} \quad \partial_2^2 S_{1,0,0}F = S_{1,2,0}F \quad (3.5) \]

for $F \in L^p(D)$, $p > 2$, in Sobolev’s sense.

3.1. Properties of the operators $S_{n,k,l}$

First, we will give some properties of operators $S_{1,0,0}$, $S_{1,1,0}$ and $S_{1,2,0}$.

**Lemma 3.1.** For $F \in L^p(D)$ where $p > 2$

\[ |S_{1,k,0}F(z)| \leq C(k, p) \| F \|_{L^p(D)} \quad (3.6) \]

for $k = 0, 1$,

\[ |S_{1,k,0}F(z_1) - S_{1,k,0}F(z_2)| \leq C(k, p) \| F \|_{L^p(D)} \begin{cases} |z_1 - z_2|^{(p-2)/p} & \text{if } k = 1, \\ |z_1 - z_2| & \text{if } k = 0 \end{cases} \quad (3.7) \]

for $z_1, z_2 \in D$ and

\[ \|S_{1,2,0}F\|_{L^p(D)} \leq C(p) \| F \|_{L^p(D)} \quad (3.8) \]

for $p > 1$. Moreover

\[ \|S_{1,2,0}F\|_{L^2(D)} \leq \| F \|_{L^2(D)} \quad (3.9) \]

holds.

**Proof.** Using the above representations for $S_{1,0,0}$, $S_{1,1,0}$ and Hölder’s inequality, it follows that for $p > 2$,

\[ |S_{1,k,0}F(z)| \leq \| F \|_{L^p(D)} \| \partial_2^k N_1(., \zeta) \|_{L^p(D)} \]

for $k = 0, 1$ and $1/p + 1/q = 1$. We deduce (3.6) from these inequalities. For the proof of (3.7) with $k = 1$, we use the technique given in [20] for the Pompeiu operator $T$, Hölder’s inequality and the fact that $\left| \frac{z - \zeta}{1 - z\bar{\zeta}} \right| \leq 1$ for $z, \zeta \in D$. For the case $k = 0$ we use the boundedness of $S_{1,1,0}$ and the mean value theorem. The proofs of (3.8) and (3.9) may be done as in [20, p. 337]. □

**Lemma 3.2.** For $F \in L^p(D)$,

\[ S_{n,k,l}F(z) = \begin{cases} S_{n-1,k-1,0}F(z), & k \geq l, \\ S_{n-k,0,l-k}F(z), & k < l \end{cases} \]

for suitable $p$. Moreover

\[ S_{n,k,l}F(z) = \overline{S_{n,l,k}F(z)} := S_{n,k,l}F(z). \quad (3.10) \]
Proof. For $k > l$, the Neumann function $N_n(z, \zeta)$ satisfies

$$\partial_z^k \partial_{\zeta}^l N_n(z, \zeta) = \partial_z^{k-l} \partial_{\zeta}^l N_n(z, \zeta) = \partial_z^{k-l} N_{n-1}(z, \zeta)$$

by (2.4). Thus, using Definition 3.1 we have $S_{n,k,l} F(z) = S_{n-1,k-1,0} F(z)$. For $k < l$, the similar arguments apply. The relation

$$\partial_z^k \partial_{\zeta}^l N_n(z, \zeta) = \partial_z^l \partial_{\zeta}^k N_n(z, \zeta)$$

for higher-order Neumann functions proves (3.10). □

From now on, we will give the properties of $S_{n,k,l}$ for $l = 0, 0 \leq k \leq 2n$ without loss of generality. Using Lemma 3.2, similar properties can be obtained for the operators $S_{n,k,l}$ with $l \neq 0$.

Lemma 3.3. For $F \in LP(\mathbb{D})$, $p > 1$

$$S_{n,0,0} F(z) = S_{1,0,0}^n F(z),$$

$$S_{n,1,0} F(z) = \partial_z S_{n,0,0} F(z) = S_{1,1,0}^n F(z),$$

$$S_{n,2,0} F(z) = \partial_z^2 S_{n,0,0} F(z) = S_{1,2,0}^n F(z)$$

hold.

Proof. The operator $S_{n,0,0}$ is given by

$$S_{n,0,0} F(z) = \frac{1}{\pi} \int_{\mathbb{D}} N_n(z, \zeta) F(\zeta) \, d\zeta \, d\eta = \frac{1}{\pi} \int_{\mathbb{D}} \left( \frac{1}{\pi} \int_{\mathbb{D}} N_1(z, \zeta) N_{n-1}(\zeta, \zeta) \, d\zeta \, d\eta \right) F(\zeta) \, d\zeta \, d\eta.$$

After changing the order of integration

$$S_{n,0,0} F(z) = S_{1,0,0} (S_{n-1,0,0} F(z)).$$

Thus inductively we get (3.11).

To prove (3.12) and (3.13) we use the fact that $S_{1,0,0} F$ has generalized derivatives given by (3.5). □

Lemma 3.4. If $F \in W^{m-p}(\mathbb{D})$ then

$$\partial_z^{m-1} S_{1,2,0} F(z) = S_{1,1,0} ((D - D_\nu)^m F(z))$$

and $\partial_z^{m-1} S_{1,2,0} F$ is in $L^p(\mathbb{D})$ where $D F(z) = \partial_z F(z), D_\nu F(z) = \partial_\nu (\bar{\zeta}^2 F(z))$ and $m \in \mathbb{N}$.

Proof. $S_{1,2,0}$ can be rewritten as

$$S_{1,2,0} F(z) = -\frac{1}{\pi^2} \int_{\mathbb{D}} \left( \frac{1}{\zeta - z} - \frac{\bar{\zeta}^2}{(1 - \zeta \bar{\zeta})^2} \right) F(\zeta) \, d\zeta \, d\eta.$$
or
\[ S_{1,2,0}F(z) = S_{1,1,0}((D - D_*)F(z)) \] (3.15)
that corresponds to the case \( m = 1 \). Using (3.15) we have
\[ \partial_z S_{1,2,0}F(z) = \partial_z S_{1,1,0}((D - D_*)F(z)) = S_{1,2,0}((D - D_*)F(z)) = S_{1,1,0}((D - D_*)^2 F(z)) \]
and differentiating iteratively, we get (3.14). \( \square \)

**Corollary 3.1.** If
\[ F \in \begin{cases} L^p(\mathbb{D}), & 1 \leq k \leq 2n - 1, \\ W^{1,p}(\mathbb{D}), & k = 2n \end{cases} \]
then
\[ S_{n,k,0}F(z) = S_{1,1,0}((D - D_*)^k S_{1,0,0}^{n-1}F(z)) \] (3.16)
and \( S_{n,k,0}F \in L^p(\mathbb{D}) \) holds.

**Proof.** By Lemma 3.3 we have
\[ S_{n,k,0}F(z) = \partial_z^k S_{n,0,0}F(z) = \partial_z^k S_{1,0,0}^{n}F(z). \]
Then by Lemmas 3.3 and 3.4
\[ S_{n,k,0}F(z) = \partial_z^{k-2} (\partial_z^2 S_{1,0,0}^{n}F(z)) \]
is obtained since \( S_{1,0,0}^{n}F(z) \in W^{k-1,p}(\mathbb{D}) \) holds with \( 1 \leq k \leq 2n - 1 \) for \( F \in L^p(\mathbb{D}) \). For the case \( k = 2n, S_{1,0,0}^{n}F(z) \in W^{k-1,p}(\mathbb{D}) \) holds if \( F \in W^{1,p}(\mathbb{D}) \). \( \square \)

The following lemma proves the boundedness of the operators \( S_{n,k,l} \).

**Lemma 3.5.** Let \( F \in L^p(\mathbb{D}), p > 2 \) and \( k + l < 2n \). Then,
\[ |S_{n,k,l}F(z)| \leq C \| F \|_{L^p(\mathbb{D})} \] (3.17)
for \( z \in \mathbb{D} \).

**Proof.** It is enough to prove this property for the operators \( S_{n,k,0} \) for \( k \leq 2n - 1 \). The case \( n = 1 \) is proved in Lemma 3.1. For \( n > 1 \) and \( k = 0 \) we have
\[ |S_{n,0,0}F(z)| = |S_{1,0,0}^{n}F(z)| \leq C \| S_{1,0,0}^{n-1}F \|_{L^p(\mathbb{D})}. \]
By iteration we get
\[ |S_{n,0,0}F(z)| \leq C^n \| F \|_{L^p(\mathbb{D})}. \]
In the case of \( 1 \leq k \leq 2n - 1, \), we can write
\[ |S_{n,k,0}F(z)| = |S_{1,1,0}((D - D_*)^k S_{1,0,0}^{n-1}F(z))| \]
by Corollary 3.1 and
\[ |S_{1,1,0}((D - D_*)^k S_{1,0,0}^{n-1}F(z))| \leq C \| (D - D_*)^k S_{1,0,0}^{n-1}F \|_{L^p(\mathbb{D})} \] (3.18)
by Lemma 3.1. It is easy to see that
\[ \| (D - D_*)S_{1,0,0}^{n-1}F \|_{L^p(\mathbb{D})} = \| S_{1,1,0}(S_{1,0,0}^{n-2}F) - 2S_{1,0,0}^{n-1}F + 2^2 S_{1,1,0}(S_{1,0,0}^{n-2}F) \|_{L^p(\mathbb{D})} \leq C(p) \| F \|_{L^p(\mathbb{D})} \]
holds by Lemma 3.4. Using the same technique iteratively, we find
\[ \| (D - D_*)^k S_{1,0,0}^{n-1}F \|_{L^p(\mathbb{D})} \leq C^n \| F \|_{L^p(\mathbb{D})} \] (3.19)
which shows that \( (D - D_*)^k S_{1,0,0}^{n-1}F \in L^p(\mathbb{D}) \). Substituting (3.19) in (3.18) we find
\[ |S_{n,k,0}F(z)| \leq C \| F \|_{L^p(\mathbb{D})} \]
which is the required result. \( \square \)
In the following we prove the uniform continuity of weakly singular integral operators $S_{n,k,l}$.

**Lemma 3.6.** Let $F \in L^p(\mathbb{D})$, $p > 2$ and $k + l < 2n$. Then for $z_1, z_2 \in \mathbb{D}$,

$$
|S_{n,k,l}F(z_1) - S_{n,k,l}F(z_2)| \leq C \|F\|_{L^p(\mathbb{D})} \left\{ \begin{array}{ll} |z_1 - z_2|^{(p-2)/p} & \text{if } k + l = 2n - 1, \\ |z_1 - z_2| & \text{otherwise.} \end{array} \right.
$$

(3.20)

**Proof.** For $n > 1$ and $0 \leq k + l \leq 2n - 2$

$$
\partial_z S_{n,k,l}F(z) = S_{n,k+1,l}F(z)
$$

and

$$
\partial_z S_{n,k,l}F(z) = S_{n,k,l+1}F(z)
$$

are bounded in $\mathbb{D}$ by Lemma 3.5. Then using the mean value theorem, the result is achieved.

For the case $k + l = 2n - 1$, using Corollary 3.1, we write

$$
S_{n,2n-1,0}F(z) = S_{1,1,0}((D - D_+)^{2n-2}S_{1,0,0}F(z)),
$$

and the result follows from Lemma 3.1 and (3.14). □

Next, the $L^p$ boundedness of the strongly singular operators will be shown.

**Lemma 3.7.** If $k + l = 2n$, then $S_{n,k,l}F \in L^p(\mathbb{D})$ for $F \in L^p(\mathbb{D})$ with $p > 1$ and

$$
\|S_{n,k,l}F\|_{L^p(\mathbb{D})} \leq C_p \|F\|_{L^p(\mathbb{D})}.
$$

(3.21)

**Particularly**

$$
\|S_{n,n+1,n-1}F\|_{L^2(\mathbb{D})} = \|S_{n,n-1,n+1}F\|_{L^2(\mathbb{D})} \leq \|F\|_{L^2(\mathbb{D})}.
$$

(3.22)

**Proof.** Eq. (3.22) can be obtained by use of Lemmas 3.1 and 3.2 iteratively. We need to prove (3.21) for the operator $S_{n,2n,0}$ for $n > 1$. In this case, by Corollary 3.1, we have

$$
S_{n,2n,0}F(z) = \partial_z S_{1,1,0}((D - D_+)^{2n-2}S_{1,0,0}F(z)) = S_{1,2,0}((D - D_+)^{2n-2}S_{1,0,0}F(z)).
$$

(3.23)

To prove the result, we use (3.23) with the $L^p$ boundedness of $S_{1,2,0}$ and $S_{1,0,0}$. □

4. Neumann problem for a generalized $n$-Poisson equation

In this section, using the properties of the operators $S_{n,k,l}$, we investigate the Neumann problem for generalized $n$-Poisson equations. Now, let us state the problem.

**Problem N.** Find $w \in W^{2n,p}(\mathbb{D})$ as a solution to the $n$th order complex differential equation

$$
\frac{\partial^{2n}w}{\partial z^n \partial \bar{z}^n} + \sum_{k+l=2n} \left( q^{(1)}_{kl}(z) \frac{\partial^{2n}w}{\partial z^k \partial \bar{z}^l} + q^{(2)}_{kl}(z) \frac{\partial^{2n}w}{\partial z^l \partial \bar{z}^k} \right) + \sum_{0 \leq k+l \leq 2n} \left[ a_{kl}(z) \frac{\partial^{k+l}w}{\partial z^k \partial \bar{z}^l} + b_{kl}(z) \frac{\partial^{k+l}w}{\partial \bar{z}^k \partial z^l} \right] = f(z) \quad \text{in } \mathbb{D}
$$

(4.24)

with Neumann conditions

$$
\bar{\partial}_\gamma \left( \partial_\gamma \partial_\zeta \right)^{n} w = \gamma_{\sigma} \quad \text{on } \partial \mathbb{D}, \quad \gamma_{\sigma} \in C(\partial \mathbb{D}) \text{ for } 0 \leq \sigma \leq n - 1,
$$

(4.25)

satisfying the normalization conditions

$$
\frac{1}{2\pi i} \int_{\bar{\partial}(\mathbb{D})} (\partial_\gamma \partial_\zeta)^n w(\zeta) \frac{d\zeta}{\zeta} = c_{\sigma}, \quad c_{\sigma} \in \mathbb{C} \text{ for } 0 \leq \sigma \leq n - 1,
$$

(4.26)

where

$$
a_{kl}, b_{kl}, f \in L^p(\mathbb{D}),
$$

(4.27)

and $q^{(1)}_{kl}$ and $q^{(2)}_{kl}$, are measurable bounded functions subject to

$$
\sum_{k+l=2n} \left( |q^{(1)}_{kl}(z)| + |q^{(2)}_{kl}(z)| \right) \leq q_0 < 1. \quad \square
$$

(4.28)
We begin by transforming the Problem N into a singular integral equation.

**Lemma 4.1.** The Neumann problem (4.24), (4.25) and (4.26) is equivalent to the singular integral equation

\[
(1 + \hat{N} + \hat{K}) g = \hat{f},
\]

if

\[
w = \varphi + S_{n,0}g,
\]

where

\[
\varphi(z) = \sum_{\mu=0}^{n-1} \left\{ \frac{1}{2} c_\mu \partial_n N_{\mu+1}(z, \zeta) - \frac{1}{4\pi i} \int_{\partial D} N_{\mu+1}(z, \zeta) \frac{d\zeta}{\zeta} \right\}
\]

and

\[
\hat{N} g = \sum_{k+l=2n, k \neq l} \left( q_{kl}^{(1)} S_{n,k,l}g + q_{kl}^{(2)} \frac{S_{n,k,l}g}{\zeta} \right),
\]

\[
\hat{K} g = \sum_{k+l=2n} \left( a_{kl} S_{n,k,l}g + b_{kl} \frac{S_{n,k,l}g}{\zeta} \right),
\]

\[
\hat{f} = f - L \varphi
\]

in which

\[
L \varphi := \sum_{k+l=2n, k \neq l} \left( q_{kl}^{(1)} (z) \frac{\partial^{2n} \varphi}{\partial z^k \partial \bar{z}^l} + q_{kl}^{(2)} (z) \frac{\partial^{2n} \varphi}{\partial z^k \partial \bar{z}^l} \right) + \sum_{0 \leq k+l < 2n} \left[ a_{kl}(z) \frac{\partial^{k+l} \varphi}{\partial z^k \partial \bar{z}^l} + b_{kl}(z) \frac{\partial^{k+l} \varphi}{\partial z^k \partial \bar{z}^l} \right].
\]

**Proof.** It is known that [7]

\[
w(z) = \sum_{\mu=0}^{n-1} \left\{ \frac{1}{2} c_\mu \partial_n N_{\mu+1}(z, \zeta) - \frac{1}{4\pi i} \int_{\partial D} N_{\mu+1}(z, \zeta) \frac{d\zeta}{\zeta} \right\} + \frac{1}{\pi} \iint_D N_n(z, \zeta) g(\zeta) d\zeta d\eta
\]

is the unique solution of the Neumann-n problem

\[
(\partial_\zeta \partial_{\bar{\zeta}})^n w = g \quad \text{in } \mathbb{D}, \quad g \in L^p(\mathbb{D}) \text{ for } 1 < p < +\infty,
\]

\[
\partial_n (\partial_\zeta \partial_{\bar{\zeta}})^\sigma w = \gamma_\sigma \quad \text{on } \partial \mathbb{D}, \quad \gamma_\sigma \in C(\partial \mathbb{D}) \text{ for } 0 \leq \sigma \leq n - 1
\]

satisfying

\[
\frac{1}{2\pi i} \int_{\partial \mathbb{D}} (\partial_\zeta \partial_{\bar{\zeta}})^n w(\zeta) \frac{d\zeta}{\zeta} = c_\sigma, \quad c_\sigma \in \mathbb{C} \text{ for } 0 \leq \sigma \leq n - 1
\]

iff

\[
\frac{1}{2\pi i} \int_{\partial \mathbb{D}} \gamma_\sigma(\zeta) \frac{d\zeta}{\zeta} = \sum_{\mu=\sigma+1}^{n-1} \alpha_{\mu-\sigma} c_\mu + \frac{1}{\pi} \iint_D \partial_\zeta N_{n-\sigma}(z, \zeta) g(\zeta) d\zeta d\eta
\]

for \(\alpha_1 = 2\) and for \(3 \leq k\)

\[
\alpha_{k-1} = -\sum_{\mu=\lfloor \frac{k}{2} \rfloor}^{k-2} \frac{\mu!^2}{(k-1)! (k-1-\mu)!^2 (2\mu-k+1)!} \alpha_\mu.
\]

Now we write \(w = \varphi + S_{n,0}g\) in the differential equation (4.24) and use the differentiation properties of the operators \(S_{n,k,l}\). Thus, singular integral equation (4.29) is obtained. □
4.1. Solvability of the problem

We decompose the strongly singular integral operator \( \hat{N} \) as the following:

\[
\hat{N} g = \sum_{k, l = 1}^{2n} (q_{kl}^{(1)} S_{n, k, l} g + q_{kl}^{(2)} S_{n, k, l} \hat{g}) + \sum_{k, l = 1}^{2n} (q_{kl}^{(1)} S_{n, h, l} g + q_{kl}^{(2)} S_{n, h, l} \hat{g}) := \hat{N}_1 g + \hat{N}_2 g.
\]

Thus, the singular integral equation (4.29) is rewritten as

\[
(I + \hat{N}_1 + \hat{N}_2 + \hat{K}) g = \hat{f}.
\]

It is known by Lemma 3.7 that \( \|S_{n, k, l}\|_{L^2(\mathbb{D})} \leq 1 \) for \( k + l = 2n, |k - l| = 2 \). Then using the ellipticity condition (4.28) and the Riesz–Thorin Interpolation Theorem, \( \|\hat{N}_1\|_{L^p(\mathbb{D})} < 1 \) holds for \( 2 < p < 2 + \epsilon \) for some \( \epsilon > 0 \). Thus \( I + \hat{N}_1 \) is an invertible operator on \( L^p(\mathbb{D}) \). Properties given in Lemmas 3.5 and 3.6 for the operators \( S_{n, k, l} \) imply that \( \hat{K} \) is a compact operator in \( L^p(\mathbb{D}) \) by the Arzela–Ascoli Theorem. Thus Fredholm alternative may be employed for the operator \( I + \hat{N}_1 + \hat{K} \). \( \hat{N}_2 \) is a bounded operator in \( L^p(\mathbb{D}) \) by Lemma 3.5. Thus, \( I + \hat{N} + \hat{K} \) is a perturbation of an index-zero Fredholm operator with a bounded operator. Employing the bounded index stability theorem [16], the following result can be stated.

**Theorem 4.1.** If the inequality

\[
q_0 \max_{k, l = 1}^{2n} \|S_{n, k, l}\|_{L^p(\mathbb{D})} \| (I + \hat{N}_1)^{-1} - K \|_{L^p(\mathbb{D})} < 1
\]

(4.33)

holds for some \( K \in K (L^p(\mathbb{D})) \), \( 0 < p - 2 < \epsilon \), then Eq. (4.24) with the boundary conditions (4.25) and normalization conditions (4.26) has a solution of the form \( w(z) = \psi(z) + S_{n, 0, 0} g(z) \), where \( g \in L^p(\mathbb{D}) \) is a solution of the singular integral equation (4.29) subject to the solvability conditions

\[
\frac{1}{2\pi} \int_{\partial \mathbb{D}} \gamma_\sigma(\zeta) \frac{d\zeta}{\zeta} = \sum_{\mu = -\sigma + 1}^{n-1} \alpha_\mu - \sigma \zeta_\mu + \frac{1}{\pi} \int_{\partial \mathbb{D}} e_{n, \sigma}(z, \zeta) \eta(z) d\zeta d\eta,
\]

(4.34)

where \( 0 \leq \sigma \leq n - 1 \), \( \alpha_1 = 2 \) and for \( 3 \leq k \)

\[
\alpha_{k-1} = -\sum_{\mu = 1}^{k-2} \frac{\mu^2}{(k-1)!} (k-1 - \mu)^2 (2\mu - k + 1) \alpha_\mu.
\]

(4.35)

**Remark 1.** For the case \( p > 2 \), the techniques given in [1] may be used to discuss the solvability of the problem defined by (4.24)–(4.26).

**References**