ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Journal of Combinatorial Theory, Series A 115 (2008) 1–31
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

Invariant theory for singular α-determinants

Kazufumi Kimoto [a,1], Masato Wakayama [b,2]

[a] *Department of Mathematical Science, University of the Ryukyus, Senbaru, Nishihara, Okinawa 903-0231, Japan*
[b] *Faculty of Mathematics, Research Center for Industrial Technology, Kyushu University, Japan*

Received 30 July 2006
Available online 19 April 2007
Communicated by George Andrews

Dedicated to Roger Howe on the occasion of his 60th birthday

Abstract

From the irreducible decompositions' point of view, the structure of the cyclic $GL_n(\mathbb{C})$-module generated by the α-determinant degenerates when $\alpha = \pm\frac{1}{k}$ ($1 \leqslant k \leqslant n-1$) (see [S. Matsumoto, M. Wakayama, Alpha-determinant cyclic modules of $\mathfrak{gl}_n(\mathbb{C})$, J. Lie Theory 16 (2006) 393–405]). In this paper, we show that $-\frac{1}{k}$-determinant shares similar properties which the ordinary determinant possesses. From this fact, one can define a new (relative) invariant called a *wreath determinant*. Using (GL_m, GL_n)-duality in the sense of Howe, we obtain an expression of a wreath determinant by a certain linear combination of the corresponding ordinary minor determinants labeled by suitable rectangular shape tableaux. Also we study a wreath determinant analogue of the Vandermonde determinant, and then, investigate symmetric functions such as Schur functions in the framework of wreath determinants. Moreover, we examine coefficients which we call (n,k)-sign appeared at the linear expression of the wreath determinant in relation with a zonal spherical function of a Young subgroup of the symmetric group $\mathfrak{S}_{nk}$.

Keywords: α-Determinants; (GL_m, GL_n)-duality; Wreath products; Partitions; Symmetric functions; Young symmetrizers; Irreducible decomposition; Zonal spherical functions

Contents

E-mail addresses: kimoto@math.u-ryukyu.ac.jp (K. Kimoto), wakayama@math.kyushu-u.ac.jp (M. Wakayama).
1 Partially supported by Grant-in-Aid for Young Scientists (B) No. 16740021.
2 Partially supported by Grant-in-Aid for Scientific Research (B) No. 15340012.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2007.03.008

1. Away from the multiplication law

There is a notion called the *α-determinant* for a square matrix in probability theory. It was first introduced in [11] and actually appeared as coefficients of the Taylor expansion of $\det(I-\alpha A)^{-1/\alpha}$. This expansion has applications, in particular, to multivariate binomial and negative binomial distributions. Moreover, recently in [9], the α-determinant is use to define a random point process through a study of the Fredholm determinants of certain integral operators.

The α-determinant $\det^{(\alpha)}(X)$ for a matrix X (see (2.1) for the definition) does not have the multiplication property which the ordinary determinant $\det(X)$ possesses. It is, however, interesting from a viewpoint of invariant theory because the α-determinant is regarded as an interpolation of the determinant ($\alpha=-1$) and permanent ($\alpha=1$)—recall that each of them generates an irreducible representation of the general linear group $GL_n(\mathbb{C})$; as representations of the special linear group $SL_n(\mathbb{C})$, the former defines the trivial representation and the latter generates the representation on the space of symmetric n-tensors of (the natural representation on) $\mathbb{C}^n$. These facts raise naturally the following question:

"*Where had the multiplication law gone when α moved away from* -1?"

The multiplication law of the determinant is equivalent to the fact that $GL_n(\mathbb{C})\cdot\det(X)\subset\mathbb{C}^\times\det(X)$. Hence, it is natural to ask the question what the smallest invariant space containing $GL_n(\mathbb{C})\cdot\det^{(\alpha)}(X)$ is. From this point of view, Matsumoto and the second author [8] have studied recently the irreducible decomposition of the cyclic module $\mathcal{U}(\mathfrak{gl}_n)\cdot\det^{(\alpha)}(X)$ and showed that the structure of the module changes drastically when α is contained in the set $\{\pm1,\pm\frac{1}{2},\dots,\pm\frac{1}{n-1}\}$. In fact, one can see that the irreducible decomposition of the cyclic module $\mathcal{U}(\mathfrak{gl}_n)\cdot\det^{(\alpha)}(X)$ degenerates when α is one of such values. More precisely, if we denote by $m^\lambda(\alpha)$ the multiplicity of the irreducible highest weight $\mathcal{U}(\mathfrak{gl}_n)$-module corresponding to a partition λ appeared in the decomposition, then, for instance, we have $m^\lambda(-\frac{1}{k})=0$ when the first component of λ is greater than k (see (3.1)). Therefore, we shall call α *singular* if $\alpha\in\{\pm1,\pm\frac{1}{2},\dots,\pm\frac{1}{n-1}\}$. This result indicates that if α is singular, then $\det^{(\alpha)}(X)$ may share some distinguished feature which explains why such a drastic change of the module structure happens. The special emphasis in this paper is laid on the study of the case $\alpha=-\frac{1}{k}$ ($k\in\mathbb{Z}_{>0}$). Actually, we first show that $\det^{(-\frac{1}{k})}(X)$ has a certain alternating property which is considered as a generalization of the alternating property of the ordinary determinant (as well as its multilinearity) in Section 2. We also show that such an alternating property characterizes the $-\frac{1}{k}$-determinants through the cyclic module $\mathcal{U}(\mathfrak{gl}_n)\cdot\det^{(-\frac{1}{k})}(X)$ by the effective use of the Young symmetrizer (Section 3). We note that a quantum analogue of the α-determinant (which we call quantum α-determinant) is introduced and studied in [6], however, it is much more difficult to describe the singular values in the quantum case.

Under these studies, one of the main purpose of the present paper is to construct an invariant, which we will call a *wreath determinant*, defined by means of a singular α-determinant. In order to obtain this new invariant for a rectangular matrix, we consider a $kn \times kn$ matrix gotten from multiplexing a given $kn \times n$ matrix A by tensoring the $1 \times k$ matrix $(1, 1, \dots, 1)$. By using the property of α-determinants developed in Section 2 for $\alpha = -\frac{1}{k}$, we show that the wreath determinant is a relative invariant for the action of the wreath product of symmetric groups $\mathfrak{S}_k \wr \mathfrak{S}_n$ (see [7]) in Section 4. Furthermore, in Section 5, we give an expression of the wreath determinant of $kn \times n$-matrix A by a linear sum of the nth minor determinants of A labeled by the corresponding rectangular shaped tableaux. In the derivation of this expression, (GL_m, GL_n)-duality in the sense of [3] provides a guiding principle. We then, beside the expression above, derive another expression of such a wreath determinant conceptually by the Frobenius reciprocity. As a corollary of the proof, we find that the wreath determinant is a relative invariant of $(\mathfrak{S}_k \wr \mathfrak{S}_n) \times GL_n$. We also give one remark on the background which explains how to get this expression and to understand a structure of the cyclic module $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^{\ell}$ for a general positive integer ℓ in the framework of (GL_m, GL_n)-duality. Note that the latter closely relates a problem for calculating a certain plethysm [4,7].

The Cauchy determinant formula (see, e.g. [12])

$$\det\left(\frac{1}{x_i + y_j}\right)_{1 \leqslant i,j \leqslant n} = \frac{\Delta_n(x)\Delta_n(y)}{\prod_{i,j=1}^{n}(x_i + y_j)}$$

can be considered as one of the most important determinant formula from the representation theoretic point of view. In Section 6, we prove an analogue of the Cauchy determinant formula for the wreath determinants. It naturally leads us to study the wreath determinant of a Vandermonde type. The aforementioned study enables us to deduce a formula for the Schur functions in terms of the $-\frac{1}{k}$-determinants of the Vandermonde type, which is regarded as a $-\frac{1}{k}$-analogue of the expression

$$s_\lambda(x_1, \dots, x_n) = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leqslant i,j \leqslant n}}{\det(x_i^{n-j})_{1 \leqslant i,j \leqslant n}}.$$

The proof is to be done first for the corresponding expressions for the monomial symmetric functions $m_\lambda(x)$, and then, it can be completed immediately by the well-known linear expression of the Schur function by $m_\lambda(x)$ using the Kostka numbers (Section 6).

We further try to understand the coefficients which we call (n,k)-*sign* appeared at the aforementioned linear expression of the wreath determinant in relation with a zonal spherical function of a Young subgroup of the symmetric group $\mathfrak{S}_{nk}$. At this point, we shall provide one conjecture about a positive definiteness of a certain symmetric matrix formed by the spherical function (see Conjecture 7.8). We do not treat the remaining singular case $\alpha = \frac{1}{k}$ $(k \in \mathbb{Z}_{>0})$. Note that, however, one can deduce the fact $m^\lambda(\frac{1}{k}) = m^{\lambda'}(-\frac{1}{k})$ from the result in [8], where λ' denotes the transposition of the partition λ as a Young diagram.

We give an α-analogue of the Laplace expansion formula for α-determinants in Appendix A.

1.1. *Conventions*

As usual, $\mathbb{N}$ is the set of positive integers and $\mathbb{C}$ is the complex number field. For $n \in \mathbb{N}$, we denote by $\mathfrak{S}_n$ the symmetric group of degree n. The cycle number of an element $\sigma \in \mathfrak{S}_n$ is written by $\nu_n(\sigma)$. Since the conjugacy classes of $\mathfrak{S}_n$ are parametrized by the cycle type, ν_n is a class function on $\mathfrak{S}_n$. In particular, we notice that $\nu_n(\sigma^{-1}) = \nu_n(\sigma)$ for any $\sigma \in \mathfrak{S}_n$ because σ and σ^{-1} are always $\mathfrak{S}_n$-conjugate.

We denote by $\mathrm{Mat}_{m,n}$ the set of $m \times n$ matrices whose entries belong to a certain commutative $\mathbb{C}$-algebra, and we put $\mathrm{Mat}_n = \mathrm{Mat}_{n,n}$. We also denote by $I_n = (\delta_{ij})_{1\leqslant i,j\leqslant n}$ the identity matrix of size n and $\mathbf{1}_n = (1)_{1\leqslant i,j\leqslant n}$ the all-one matrix of size n. For a permutation $\sigma \in \mathfrak{S}_n$, $P(\sigma) = (\delta_{i\sigma(j)})_{1\leqslant i,j\leqslant n}$ is the permutation matrix for σ.

The (complex) general linear group $GL_n(\mathbb{C})$ is the group consisting of invertible matrices in $\mathrm{Mat}_n(\mathbb{C})$. We exclusively deal with the complex vector spaces so that we often omit the symbol $\mathbb{C}$ and simply write GL_n instead of writing $GL_n(\mathbb{C})$.

Let us put $[N] := \{1, 2, \dots, N\}$ for $N \in \mathbb{N}$. For a given partition (or Young diagram) λ of size N, we denote by $\mathrm{SSTab}_N(\lambda)$ the set of all semistandard tableaux with shape λ whose entries are in $[N]$, and we also denote by $\mathrm{STab}(\lambda)$ the set of all standard tableaux with shape λ. For a semistandard tableau $T \in \mathrm{SSTab}_N(\lambda)$, we associate a sequence $\mathrm{wt}(T) := (\mu_1, \mu_2, \dots, \mu_N)$ of nonnegative integers where $\mu_k = |\{t_{ij} = k\}|$ is the number of entries in T which is equal to k. We call $\mathrm{wt}(T)$ the *weight* of T. Notice that a semistandard tableau $T \in \mathrm{SSTab}_N(\lambda)$ is standard if and only if $\mathrm{wt}(T) = (1, 1, \dots, 1)$. For a given partition $\lambda, \mu \vdash N$ of the same size N, we denote by $K_{\lambda\mu}$ the number of semistandard tableaux T with shape λ such that $\mathrm{wt}(T) = \mu$. Namely,

$$K_{\lambda\mu} = \bigl|\bigl\{T \in \mathrm{SSTab}_N(\lambda) \bigm| \mathrm{wt}(T) = \mu\bigr\}\bigr|.$$

We call $K_{\lambda\mu}$ the *Kostka number*. We also put $f^\lambda = |\mathrm{STab}(\lambda)| = K_{\lambda,(1,\dots,1)}$, and denote by $\ell(\lambda)$ the depth of the diagram λ. See [1,7] for detailed information on partitions and tableaux.

The irreducible polynomial representations of GL_m are highest weight modules and the highest weights are identified with partitions such that $\ell(\lambda) \leqslant m$. We denote by $\mathcal{M}_m^\lambda$ the irreducible GL_m-module corresponding to the partition λ. The irreducible representations of $\mathfrak{S}_n$ are also parametrized by partitions of n. We denote by $\mathcal{J}_n^\lambda$ the irreducible $\mathfrak{S}_n$-module corresponding to the partition $\lambda \vdash n$. See [12] (or [1]) for detailed information on representation theory of GL_m and $\mathfrak{S}_n$.

2. Basic properties of general α-determinants

Let α be a complex parameter. The α-determinant $\det^{(\alpha)} A$ of a square matrix $A = (a_{ij})_{1\leqslant i,j\leqslant n} \in \mathrm{Mat}_n$ is defined by

$$\det^{(\alpha)} A := \sum_{w\in\mathfrak{S}_n} \alpha^{n-\nu_n(w)} a_{w(1)1} \cdots a_{w(n)n}. \tag{2.1}$$

We note that $\det^{(\alpha)}({}^tA) = \det^{(\alpha)}(A)$ because $\nu_n(w^{-1}) = \nu_n(w)$ for any $w \in \mathfrak{S}_n$. We also notice that $\det^{(\alpha)}$ is *multilinear* with respect to the column (and/or row) vectors. We mainly deal with the $-\frac{1}{k}$-determinants for $k \in \mathbb{N}$ below, so it is convenient to put

$$\det_k A = |A|_k := \det^{(-1/k)} A.$$

We note that $\det_1 = \det^{(-1)}$ is the ordinary determinant.

The α-determinant of the all-one matrix $\mathbf{1}_n$ (i.e. every element equals 1) is calculated as

$$\det^{(\alpha)} \mathbf{1}_n = \sum_{w\in\mathfrak{S}_n} \alpha^{n-\nu_n(w)} = \prod_{1\leqslant i<n} (1 + i\alpha). \tag{2.2}$$

We note that this is the generating function of the Stirling numbers of the first kind (see, e.g. [10]). The following lemma is the (shifted) partial sum generalization of the identity above.

Lemma 2.1. *For a subset I of $[n]=\{1,2,\dots,n\}$, put*

$$\mathfrak{S}_n(I):=\{w\in\mathfrak{S}_n \mid x\notin I\Rightarrow w(x)=x\}.$$

Then, for any $g\in\mathfrak{S}_n$, there exists a nonnegative integer $m(g,I)$ such that

$$\sum_{w\in\mathfrak{S}_n(I)}\alpha^{n-\nu_n(gw)}=\alpha^{m(g,I)}\prod_{1\leqslant i<k}(1+i\alpha),$$

where $k=|I|$. The integer $m(g,I)$ is given by $n-\nu_n(gw_0)$ where $w_0\in\mathfrak{S}_n(I)$ is the unique element such that $\nu_n(gw_0)\geqslant\nu_n(gw)$ for any $w\in\mathfrak{S}_n(I)$.

Proof. Take an element $h\in\mathfrak{S}_n$ such that $h\cdot I=[k]$. We identify $\mathfrak{S}_k$ and $\mathfrak{S}_n([k])$ naturally. Since $w\in\mathfrak{S}_n(I)$ if and only if $hwh^{-1}\in\mathfrak{S}_k$, it follows that

$$\sum_{w\in\mathfrak{S}_n(I)}\alpha^{n-\nu_n(gw)}=\sum_{w\in\mathfrak{S}_k}\alpha^{n-\nu_n(gh^{-1}wh)}=\sum_{w\in\mathfrak{S}_k}\alpha^{n-\nu_n(gh^{-1}(hw_0h^{-1})wh)}$$
$$=\sum_{w\in\mathfrak{S}_k}\alpha^{n-\nu_n(g'w)}$$

where $g'=hgw_0h^{-1}$. By the definition of w_0 and g', it is easy to see that

$$\nu_n(g')\geqslant\nu_n(g'w)\quad(w\in\mathfrak{S}_k).\tag{2.3}$$

Assume that g' contains a cycle of the form $(\boldsymbol{j}_2,i_2,\boldsymbol{j}_1,i_1)$ ($i_1,i_2\in\{1,2,\dots,k\}$, $i_1\neq i_2$ and $\boldsymbol{j}_1,\boldsymbol{j}_2$ stand for certain disjoint strings in $\{1,2,\dots,n\}$ which are possibly empty). Then it follows that $\nu_n(g'\cdot(i_1,i_2))=\nu_n(g')+1$ because

$$(\boldsymbol{j}_2,i_2,\boldsymbol{j}_1,i_1)\cdot(i_2,i_1)=(\boldsymbol{j}_2,i_2)\cdot(\boldsymbol{j}_1,i_1).$$

This contradicts the inequality (2.3). Therefore, each cycle in the cycle decomposition of g' contains at most one element in $\{1,2,\dots,k\}$. Namely, g' is of the form

$$g'=(\boldsymbol{j}_k,k)\cdot\cdots\cdot(\boldsymbol{j}_2,2)\cdot(\boldsymbol{j}_1,1)\cdot h$$

for certain (possibly empty) disjoint strings $\boldsymbol{j}_1,\dots,\boldsymbol{j}_k$ in $\{k+1,\dots,n\}$ and $h\in\mathfrak{S}_n(\{k+1,\dots,n\})$.

For distinct elements $i_1,\dots,i_l\in\{1,2,\dots,k\}$, we have

$$(\boldsymbol{j}_{i_l},i_l)\cdot\cdots\cdot(\boldsymbol{j}_{i_2},i_2)\cdot(\boldsymbol{j}_{i_1},i_1)\cdot(i_l,\dots,i_2,i_1)=(\boldsymbol{j}_{i_l},i_l,\dots,\boldsymbol{j}_{i_2},i_2,\boldsymbol{j}_{i_1},i_1).$$

This implies that l distinct cycles in g' turn into one cycle in $g'\cdot(i_l,\dots,i_2,i_1)$, that is,

$$\nu_n(g')-\nu_n\bigl(g'\cdot(i_l,\dots,i_2,i_1)\bigr)=l-1.$$

Hence, if $w\in\mathfrak{S}_k$ is of the type $1^{r_1}2^{r_2}\cdots k^{r_k}$, then we have

$$\nu_n(g')-\nu_n(g'w)=\sum_{l=1}^{k}r_l(l-1)=k-\nu_k(w).$$

Therefore it follows that

$$\sum_{w\in\mathfrak{S}_k}\alpha^{n-\nu_n(g'w)}=\alpha^{n-\nu_n(g')}\sum_{w\in\mathfrak{S}_k}\alpha^{k-\nu_k(w)}=\alpha^{n-\nu_n(gw_0)}\prod_{1\leqslant i<k}(1+i\alpha).$$

This completes the proof. □

Let us define the left action of $\mathfrak{S}_m$ (respectively the right action of $\mathfrak{S}_n$) on the set $\mathrm{Mat}_{m,n}$ as permutations of row (respectively column) vectors:

$$\sigma \cdot (a_{ij})_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} := \big(a_{\sigma^{-1}(i)j}\big)_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \quad (\sigma \in \mathfrak{S}_m),$$
$$(a_{ij})_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \cdot \tau := \big(a_{i\tau(j)}\big)_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \quad (\tau \in \mathfrak{S}_n).$$

Notice that $\sigma \cdot A = P(\sigma)A$ and $A \cdot \tau = AP(\tau)$ for $\sigma \in \mathfrak{S}_m$, $\tau \in \mathfrak{S}_n$ and $A \in \mathrm{Mat}_{m,n}$. If $m = n$, then we have

$$\begin{aligned}\det{}^{(\alpha)}(w \cdot A) = \det{}^{(\alpha)}(a_{w^{-1}(i)j}) &= \sum_{g\in\mathfrak{S}_n} \alpha^{n-\nu_n(g)} \prod_{i=1}^{n} a_{w^{-1}g(i)i} \\ &= \sum_{g\in\mathfrak{S}_n} \alpha^{n-\nu_n(wgw^{-1})} \prod_{i=1}^{n} a_{g(i)w(i)} = \det{}^{(\alpha)}(a_{iw(j)}) = \det{}^{(\alpha)}(A \cdot w)\end{aligned}$$

for any $w \in \mathfrak{S}_n$ and any $A = (a_{ij}) \in \mathrm{Mat}_n$.

Lemma 2.2. *The equality*

$$\sum_{w\in\mathfrak{S}_n(I)} \det{}^{(\alpha)}(A \cdot w) = \prod_{1\leqslant i<k} (1 + i\alpha) \sum_{g\in\mathfrak{S}_n} \alpha^{m(g,I)} \prod_{i=1}^{n} a_{g(i)i}$$

holds for $A = (a_{ij})_{1\leqslant i,j\leqslant n} \in \mathrm{Mat}_n$ *and* $I \subset [n]$ *such that* $|I| = k$.

Proof. Using Lemma 2.1, we have

$$\begin{aligned}\sum_{w\in\mathfrak{S}_n(I)} \det{}^{(\alpha)}(A \cdot w) &= \sum_{w\in\mathfrak{S}_n(I)} \sum_{g\in\mathfrak{S}_n} \alpha^{n-\nu_n(g)} \prod_{i=1}^{n} a_{g(i)w(i)} \\ &= \sum_{g\in\mathfrak{S}_n} \sum_{w\in\mathfrak{S}_n(I)} \alpha^{n-\nu_n(g)} \prod_{i=1}^{n} a_{gw^{-1}(i)i} \\ &= \sum_{g\in\mathfrak{S}_n} \Bigg\{ \sum_{w\in\mathfrak{S}_n(I)} \alpha^{n-\nu_n(gw)} \Bigg\} \prod_{i=1}^{n} a_{g(i)i} \\ &= \prod_{1\leqslant i<k} (1 + i\alpha) \sum_{g\in\mathfrak{S}_n} \alpha^{m(g,I)} \prod_{i=1}^{n} a_{g(i)i}\end{aligned}$$

as we desired. □

As a corollary, we have the following lemma.

Lemma 2.3. *For* $I \subset [n]$ *such that* $|I| > k$ *and* $A \in \mathrm{Mat}_n$, *the equalities*

$$\sum_{w\in\mathfrak{S}_n(I)} \det{}_k(A \cdot w) = \sum_{w\in\mathfrak{S}_n(I)} \det{}_k(w \cdot A) = 0$$

hold. In particular, if $k + 1$ *column* (*row*) *vectors in* A *are equal, then* $\det_k A = 0$.

Lemma 2.3 and the multilinearity of $\det_k$ yield immediately the

Lemma 2.4. *Let $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) \in \mathrm{Mat}_n$. If $\boldsymbol{a}_{i_1} = \cdots = \boldsymbol{a}_{i_k} = \boldsymbol{b}$ for some $1 \leqslant i_1 < \cdots < i_k \leqslant n$, then*

$$\det_k(\boldsymbol{a}_1, \dots, \boldsymbol{a}_j + \boldsymbol{b}, \dots, \boldsymbol{a}_n) = \det_k(\boldsymbol{a}_1, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_n)$$

for any $j \in [n] \setminus \{i_1, \dots, i_k\}$.

When we regard $\sigma \in \mathfrak{S}_n$ as an element in $\mathfrak{S}_{n+m}$ ($m \in \mathbb{N}$) in natural way, we notice that $\nu_{n+m}(\sigma) = \nu_n(\sigma) + m$. Further, if we take a permutation $\tau \in \mathfrak{S}_m$ and regard τ as an element in $\mathfrak{S}_{n+m}$ which leave each letter in $[n]$ invariant, then $\nu_{n+m}(\sigma\tau) = \nu_n(\sigma) + \nu_m(\tau)$. This fact readily implies the following simple consequence which will be used in the proof of Lemma 4.6 (see also Appendix A).

Lemma 2.5. *The equality*

$$\det^{(\alpha)}\begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix} = \det^{(\alpha)}(A_{11}) \det^{(\alpha)}(A_{22})$$

holds. In particular, $\det^{(\alpha)}(A_{11} \oplus A_{22}) = \det^{(\alpha)}(A_{11}) \det^{(\alpha)}(A_{22})$.

Proof. Suppose that $A = (a_{ij}) \in \mathrm{Mat}_{n+m}$ and $A_{11} = (a_{ij})_{1 \leqslant i,j \leqslant n}$, $A_{22} = (a_{ij})_{n+1 \leqslant i,j \leqslant n+m}$. We also assume that $a_{ij} = 0$ if $n+1 \leqslant i \leqslant n+m$ and $1 \leqslant j \leqslant n$. Then it follows that

$$\begin{aligned}
\det^{(\alpha)} A &= \sum_{\sigma \in \mathfrak{S}_{n+m}} \alpha^{n+m-\nu_{n+m}(\sigma)} \prod_{i=1}^{n+m} a_{i\sigma(i)} \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_{n+m}([n]) \\ \tau \in \mathfrak{S}_{n+m}(n+[m])}} \alpha^{n+m-\nu_{n+m}(\sigma\tau)} \prod_{i=1}^{n} a_{i\sigma(i)} \prod_{i=1}^{m} a_{n+i,\tau(n+i)} \\
&= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \tau \in \mathfrak{S}_m}} \alpha^{n+m-\nu_n(\sigma)-\nu_m(\tau)} \prod_{i=1}^{n} a_{i\sigma(i)} \prod_{i=1}^{m} a_{n+i,n+\tau(i)} = \det^{(\alpha)}(A_{11}) \det^{(\alpha)}(A_{22}).
\end{aligned}$$

This proves the claim. □

3. Characterization of $-\frac{1}{k}$-determinants

In Lemma 2.3, we prove that $\det_k$ has an alternating property among $k+1$ column (and/or row) vectors. In this section, we show, conversely, this property essentially characterizes $\det_k$.

We denote by $\mathcal{P}(\mathrm{Mat}_n(\mathbb{C}))$ the commutative $\mathbb{C}$-algebra consisting of polynomial functions on $\mathrm{Mat}_n(\mathbb{C})$. The Lie algebra of GL_n is denoted by $\mathfrak{gl}_n$, and its universal enveloping algebra is denoted by $\mathcal{U}(\mathfrak{gl}_n)$. The algebra $\mathcal{P}(\mathrm{Mat}_n(\mathbb{C}))$ has a $\mathcal{U}(\mathfrak{gl}_n) \times \mathfrak{S}_n$-module structure by defining

$$(E_{ij} \cdot f)(X) = \sum_{k=1}^{n} x_{ik} \frac{\partial f}{\partial x_{jk}}(X) \quad (1 \leqslant i, j \leqslant n), \qquad (\sigma \cdot f)(X) = f(X \cdot \sigma) \quad (\sigma \in \mathfrak{S}_n)$$

for $f \in \mathcal{P}(\mathrm{Mat}_n(\mathbb{C}))$ where E_{ij} are the standard basis of $\mathfrak{gl}_n$ and x_{ij} are the standard coordinate functions on $\mathrm{Mat}_n(\mathbb{C})$. We note that this action of $\mathcal{U}(\mathfrak{gl}_n)$ is obtained as the differential representation of GL_n given by $(g \cdot f)(X) = f({}^t g X)$ for $g \in GL_n$, which is the contragradient representation of the left regular representation on $\mathcal{P}(\mathrm{Mat}_n(\mathbb{C}))$. Here ${}^t g$ denotes the transposed matrix of g.

Let ML_n be a subspace of $\mathcal{P}(\mathrm{Mat}_n(\mathbb{C}))$ consisting of functions which are multilinear with respect to *column* vectors. Clearly, we have

$$\mathrm{ML}_n = \bigoplus_{1 \leqslant i_1, \dots, i_n \leqslant n} \mathbb{C} \cdot x_{i_1 1} \cdots x_{i_n n}.$$

The subspace ML_n is a $\mathcal{U}(\mathfrak{gl}_n) \times \mathfrak{S}_n$-submodule of $\mathcal{P}(\mathrm{Mat}_n(\mathbb{C}))$. For each $k \in \mathbb{N}$, we put

$$\mathrm{AL}_n^k := \left\{ f \in \mathrm{ML}_n \;\middle|\; I \subset [n],\ |I| > k \Rightarrow \sum_{\tau \in \mathfrak{S}_n(I)} f(X \cdot \tau) = 0 \right\}$$

where $X = (x_{ij})_{1 \leqslant i,j \leqslant n}$. This subspace AL_n^k is also $\mathcal{U}(\mathfrak{gl}_n)$-invariant because the actions of $\mathcal{U}(\mathfrak{gl}_n)$ and $\mathfrak{S}_n$ on $\mathcal{P}(\mathrm{Mat}_n(\mathbb{C}))$ commute each other. We also see that AL_n^k is $\mathfrak{S}_n$-invariant since

$$\sum_{\tau \in \mathfrak{S}_n(I)} (\sigma \cdot f)(X \cdot \tau) = \sum_{\tau \in \mathfrak{S}_n(I)} f(X \cdot \tau\sigma) = \sum_{\tau \in \mathfrak{S}_n(\sigma^{-1} I)} f(Y \cdot \tau)\Big|_{Y = X \cdot \sigma} = 0$$

for any $I \subset [n]$, $|I| > k$ if $f \in \mathrm{AL}_n^k$ and $\sigma \in \mathfrak{S}_n$. Since $\det_k \in \mathrm{AL}_n^k$ by Lemma 2.3, it follows that $\mathrm{AL}_n^k \supset \mathcal{U}(\mathfrak{gl}_n) \cdot \det_k(X)$.

Theorem 3.1. *The equality* $\mathrm{AL}_n^k = \mathcal{U}(\mathfrak{gl}_n) \cdot \det_k(X)$ *holds for* $k = 1, 2, \dots, n-1$.

Proof. In [8], it is shown that

$$\mathcal{U}(\mathfrak{gl}_n) \cdot \det_k(X) \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda_1 \leqslant k}} (\mathcal{M}_n^\lambda)^{\oplus f^\lambda}, \tag{3.1}$$

where $\mathcal{M}_n^\lambda$ denotes the highest weight $\mathcal{U}(\mathfrak{gl}_n)$-module of highest weight λ, which is the differential representation of $\mathcal{M}_n^\lambda$ and we use the same symbol to indicate it. The irreducible module $\mathcal{M}_n^\lambda$ is realized in $\mathcal{U}(\mathfrak{gl}_n) \cdot \det_k(X)$ as an image of the Young symmetrizer

$$c_T = \sum_{\substack{q \in C(T) \\ p \in R(T)}} \mathrm{sgn}(q) q p \in \mathbb{C}[\mathfrak{S}_n] \quad \big(T \in \mathrm{STab}(\lambda)\big).$$

Here $C(T)$ and $R(T)$ are the column group and row group of T respectively (see, e.g. [12]). Hence, to prove the opposite inclusion $\mathrm{AL}_n^k \subset \mathcal{U}(\mathfrak{gl}_n) \cdot \det_k(X)$, it is enough to show that each element f in AL_n^k is killed by the Young symmetrizer c_T when $T \in \mathrm{STab}(\lambda)$ and $\lambda_1 > k$. We now prove this. The image $c_T \cdot f$ of $f \in \mathrm{AL}_n^k$ by c_T is calculated as

$$(c_T \cdot f)(X) = \sum_{q \in C(T)} \mathrm{sgn}(q) \sum_{p \in R(T)} f\big(X \cdot q p q^{-1} q\big) = \sum_{q \in C(T)} \mathrm{sgn}(q) \sum_{p \in R(qT)} f(X \cdot p q).$$

For each $q \in C(T)$, we see that

$$\sum_{p \in R(qT)} f(X \cdot pq) = \sum_{p' \in R_1'(qT)} \left\{ \sum_{p \in R_1(qT)} (p' q \cdot f)(X \cdot p) \right\} = 0$$

since $p'q \cdot f \in \mathrm{AL}_n^k$ by $\mathfrak{S}_n$-invariance of AL_n^k. Here $R_1(qT)$ is the subgroup of $R(qT)$ consisting of permutations which moves only the entries in the first row of qT, and $R_1'(qT)$ is the subgroup of $R(qT)$ which leaves the first row of qT invariant so that $R(qT) = R_1(qT) \times R_1'(qT)$. This completes the proof. □

4. Determinants from a variation on wreath product groups

Let $m, n, k \in \mathbb{N}$. For a matrix $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) \in \mathrm{Mat}_{m,n}$, we define the column k-plexing $A^{[k]} \in \mathrm{Mat}_{m,kn}$ of A by

$$A^{[k]} := (\overbrace{\boldsymbol{a}_1, \dots, \boldsymbol{a}_1}^{k}, \dots, \overbrace{\boldsymbol{a}_n, \dots, \boldsymbol{a}_n}^{k}).$$

This is nothing but the Kronecker product matrix $A \otimes (1, \dots, 1)$ of A and $(1, \dots, 1) \in \mathrm{Mat}_{1,k}$. The row k-plexing $A_{[k]} \in \mathrm{Mat}_{km,n}$ of A is also defined in a similar way.

Example 4.1. If

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \in \mathrm{Mat}_{3,2},$$

then

$$A^{[2]} = \begin{pmatrix} a_1 & a_1 & b_1 & b_1 \\ a_2 & a_2 & b_2 & b_2 \\ a_3 & a_3 & b_3 & b_3 \end{pmatrix} \in \mathrm{Mat}_{3,4},$$
$$A^{[3]} = \begin{pmatrix} a_1 & a_1 & a_1 & b_1 & b_1 & b_1 \\ a_2 & a_2 & a_2 & b_2 & b_2 & b_2 \\ a_3 & a_3 & a_3 & b_3 & b_3 & b_3 \end{pmatrix} \in \mathrm{Mat}_{3,6}.$$

We notice that

$$A^{[k]} = A \cdot (I_n)^{[k]}, \qquad A_{[k]} = (I_m)_{[k]} \cdot A$$

for $A \in \mathrm{Mat}_{m,n}$. Hence one has the

Lemma 4.2. *Let $A \in \mathrm{Mat}_{m,n}$. Then the equalities*

$$(PA)^{[k]} = P \cdot A^{[k]}, \qquad (AQ)_{[k]} = A_{[k]} \cdot Q$$

hold for $P \in \mathrm{Mat}_m$, $Q \in \mathrm{Mat}_n$. In particular, we have

$$\sigma \cdot A^{[k]} = (\sigma \cdot A)^{[k]}, \qquad A_{[k]} \cdot \tau = (A \cdot \tau)_{[k]}$$

for $\sigma \in \mathfrak{S}_m$, $\tau \in \mathfrak{S}_n$.

Definition 4.3. For a rectangular matrix $A = (a_{ij})_{\substack{1 \leqslant i \leqslant kn \\ 1 \leqslant j \leqslant n}} \in \mathrm{Mat}_{kn,n}$, we define the kth *wreath determinant* of A by

$$\mathrm{wrdet}_k A := \det_k\bigl(A^{[k]}\bigr) = \sum_{\sigma \in \mathfrak{S}_{kn}} \left(-\frac{1}{k}\right)^{kn - \nu_{kn}(\sigma)} \prod_{p=1}^{n} \prod_{l=1}^{k} a_{\sigma((p-1)k+l),p}.$$

By Lemma 2.4, it is immediate to see that the equalities

$$\begin{aligned}&\operatorname{wrdet}_k(\boldsymbol{a}_1,\dots,\boldsymbol{a}_{i-1},\boldsymbol{a}_i+c\boldsymbol{a}_j,\boldsymbol{a}_{i+1},\dots,\boldsymbol{a}_n)\\&\quad=\operatorname{wrdet}_k(\boldsymbol{a}_1,\dots,\boldsymbol{a}_{i-1},\boldsymbol{a}_i,\boldsymbol{a}_{i+1},\dots,\boldsymbol{a}_n)\quad(i\neq j),\\&\operatorname{wrdet}_k(\boldsymbol{a}_1,\dots,c\boldsymbol{a}_i,\dots,\boldsymbol{a}_n)=c^k\operatorname{wrdet}_k(\boldsymbol{a}_1,\dots,\boldsymbol{a}_i,\dots,\boldsymbol{a}_n)\end{aligned}$$

hold for $A=(\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)\in\mathrm{Mat}_{kn,n}$ and $c\in\mathbb{C}$. Then it also follows that

$$\operatorname{wrdet}_k(A\cdot\sigma)=(\operatorname{sgn}\sigma)^k\operatorname{wrdet}_k A\quad(\sigma\in\mathfrak{S}_n).\tag{4.1}$$

In general, we have the

Lemma 4.4. *If $A\in\mathrm{Mat}_{kn,n}$ and $P\in\mathrm{Mat}_n$, then*

$$\operatorname{wrdet}_k(AP)=(\det P)^k\operatorname{wrdet}_k(A).$$

Namely, wrdet_k *is a relative invariant of GL_n in $\mathcal{P}(\mathrm{Mat}_{kn,n}(\mathbb{C}))$ with respect to the (right) regular representation (see also Section 5).*

Example 4.5. Lemma 4.4 says that the equality

$$\det{}^{(\alpha)}\big((AP)^{[k]}\big)=(\det P)^k\det{}^{(\alpha)}\big(A^{[k]}\big)\tag{4.2}$$

holds when $\alpha=-1/k$. When $k=1$ and $\alpha=-1$, this is nothing but the multiplicativity of the ordinary determinant. We also notice that (4.2) becomes trivial when $\alpha=-1,-1/2,\dots,-1/(k-1)$. Actually, because of Lemma 2.3, each side of (4.2) vanishes for such values. Further, we notice that (4.2) holds only if $\alpha=-1,-1/2,\dots,-1/k$. Actually, if $\det^{(\alpha)}(X^{[k]})$ satisfies (4.2), then the ratio $\det^{(\alpha)}(X^{[k]})/\operatorname{wrdet}_k(X)$ gives an absolute invariant of GL_n, which must be a constant. If the constant is 0, then it follows from (2.2) that $\alpha=-1,-1/2,\dots,-1/(k-1)$. If the constant is not 0, then we immediately have $\alpha=-1/k$. Here we give a simple and direct example. When $n=k=2$ and $P=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, we have

$$\begin{aligned}&\det{}^{(\alpha)}\big((AP)^{[2]}\big)-(\det P)^2\det{}^{(\alpha)}\big(A^{[2]}\big)\\&\quad=(1+\alpha)(1+2\alpha)\big((1+3\alpha)a_{11}a_{21}a_{31}a_{41}+2\alpha(a_{12}a_{21}+a_{11}a_{22})a_{31}a_{41}\\&\qquad+(1+\alpha)a_{11}a_{21}(a_{32}a_{41}+a_{31}a_{42})\big)\end{aligned}$$

which is identically zero only if $\alpha=-1,-\frac{1}{2}$. See also Corollary 5.8.

Lemma 4.6. *If $A\in\mathrm{Mat}_n$, then the equality*

$$\det{}_k\big(A^{[k]}_{[k]}\big)=\operatorname{wrdet}_k(A_{[k]})=\left(\frac{k!}{k^k}\right)^n(\det A)^k$$

holds for any $k\in\mathbb{N}$.

Proof. By Lemmas 4.2 and 4.4, we have

$$\begin{aligned}\det{}_k\big(A^{[k]}_{[k]}\big)&=\operatorname{wrdet}_k(A_{[k]})=\operatorname{wrdet}_k\big((I_n)_{[k]}\cdot A\big)\\&=\operatorname{wrdet}_k\big((I_n)_{[k]}\big)\cdot(\det A)^k=\det{}_k\big((I_n)^{[k]}_{[k]}\big)\cdot(\det A)^k.\end{aligned}$$

Since $(I_n)^{[k]}_{[k]} = \overbrace{\mathbf{1}_k \oplus \cdots \oplus \mathbf{1}_k}^{n}$ and $\det_k(\mathbf{1}_k) = \prod_{1 \leqslant i < k}(1 - \frac{i}{k}) = \frac{k!}{k^k}$, we have $\det_k((I_n)^{[k]}_{[k]}) = (\frac{k!}{k^k})^n$ by Lemma 2.5. This completes the proof. □

This lemma will be used in Section 7.

We consider the two injective homomorphisms $\phi : \mathfrak{S}_k^n \to \mathfrak{S}_{kn}$ and $\psi : \mathfrak{S}_n \to \mathfrak{S}_{kn}$ defined as

$$\phi(\sigma_1, \dots, \sigma_n) : [kn] \ni (i-1)k + j \mapsto (i-1)k + \sigma_i(j) \in [kn] \quad (1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k),$$
$$\psi(\tau) : [kn] \ni (i-1)k + j \mapsto \big(\tau(i) - 1\big)k + j \in [kn] \quad (1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k)$$

for $(\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_k^n$ and $\tau \in \mathfrak{S}_n$. To avoid the confusion, we put $S_k^n := \phi(\mathfrak{S}_k^n)$ and $S_n := \psi(\mathfrak{S}_n)$. We note that S_k^n is the Young subgroup $\mathfrak{S}_{(k^n)}$ of $\mathfrak{S}_{kn}$ corresponding to the partition $(k^n) \vdash kn$.

By the definition of k-plexing, one finds that $A^{[k]} \cdot \sigma = A^{[k]}$ for $A \in \mathrm{Mat}_{kn,n}$ and $\sigma \in S_k^n$, whence it follows that

$$\mathrm{wrdet}_k(\sigma \cdot A) = \det_k\big(\sigma \cdot A^{[k]}\big) = \det_k\big(A^{[k]} \cdot \sigma\big) = \det_k\big(A^{[k]}\big) = \mathrm{wrdet}_k A \quad \big(\sigma \in S_k^n\big).$$

We also see that $A^{[k]} \cdot \psi(\tau) = (A \cdot \tau)^{[k]}$ for any $\tau \in \mathfrak{S}_n$. Hence we have

$$\begin{aligned}\mathrm{wrdet}_k\big(\psi(\tau) \cdot A\big) &= \det_k\big(\psi(\tau) \cdot A^{[k]}\big) = \det_k\big(A^{[k]} \cdot \psi(\tau)\big) \\ &= \mathrm{wrdet}_k(A \cdot \tau) = (\operatorname{sgn} \tau)^k \,\mathrm{wrdet}_k A \quad (\tau \in \mathfrak{S}_n)\end{aligned}$$

by (4.1). Consequently, we obtain the

Lemma 4.7. *If $A \in \mathrm{Mat}_{kn,n}$, then*

$$\mathrm{wrdet}_k(g \cdot A) = \chi_{n,k}(g)^k \,\mathrm{wrdet}_k A$$

for any $g \in \mathfrak{S}_k \wr \mathfrak{S}_n$. In other words, $\mathbb{C} \cdot \mathrm{wrdet}_k \subset \mathcal{P}(\mathrm{Mat}_{kn,n})$ defines a one-dimensional representation of $\mathfrak{S}_k \wr \mathfrak{S}_n$. Here $\mathfrak{S}_k \wr \mathfrak{S}_n := S_k^n \rtimes S_n$ is the wreath product group (see [7]). The character $\chi_{n,k}$ of $\mathfrak{S}_k \wr \mathfrak{S}_n$ is defined by

$$\chi_{n,k}(g) = \operatorname{sgn} \tau$$

for $g = (\phi(\sigma_1, \dots, \sigma_n); \psi(\tau))$ $(\sigma_i \in \mathfrak{S}_k, \tau \in \mathfrak{S}_n)$.

5. Expressions of wreath determinants and (GL_{kn}, GL_n)-duality

For given two linear spaces V and W, as a $GL(V) \times GL(W)$-module, the multiplicity-free decomposition

$$\mathcal{S}(V \otimes W) \cong \bigoplus_{\lambda} \mathcal{M}_V^{\lambda} \boxtimes \mathcal{M}_W^{\lambda} \tag{5.1}$$

of the symmetric algebra $\mathcal{S}(V \otimes W)$ holds. Here λ runs over the partitions such that $\ell(\lambda) \leqslant \min\{\dim V, \dim W\}$. This fact is referred as $(GL(V), GL(W))$-*duality* (see [3] and [12]).

The algebra $\mathcal{P}(\mathrm{Mat}_{kn,n})$ has a $GL_{kn} \times GL_n$-module structure given by

$$\big((g,h).f\big)(A) := f\big({}^t g A h\big) \quad (g \in GL_{kn},\ h \in GL_n,\ A \in \mathrm{Mat}_{kn,n}),$$

where ${}^t g$ denotes the transposition of g with respect to the standard coordinate. We see that

$$\mathcal{P}(\mathrm{Mat}_{kn,n}) \cong \mathcal{P}\big((\mathbb{C}^{kn})^* \otimes (\mathbb{C}^n)^*\big) \cong \mathcal{S}\big(\mathbb{C}^{kn} \otimes \mathbb{C}^n\big)$$

as $GL_{kn} \times GL_n$-module. Here V^* indicates the contragradient representation of V. We notice that if (ρ, V) is a representation of GL_m, then $\tilde{\rho}(g) = \rho({}^t g^{-1})$ $(g \in GL_m)$ defines a representation on V which is equivalent to V^*.

Remark 5.1. It is standard to define a representation of $GL_{kn} \times GL_n$ on the algebra $\mathcal{P}(\mathrm{Mat}_{kn,n}(\mathbb{C}))$ by

$$\big((g,h).f\big)(A) := f\big(g^{-1}Ah\big) \quad (g \in GL_{kn},\ h \in GL_n,\ A \in \mathrm{Mat}_{kn,n}),$$

which is a combination of the left regular action of GL_{kn} and the right regular action of GL_n. If we adopt this one, however, then it is no longer a *polynomial* representation. Instead, in our argument, we adopt the contragradient of the left regular action of GL_{kn} so that each (irreducible) factor of the $GL_{kn} \times GL_n$-module $\mathcal{P}(\mathrm{Mat}_{kn,n}(\mathbb{C}))$ is polynomial.

By (GL_{kn}, GL_n)-duality, one has the multiplicity-free decomposition of $\mathcal{P}(\mathrm{Mat}_{kn,n})$:

$$\mathcal{P}(\mathrm{Mat}_{kn,n}) \cong \bigoplus_{\ell(\lambda) \leqslant n} \mathcal{M}_{kn}^{\lambda} \boxtimes \mathcal{M}_n^{\lambda}.$$

If we look at the det-eigenspace with respect to the left action of the diagonal torus $T_{kn} \cong (\mathbb{C}^\times)^{kn}$ of GL_{kn}, then we have

$$\mathcal{P}(\mathrm{Mat}_{kn,n})^{T_{kn},\det} \cong \bigoplus_{\ell(\lambda) \leqslant n} \big(\mathcal{M}_{kn}^{\lambda}\big)^{T_{kn},\det} \boxtimes \mathcal{M}_n^{\lambda}.$$

Here, for a GL_{kn}-module V, we denote by $V^{T_{kn},\det}$ the det-eigenspace

$$V^{T_{kn},\det} = \big\{ v \in V \;\big|\; t.v = \det(t) v \ (t \in T_{kn}) \big\}$$

with respect to T_{kn}. Since the symmetric group $\mathfrak{S}_{kn}$ is the normalizer of T_{kn} in GL_{kn}, each det-eigenspace $(\mathcal{M}_{kn}^{\lambda})^{T_{kn},\det}$ becomes a $\mathfrak{S}_{kn}$-module. It is known that the equivalence $(\mathcal{M}_{kn}^{\lambda})^{T_{kn},\det} \cong \mathcal{J}_{kn}^{\lambda}$ holds as $\mathfrak{S}_{kn}$-modules if λ is a partition of kn (see, e.g. [3]).

Let us denote by $M_{n,k}$ the irreducible $GL_{kn} \times GL_n$-submodule of $\mathcal{P}(\mathrm{Mat}_{kn,n})$ corresponding to the partition (k^n), that is, $M_{n,k} \cong \mathcal{M}_{kn}^{(k^n)} \boxtimes \mathcal{M}_n^{(k^n)}$. As $\mathfrak{S}_{kn}$-modules, we have the equivalence

$$M_{n,k}^{T_{kn},\det} \cong \big(\mathcal{M}_{kn}^{(k^n)}\big)^{T_{kn},\det} \boxtimes \mathcal{M}_n^{(k^n)} \cong \big(\mathcal{M}_{kn}^{(k^n)}\big)^{T_{kn},\det} \cong \mathcal{J}_{kn}^{(k^n)}$$

since the multiplicity space $\mathcal{M}_n^{(k^n)}$ is of dimension one. In particular, we have $\dim M_{n,k}^{T_{kn},\det} = f^{(k^n)}$.

By Lemma 4.4 and (GL_{kn}, GL_n)-duality, it follows that $\mathrm{wrdet}_k \in M_{n,k}$. Moreover, since

$$\big(\mathrm{diag}(c_1, \dots, c_{kn}).\,\mathrm{wrdet}_k\big)(A) = \mathrm{wrdet}_k\big({}^t\mathrm{diag}(c_1, \dots, c_{kn})A\big) = \left(\prod_{i=1}^{kn} c_i\right) \mathrm{wrdet}_k A,$$

it follows that wrdet_k belongs to $M_{n,k}^{T_{kn},\det}$.

For each standard tableau $T = (t_{ij})_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant k}} \in \mathrm{STab}((k^n))$, we define the function $\det_T$ on $\mathrm{Mat}_{kn,n}$ by

$$\det_T(A) := \prod_{l=1}^{k} \det(a_{t_{il},j})_{1\leqslant i,j\leqslant n} \quad \bigl(A = (a_{ij})_{\substack{1\leqslant i\leqslant kn\\ 1\leqslant j\leqslant n}} \in \mathrm{Mat}_{kn,n}\bigr).$$

We also define the matrix $I(T) \in \mathrm{Mat}_{kn,n}$ so that t_{ij}th row vector of $I(T)$ is equal to the ith fundamental row vector $\boldsymbol{e}_i = (0,\dots,0,\overset{i\text{th}}{1},0,\dots,0)$ for each $i = 1,\dots,n$ and $j = 1,\dots,k$. In other words, if we define $g(T) \in \mathfrak{S}_{kn}$ for $T \in \mathrm{STab}((k^n))$ by

$$g(T)\bigl((i-1)k + j\bigr) = t_{ij} \quad (1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k), \tag{5.2}$$

then $I(T) = g(T) \cdot (I_n)_{[k]}$. Denote by T_0 the standard tableau with shape (k^n) whose (i,j)-entry is $(i-1)k + j$. We note that $g(T) \in \mathfrak{S}_{kn}$ is the permutation determined by $g(T) \cdot T_0 = T$ for each $T \in \mathrm{STab}((k^n))$.

Lemma 5.2. *For $T, U \in \mathrm{STab}((k^n))$, the equality*

$$\det_T\bigl(I(U)\bigr) = \begin{cases} 1 & T = U, \\ 0 & T \neq U \end{cases}$$

holds.

Proof. When $T = U$, the t_{il}th row vector $I(T)_{t_{il}}$ of $I(T)$ is equal to $\boldsymbol{e}_i$ if $i \in [n]$ and $l \in [k]$, and hence $\det_T(I(T)) = 1$. When $T = (t_{ij})$ and $U = (u_{ij})$ are distinct standard tableaux of shape (k^n), there exists a pair (s_1, s_2) of distinct elements in $[kn]$ such that s_1 and s_2 are in the same column of T and in the same row of U, say $s_1 = t_{i_1c} = u_{rj_1}$ and $s_2 = t_{i_2c} = u_{rj_2}$ $(i_1 \neq i_2, j_1 \neq j_2)$. Then we have

$$I(U)_{t_{i_1}} = I(U)_{t_{i_2}} = \boldsymbol{e}_r,$$

which implies that $\det(I(U)_{t_{ic},j})_{1\leqslant i,j\leqslant n} = 0$, and hence $\det_T(I(U)) = 0$. □

Theorem 5.3. *The wreath determinant* $\mathrm{wrdet}_k A$ *of a matrix* $A \in \mathrm{Mat}_{kn,n}$ *is expressed as a linear combination*

$$\mathrm{wrdet}_k A = \sum_{T\in\mathrm{STab}((k^n))} \mathrm{wrdet}_k I(T) \cdot \det_T(A)$$

of $\det_T(A)$ *for* $T \in \mathrm{STab}((k^n))$. *The coefficient* $\mathrm{wrdet}_k I(T)$ *is given by the sum*

$$\mathrm{wrdet}_k I(T) = \sum_{\sigma\in S_k^n} \left(-\frac{1}{k}\right)^{kn-\nu_{kn}(g(T)\sigma)},$$

where $g(T) \in \mathfrak{S}_{kn}$ *is a permutation defined by* (5.2).

Proof. We observe that $\det_T(A)$ is a homogeneous polynomial in a_{ij} of degree kn satisfying the condition that $\det_T(AP) = (\det P)^k \det_T(A)$ for any $P \in \mathrm{Mat}_n$. We also see that

$$\bigl(\mathrm{diag}(c_1,\dots,c_{kn}).\det_T\bigr)(A) = \det_T\bigl({}^t\mathrm{diag}(c_1,\dots,c_{kn})A\bigr) = \left(\prod_{i=1}^{kn} c_i\right)\det_T A.$$

Thus, it follows that every $\det_T$ belongs to $M_{n,k}^{T_{kn},\det}$ by (GL_{kn}, GL_n)-duality.

We show that $\{\det_T\}_{T\in\mathrm{STab}((k^n))}$ are linearly independent. Suppose that

$$\sum_{T\in\mathrm{STab}((k^n))} C_T \det_T(A) = 0$$

for any $A \in \mathrm{Mat}_{kn,n}$. Then, by Lemma 5.2, we have

$$0 = \sum_{T\in\mathrm{STab}((k^n))} C_T \det_T\bigl(I(U)\bigr) = C_U$$

for each $U \in \mathrm{STab}((k^n))$, which assures the linear independence of $\{\det_T\}_{T\in\mathrm{STab}((k^n))}$. Since $\dim M_{n,k}^{T_{kn},\det} = f^{(k^n)}$, it follows that $\{\det_T\}_{T\in\mathrm{STab}((k^n))}$ is a basis of $M_{n,k}^{T_{kn},\det}$. Hence wrdet_k is written as

$$\mathrm{wrdet}_k A = \sum_{T\in\mathrm{STab}((k^n))} C'_T \det_T(A) \quad (A \in \mathrm{Mat}_{kn,n}).$$

By Lemma 5.2 again, the coefficient C'_U for $U \in \mathrm{STab}((k^n))$ is calculated as

$$\mathrm{wrdet}_k I(U) = \sum_T C'_T \det_T\bigl(I(U)\bigr) = C'_U \det_U\bigl(I(U)\bigr) = C'_U.$$

This completes the proof of the theorem. (The coefficient $\mathrm{wrdet}_k I(U)$ is calculated later in Section 7.) □

Example 5.4. When $n = 3$ and $k = 2$, there are five standard tableaux with shape (2^3):

$$U_1 = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline\end{array}, \quad U_2 = \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&6\\\hline\end{array}, \quad U_3 = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline\end{array}, \quad U_4 = \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline\end{array}, \quad U_5 = \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}.$$

(We remark that $T_0 = U_1$ in this case.) The corresponding matrices $I(U_p)$ are given by

$$I(U_1) = \begin{pmatrix}1&0&0\\1&0&0\\0&1&0\\0&1&0\\0&0&1\\0&0&1\end{pmatrix}, \quad I(U_2) = \begin{pmatrix}1&0&0\\1&0&0\\0&1&0\\0&0&1\\0&1&0\\0&0&1\end{pmatrix}, \quad I(U_3) = \begin{pmatrix}1&0&0\\0&1&0\\1&0&0\\0&1&0\\0&0&1\\0&0&1\end{pmatrix},$$

$$I(U_4) = \begin{pmatrix}1&0&0\\0&1&0\\1&0&0\\0&0&1\\0&1&0\\0&0&1\end{pmatrix}, \quad I(U_5) = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\1&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

and their 2-wreath determinants are calculated as

$$\mathrm{wrdet}_2 I(U_1) = \frac{1}{8}, \qquad \mathrm{wrdet}_2 I(U_2) = \mathrm{wrdet}_2 I(U_3) = -\frac{1}{16},$$
$$\mathrm{wrdet}_2 I(U_4) = \mathrm{wrdet}_2 I(U_5) = \frac{1}{32}.$$

Thus we have

$$\operatorname{wrdet}_2 A = \frac{1}{8}\det_{U_1}(A) - \frac{1}{16}\det_{U_2}(A) - \frac{1}{16}\det_{U_3}(A) + \frac{1}{32}\det_{U_4}(A) + \frac{1}{32}\det_{U_5}(A)$$

for $A \in \mathrm{Mat}_{6,3}$.

As a corollary of the theorem, we obviously have the

Corollary 5.5. *For $A \in \mathrm{Mat}_{p,n}$ and $B \in \mathrm{Mat}_{q,n}$, we denote by $A \boxplus B \in \mathrm{Mat}_{p+q,n}$ the matrix obtained by piling A on B. If $A_1, \dots, A_k \in \mathrm{Mat}_{n,n}$, then the equality*

$$\operatorname{wrdet}_k(A_1 \boxplus \cdots \boxplus A_k) = \sum_{T \in \mathrm{STab}((k^n))} \operatorname{wrdet}_k I(T) \prod_{i=1}^{k} \det B_i(T)$$

holds, where $B_j(T)$ is a matrix whose ith row is equal to the t_{ij}th row of $A_1 \boxplus \cdots \boxplus A_k$.

Example 5.6. If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

then we have

$$\begin{aligned} \operatorname{wrdet}_2(A \boxplus B) &= \operatorname{wrdet}_2 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= \frac{1}{4}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} - \frac{1}{8}\begin{vmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} \end{vmatrix}\begin{vmatrix} a_{21} & a_{22} \\ b_{21} & b_{22} \end{vmatrix}. \end{aligned}$$

Recall that the wreath determinant wrdet_k is S_k^n-invariant. By the Frobenius reciprocity, it follows that

$$\dim\bigl(M_{n,k}^{T_{kn},\det}\bigr)^{S_k^n} = \bigl\langle \operatorname{res}^{S_k^n}_{\mathfrak{S}_{kn}}\bigl(M_{n,k}^{T_{kn},\det}\bigr), 1_{S_k^n}\bigr\rangle_{S_k^n} = \bigl\langle M_{n,k}^{T_{kn},\det}, \operatorname{ind}^{\mathfrak{S}_{kn}}_{S_k^n} 1_{S_k^n}\bigr\rangle_{\mathfrak{S}_{kn}} = K_{(k^n)(k^n)} = 1,$$

where $\langle V, W\rangle_G$ denotes the intertwining number of two G-modules V and W, and 1_G is the trivial representation of G. Hence we have

$$\bigl(M_{n,k}^{T_{kn},\det}\bigr)^{S_k^n} = \mathbb{C} \cdot \operatorname{wrdet}_k(X). \tag{5.3}$$

This fact implies that $\sum_{\sigma \in S_k^n} f(\sigma \cdot X)$ is proportional to $\operatorname{wrdet}_k(X)$ for any $f \in M_{n,k}^{T_{kn},\det}$. Therefore, we have

$$\sum_{\sigma \in S_k^n} \det_{T_0}(\sigma \cdot X) = C \operatorname{wrdet}_k(X)$$

for a certain constant C. If we set $X = (I_n)_{[k]}$, then we have

$$C = \frac{1}{\operatorname{wrdet}_k(I_n)_{[k]}} \sum_{\sigma \in S_k^n} \det_{T_0}\bigl(\sigma \cdot (I_n)_{[k]}\bigr) = \left(\frac{k^k}{k!}\right)^n \sum_{\sigma \in S_k^n} 1 = k^{kn}.$$

Consequently, we obtain another (symmetric) expression of $\operatorname{wrdet}_k(X)$ as follows.

Corollary 5.7. *The equality*

$$\mathrm{wrdet}_k(A) = \frac{1}{k^{kn}} \sum_{\sigma \in S_k^n} \det_{T_0}(\sigma \cdot A)$$

holds for any $A \in \mathrm{Mat}_{kn,n}$.

As a corollary of the discussion above, we obtain the

Corollary 5.8 *(Characterization of the wreath determinant). Put*

$$\mathcal{P}(\mathrm{Mat}_{kn,n})^{\chi_{n,k}^k, \det^k} = \bigl\{ f \in \mathcal{P}(\mathrm{Mat}_{kn,n}) \bigm| f(\sigma X P) = \chi_{n,k}(\sigma)^k (\det P)^k f(X),\ \sigma \in \mathfrak{S}_k \wr \mathfrak{S}_n,\ P \in GL_n \bigr\}.$$

Then $\mathcal{P}(\mathrm{Mat}_{kn,n})^{\chi_{n,k}^k, \det^k}$ *is a one-dimensional subspace spanned by* wrdet_k. *Namely, the equality*

$$\mathcal{P}(\mathrm{Mat}_{kn,n})^{\chi_{n,k}^k, \det^k} = \mathbb{C} \cdot \mathrm{wrdet}_k(X)$$

holds.

Corollary 5.8 and Example 4.5 suggest the following problem: Describe the irreducible decomposition and singular values of the cyclic module $\mathcal{U}(\mathfrak{gl}_{kn}) \cdot \det^{(\alpha)}(X^{[k]}) \subset \mathcal{P}(\mathrm{Mat}_{kn,n})$ ($X = (x_{ij})_{1 \leqslant i \leqslant kn, 1 \leqslant j \leqslant n}$). This is solved in the following way. If $\alpha = 0$, then we see that

$$\mathcal{U}(\mathfrak{gl}_{kn}) \cdot \det^{(0)}(X^{[k]}) \cong \mathcal{S}^k(\mathbb{C}^{kn})^{\otimes n} \cong \bigoplus_{\lambda \vdash kn} (\mathcal{M}_{kn}^{\lambda})^{\oplus K_{\lambda,(k^n)}}$$

by a similar discussion in [5] (we also refer to [8] for the case where $k = 1$). By [8], the λ-isotypic component of the module $\mathcal{U}(\mathfrak{gl}_{kn}) \cdot \det^{(\alpha)}(\widetilde{X}) \subset \mathcal{P}(\mathrm{Mat}_{kn})$ does have a positive multiplicity if and only if $f_\lambda(\alpha) \neq 0$ and is given by $\mathcal{U}(\mathfrak{gl}_{kn}) \cdot \mathrm{Imm}_\lambda(\widetilde{X})$ (we put $\widetilde{X} = (x_{ij})_{1 \leqslant i,j \leqslant kn}$ in order to avoid confusion). Here $\mathrm{Imm}_\lambda(\widetilde{X})$ is the *immanant* of $\widetilde{X}$ for λ and $f_\lambda(\alpha) := \prod_{(i,j) \in \lambda} (1 + (j-i)\alpha)$ is the (modified) content polynomial for λ. Since the map $\mathcal{P}(\mathrm{Mat}_{kn}) \ni f(\widetilde{X}) \mapsto f(X^{[k]}) \in \mathcal{P}(\mathrm{Mat}_{kn,n})$ defines a GL_{kn}-intertwiner, we see that

$$\text{the } \lambda\text{-isotypic component of } \mathcal{U}(\mathfrak{gl}_{kn}) \cdot \det^{(\alpha)}(X^{[k]}) \cong \begin{cases} \mathcal{U}(\mathfrak{gl}_{kn}) \cdot \mathrm{Imm}_\lambda(X^{[k]}) & f_\lambda(\alpha) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda \vdash kn$. Thus it follows that $\mathcal{U}(\mathfrak{gl}_{kn}) \cdot \mathrm{Imm}_\lambda(X^{[k]}) \cong (\mathcal{M}_{kn}^{\lambda})^{\oplus K_{\lambda,(k^n)}}$. Hence we obtain the following theorem which is regarded as a generalization of the result in [8].

Theorem 5.9. *The irreducible decomposition of the cyclic module generated by* $\det^{(\alpha)}(X^{[k]})$ *is given by*

$$\mathcal{U}(\mathfrak{gl}_{kn}) \cdot \det^{(\alpha)}(X^{[k]}) \cong \bigoplus_{\substack{\lambda \vdash kn \\ f_\lambda(\alpha) \neq 0}} (\mathcal{M}_{kn}^{\lambda})^{\oplus K_{\lambda,(k^n)}}.$$

In particular, the singular values are given as roots of the content polynomials.

5.1. Remarks on this section

Let $\mathcal{S}(\mathbb{C}^n)=\sum_{k\geqslant 0}\mathcal{S}^k(\mathbb{C}^n)$ be the homogeneous decomposition of $\mathcal{S}(\mathbb{C}^n)$. Each symmetric power $\mathcal{S}^k(\mathbb{C}^n)$, that is, the space of kth symmetric tensors defines an irreducible $GL_n(\mathbb{C})$-module [1]. We see that the eigenspace decomposition of the $GL_m\times GL_n$-module $\mathcal{S}(\mathbb{C}^m\otimes\mathbb{C}^n)$ with respect to the diagonal torus T_m of $GL_m(\mathbb{C})$ is given by

$$\mathcal{S}\big(\mathbb{C}^m\otimes\mathbb{C}^n\big)\cong\bigoplus_{k_1,\dots,k_m\geqslant 0}\mathcal{S}^{k_1}\big(\mathbb{C}^n\big)\otimes\cdots\otimes\mathcal{S}^{k_m}\big(\mathbb{C}^n\big).$$

Hence the mth tensor product $\mathcal{S}^k(\mathbb{C}^n)^{\otimes m}$ can be identified to the $\det^k$-eigenspace

$$\mathcal{S}\big(\mathbb{C}^m\otimes\mathbb{C}^n\big)^{T_m,\det^k}=\big\{v\in\mathcal{S}\big(\mathbb{C}^m\otimes\mathbb{C}^n\big)\bigm| t.v=(\det t)^k v\ (t\in T_m)\big\}$$

for T_m [3]. By (GL_m,GL_n)-duality (5.1), we see that

$$\mathcal{S}^k\big(\mathbb{C}^n\big)^{\otimes m}\cong\mathcal{S}\big(\mathbb{C}^m\otimes\mathbb{C}^n\big)^{T_m,\det^k}\cong\sum_{\ell(\lambda)\leqslant\min\{m,n\}}\big(\mathcal{M}_m^\lambda\big)^{T_m,\det^k}\boxtimes\mathcal{M}_n^\lambda.\tag{5.4}$$

We notice that $(\mathcal{M}_m^\lambda)^{T_m,\det^k}=\{0\}$ unless $\lambda\vdash km$, and hence the last sum (5.4) is effectively over the partitions of km. Note also that $\dim(\mathcal{M}_m^\lambda)^{T_m,\det^k}=K_{\lambda(k^m)}$ and $(\mathcal{M}_m^\lambda)^{T_m,\det^k}$ is stable under the action of the Weyl group $\mathfrak{S}_m$ of $GL_m(\mathbb{C})$. We note that the decomposition (5.4) for $k=1$ gives $(\mathfrak{S}_m,GL_n)$-duality (Schur duality)

$$\big(\mathbb{C}^n\big)^{\otimes m}\cong\sum_{\lambda\vdash m,\ \ell(\lambda)\leqslant n}\mathcal{J}_m^\lambda\boxtimes\mathcal{M}_n^\lambda.\tag{5.5}$$

Suppose now $\lambda\vdash km$. The group $\mathfrak{S}_k\wr\mathfrak{S}_m$ acts on the weight space $(\mathcal{M}_m^\lambda)^{T_m,\det^k}$ because the wreath product $\mathfrak{S}_k\wr\mathfrak{S}_m=S_k^m\rtimes S_m$ is obviously acting on the space $\mathcal{S}^k(\mathbb{C}^n)^{\otimes m}$. Since $\mathfrak{S}_k$ acts on $\mathcal{S}^k(\mathbb{C}^n)^{\otimes m}$ trivially, its action on the weight space $(\mathcal{M}_m^\lambda)^{T_m,\det^k}$ is also trivial. Hence, (5.4) does not provide the irreducible decomposition as a bi-module of $(\mathfrak{S}_k\wr\mathfrak{S}_m,GL_n(\mathbb{C}))$. Then, the question how the space $(\mathcal{M}_m^\lambda)^{T_m,\det^k}$ decomposes as a $\mathfrak{S}_m$-module comes into being. Now we establish this question in a concrete way. From Schur duality, as a $\mathfrak{S}_m\times GL(\mathcal{S}^k(\mathbb{C}^n))$-module, we obtain

$$\mathcal{S}^k\big(\mathbb{C}^n\big)^{\otimes m}\cong\sum_{\mu\vdash m,\ \ell(\mu)\leqslant N}\mathcal{J}_m^\mu\boxtimes\mathcal{M}_N^\mu,$$

where $N=\dim\mathcal{S}^k(\mathbb{C}^n)=\binom{n+k-1}{k}\geqslant n$. Decompose the module $\mathcal{M}_N^\lambda$ of $GL(\mathcal{S}^k(\mathbb{C}^n))$ into irreducible ones as a representation of the subgroup $GL_n(\mathbb{C})$ of $GL(\mathcal{S}^k(\mathbb{C}^n))$:

$$\mathcal{M}_N^\mu\big|_{GL_n(\mathbb{C})}\cong\sum_{\lambda,\ell(\lambda)\leqslant n}\big(\mathcal{M}_n^\lambda\big)^{\oplus m_\lambda(\mu)},$$

$m_\lambda(\mu)$ being the multiplicity of $\mathcal{M}_n^\lambda$ in the irreducible summands of the restriction. Then we have

$$\mathcal{S}^k\big(\mathbb{C}^n\big)^{\otimes m}\cong\sum_{\lambda,\ell(\lambda)\leqslant n}\sum_{\mu\vdash m}\big(\mathcal{J}_m^\mu\boxtimes\mathcal{M}_n^\lambda\big)^{\oplus m_\lambda(\mu)}.$$

Therefore, it follows from (5.4) that

$$\sum_{\mu\vdash m}\bigl(\mathcal{J}_m^{\mu}\bigr)^{\oplus m_\lambda(\mu)} \cong \bigl(\mathcal{M}_m^{\lambda}\bigr)^{T_m,\det^k}. \tag{5.6}$$

The procedure explained above is a special case of the problem for computing *plethysm* (or the functorial composition of operations $\lambda \mapsto \mathcal{M}^\lambda$) (see [4,7]). Note also that the problem for describing the decomposition (5.6) for $\lambda \vdash km$ explicitly comes up naturally when one wants to know the structure of the cyclic $GL_n(\mathbb{C})$-module generated by $\det^{(\alpha)}(X)^k$ $(k = (1,)2,3,\ldots)$ (see [5]).

6. Formulas for wreath determinants à la Cauchy et van der Monde

We give an analogue of the Cauchy determinant formula in the context of wreath determinants developed in the previous sections.

Proposition 6.1. *Let $n,k \in \mathbb{N}$ and $x_1,\ldots,x_{kn},y_1,\ldots,y_n$ be commutative variables. Put*

$$C_{n,k}(x,y) = \left(\frac{1}{x_i+y_j}\right)_{\substack{1\leqslant i\leqslant kn\\ 1\leqslant j\leqslant n}}, \qquad V_{n,k}(x) = \bigl(x_i^{n-j}\bigr)_{\substack{1\leqslant i\leqslant kn\\ 1\leqslant j\leqslant n}}.$$

Then we have

$$\operatorname{wrdet}_k C_{n,k}(x,y) = \frac{\Delta_n(y)^k}{\prod_{\substack{1\leqslant i\leqslant kn\\ 1\leqslant j\leqslant n}}(x_i+y_j)} \operatorname{wrdet}_k V_{n,k}(x). \tag{6.1}$$

Here $\Delta_n(y)$ denotes the difference product

$$\Delta_n(y) = \prod_{1\leqslant i<j\leqslant n}(y_i-y_j).$$

Proof. For a rational function $f(t)$ in variable t, we write

$$f(x_\star) := \begin{pmatrix} f(x_1)\\ \vdots \\ f(x_{kn})\end{pmatrix} \in \mathrm{Mat}_{kn,1}.$$

Using this convention, we have

$$C_{n,k}(x,y) = \left(\frac{1}{x_\star+y_1},\ldots,\frac{1}{x_\star+y_n}\right), \qquad V_{n,k}(x) = \bigl(x_\star^{n-1},\ldots,x_\star,1\bigr).$$

By Lemma 4.4, we have

$$\begin{aligned}
&\operatorname{wrdet}_k\left(\frac{1}{x_\star+y_1},\ldots,\frac{1}{x_\star+y_n}\right)\\
&\quad= \operatorname{wrdet}_k\left(\frac{1}{x_\star+y_1},\frac{1}{x_\star+y_2}-\frac{1}{x_\star+y_1},\ldots,\frac{1}{x_\star+y_n}-\frac{1}{x_\star+y_1}\right)\\
&\quad= \operatorname{wrdet}_k\left(\frac{1}{x_\star+y_1},\frac{y_1-y_2}{(x_\star+y_1)(x_\star+y_2)},\ldots,\frac{y_1-y_n}{(x_\star+y_1)(x_\star+y_n)}\right)\\
&\quad= (y_1-y_2)^k\cdots(y_1-y_n)^k
\end{aligned}$$

$$\times \operatorname{wrdet}_k\left(\frac{1}{x_\star + y_1}, \frac{1}{(x_\star + y_1)(x_\star + y_2)}, \dots, \frac{1}{(x_\star + y_1)(x_\star + y_n)}\right).$$

Iterating this procedure, we reach to the expression

$$\operatorname{wrdet}_k\left(\frac{1}{x_\star + y_1}, \dots, \frac{1}{x_\star + y_n}\right)$$
$$= \Delta_n(y)^k \operatorname{wrdet}_k\left(\frac{1}{x_\star + y_1}, \frac{1}{(x_\star + y_1)(x_\star + y_2)}, \dots, \prod_{j=1}^n \frac{1}{(x_\star + y_j)}\right).$$

Using the multilinearity of $\det_k$ with respect to the *row* vectors, we have

$$\operatorname{wrdet}_k\left(\frac{1}{x_\star + y_1}, \frac{1}{(x_\star + y_1)(x_\star + y_2)}, \dots, \prod_{j=1}^n \frac{1}{(x_\star + y_j)}\right)$$
$$= \prod_{\substack{1\leqslant i\leqslant kn\\ 1\leqslant j\leqslant n}} \frac{1}{x_i + y_j} \operatorname{wrdet}_k\left(\prod_{j=2}^n (x_\star + y_j), \prod_{j=3}^n (x_\star + y_j), \dots, (x_\star + y_n), 1\right).$$

The last wreath determinant is equal to $\operatorname{wrdet}_k(x_\star^{n-1}, \dots, x_\star, 1) = \operatorname{wrdet}_k V_{n,k}(x)$ by Lemma 4.4. This completes the proof. □

We note that the proof above is exactly a wreath-analogue of the one of the Cauchy formula [12].

Example 6.2 *($k = 1$).* When $k = 1$, formula (6.1) is nothing but the ordinary Cauchy determinant formula

$$\det\left(\frac{1}{x_i + y_j}\right)_{1\leqslant i,j\leqslant n} = \frac{\Delta_n(x)\Delta_n(y)}{\prod_{i,j=1}^n (x_i + y_j)}.$$

Example 6.3 *($k = 2$).* When $k = 2$, (6.1) gives the formula

$$\operatorname{wrdet}_2\begin{pmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n}\\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{x_{2n}+y_1} & \frac{1}{x_{2n}+y_2} & \cdots & \frac{1}{x_{2n}+y_n} \end{pmatrix}$$
$$= \frac{\prod_{1\leqslant i<j\leqslant n}(y_i - y_j)^2}{\prod_{\substack{1\leqslant i\leqslant 2n\\ 1\leqslant j\leqslant n}} (x_i + y_j)} \operatorname{wrdet}_2\begin{pmatrix} x_1^{n-1} & \dots & x_1 & 1\\ x_2^{n-1} & \dots & x_2 & 1\\ \vdots & \ddots & \vdots & \vdots\\ x_{2n}^{n-1} & \dots & x_{2n} & 1 \end{pmatrix}.$$

We notice that the other variant of this Cauchy-type identity also follows immediately from (6.1). Indeed, we have

$$\operatorname{wrdet}_k\left(\frac{1}{1 - x_\star y_1}, \dots, \frac{1}{1 - x_\star y_n}\right) = \frac{\Delta_n(y)^k}{\prod_{\substack{1\leqslant i\leqslant kn\\ 1\leqslant j\leqslant n}} (1 - x_i y_j)} \operatorname{wrdet}_k V_{n,k}(x),$$

which is a wreath determinant analogue of the formula

$$\det\left(\frac{1}{1-x_i y_j}\right)_{1\leqslant i,j\leqslant n} = \frac{\Delta_n(x)\Delta_n(y)}{\prod_{i,j=1}^{n}(1-x_i y_j)}.$$

As a corollary of Theorem 5.3, we have the

Theorem 6.4. *The wreath Vandermonde determinant* $\mathrm{wrdet}_k\, V_{n,k}(x)$ *is given by*

$$\mathrm{wrdet}_k\, V_{n,k}(x) = \sum_{T\in \mathrm{STab}((k^n))} \mathrm{wrdet}_k\, I(T)\cdot \Delta_T(x),$$

where $\Delta_T(x)$ *is the Specht polynomial for a standard tableau* $T=(t_{ij})\in \mathrm{STab}((k^n))$ *defined by the product*

$$\Delta_T(x) := \prod_{i=1}^{k} \Delta_n(x_{t_{1i}},\dots,x_{t_{ni}})$$

of difference products.

Another (symmetric) expression for $\mathrm{wrdet}_k\, V_{n,k}(x)$ also follows from Corollary 5.7.

Theorem 6.5. *The equality*

$$\mathrm{wrdet}_k\, V_{n,k}(x) = \frac{1}{k^{kn}} \sum_{\sigma\in S_k^n} \sigma\cdot \Delta_{n,k}(x)$$

holds where $\Delta_{n,k}(x)$ *is given by*

$$\Delta_{n,k}(x) := \prod_{l=1}^{k} \Delta_n(x_l, x_{l+k},\dots,x_{l+(n-1)k}) = \Delta_{T_0}(x).$$

For a partition $\lambda=(\lambda_1,\dots,\lambda_N)$ of depth at most N, the Schur function $s_\lambda(x_1,\dots,x_N)$ of N variables is defined as the ratio of the Vandermonde-type determinants as

$$s_\lambda(x_1,\dots,x_N) = \frac{\det(x_i^{\lambda_j+N-j})_{1\leqslant i,j\leqslant N}}{\det(x_i^{N-j})_{1\leqslant i,j\leqslant N}}.$$

An arbitrary symmetric function can be written as a linear combination of the Schur functions. We show that any symmetric function in kn variables can be written as a linear combination of the ratios of the Vandermonde type $-\frac{1}{k}$-determinants analogously.

We recall the Cauchy identity concerning the Schur functions (see, e.g. [7,12]).

Lemma 6.6. *For* $m,n\in\mathbb{N}$*, the equality*

$$\prod_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \frac{1}{1-x_i y_j} = \sum_{\ell(\lambda)\leqslant \min\{m,n\}} s_\lambda(x_1,\dots,x_m)s_\lambda(y_1,\dots,y_n)$$

holds.

By the multilinearity of $\det_k$ with respect to *column* vectors, we have the following expansion formula

$$\begin{aligned}
&\operatorname{wrdet}_k\left(\frac{1}{1-x_\star y_1},\dots,\frac{1}{1-x_\star y_n}\right)\\
&\quad=\det_k\left(\sum_{i_{11}\geqslant 0}(x_\star y_1)^{i_{11}},\sum_{i_{21}\geqslant 0}(x_\star y_2)^{i_{21}},\dots,\sum_{i_{kn}\geqslant 0}(x_\star y_n)^{i_{kn}}\right)\\
&\quad=\sum_{i_{11},i_{21},\dots,i_{kn}\geqslant 0} y_1^{i_{11}+\cdots+i_{k1}}\cdots y_n^{i_{1n}+\cdots+i_{kn}}\det_k\bigl(x_\star^{i_{11}},x_\star^{i_{21}},\dots,x_\star^{i_{kn}}\bigr).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\sum_{i_{11},i_{21},\dots,i_{kn}\geqslant 0} y_1^{i_{11}+\cdots+i_{k1}}\cdots y_n^{i_{1n}+\cdots+i_{kn}}\det_k\bigl(x_\star^{i_{11}},x_\star^{i_{21}},\dots,x_\star^{i_{kn}}\bigr)\\
&\quad=\Delta_n(y)^k\operatorname{wrdet}_k V_{n,k}(x)\sum_{\ell(\lambda)\leqslant n}s_\lambda(x_1,\dots,x_{kn})s_\lambda(y_1,\dots,y_n).
\end{aligned}\tag{6.2}$$

Comparing the homogeneous terms in (6.2), we have the

Lemma 6.7. *Put*

$$H_{n,k}^d(x,y):=\sum_{\substack{i_{11},i_{21},\dots,i_{kn}\geqslant 0\\ i_{11}+\cdots+i_{kn}=d+\frac{kn(n-1)}{2}}} y_1^{i_{11}+\cdots+i_{k1}}\cdots y_n^{i_{1n}+\cdots+i_{kn}}\det_k\bigl(x_\star^{i_{11}},x_\star^{i_{21}},\dots,x_\star^{i_{kn}}\bigr).$$

Then, the equalities

$$\operatorname{wrdet}_k\left(\frac{1}{1-x_\star y_1},\dots,\frac{1}{1-x_\star y_n}\right)=\sum_{d=0}^{\infty}H_{n,k}^d(x,y)$$

and

$$H_{n,k}^d(x,y)=\Delta_n(y)^k\operatorname{wrdet}_k V_{n,k}(x)\sum_{\substack{\ell(\lambda)\leqslant n\\ |\lambda|=d}}s_\lambda(x)s_\lambda(y)$$

hold.

Since the Schur functions of n variables are the irreducible characters of the unitary group $U(n)$, it follows from (6.2) that

$$\begin{aligned}
s_\lambda(x_1,\dots,x_{kn})=&\sum_{\substack{i_{11},i_{21},\dots,i_{kn}\geqslant 0\\ i_{11}+\cdots+i_{kn}=|\lambda|+\frac{kn(n-1)}{2}}}\left\{\int_{T_n}\frac{y_1^{i_{11}+\cdots+i_{k1}}\cdots y_n^{i_{1n}+\cdots+i_{kn}}s_\lambda(y)}{\Delta_n(y)^k}\,dg(y)\right\}\\
&\times\frac{\det_k(x_\star^{i_{11}},x_\star^{i_{21}},\dots,x_\star^{i_{kn}})}{\operatorname{wrdet}_k V_{n,k}(x)},
\end{aligned}$$

where T_n is the n-torus in $U(n)$ and dg is its normalized Haar measure. Thus implicitly, we find the Schur function $s_\lambda(x_1,\dots,x_{kn})$ can be written as a linear combination of the ratios $\det_k(x_\star^{i_{11}},x_\star^{i_{21}},\dots,x_\star^{i_{kn}})/\operatorname{wrdet}_k V_{n,k}(x)$ of Vandermonde type $-\frac{1}{k}$-determinants. Actually, we have the following expression.

Proposition 6.8. *For a given sequence $\boldsymbol{a}=(a_1,\dots,a_{kn})\in\mathbb{Z}_{\geqslant 0}^{kn}$ of nonnegative integers, put*

$$D_{n,k}(x;\boldsymbol{a})=\det_k\big(x_i^{a_j}\big)_{1\leqslant i,j\leqslant kn}.$$

Let us also define $\boldsymbol{e}_i,\boldsymbol{\delta}_{n,k}\in\mathbb{Z}_{\geqslant 0}^{kn}$ by

$$\boldsymbol{e}_i=(0,\dots,0,\overset{i\text{th}}{1},0,\dots,0),\qquad \boldsymbol{\delta}_{n,k}=\sum_{j=1}^{kn}\left(n-1-\left\lfloor\frac{j-1}{k}\right\rfloor\right)\boldsymbol{e}_j.$$

Then, the Schur function $s_\lambda(x)$ is written as

$$s_\lambda(x)=\frac{1}{\operatorname{wrdet}_k V_{n,k}(x)}\sum_{\substack{\mu\leqslant\lambda\\|\mu|=|\lambda|}}\sum_{\sigma\in\mathfrak{S}_{kn}/\mathfrak{S}_\mu}K_{\lambda\mu}\cdot D_{n,k}\left(x;\boldsymbol{\delta}_{n,k}+\sum_{i=1}^{kn}\mu_{\sigma(i)}\boldsymbol{e}_i\right).$$

We notice that $\operatorname{wrdet}_k V_{n,k}(x)=D_{n,k}(x;\boldsymbol{\delta}_{n,k})$.

For a partition $\lambda=(\lambda_1,\dots,\lambda_{kn})$ whose depth is at most kn, the monomial symmetric function $m_\lambda(x)$ is defined by

$$m_\lambda(x)=\sum_{\sigma\in\mathfrak{S}_{kn}/\mathfrak{S}_\lambda}\prod_{i=1}^{kn}x_i^{\lambda_{\sigma(i)}}.$$

Here $\mathfrak{S}_\lambda$ is the stabilizer of λ, that is, $\mathfrak{S}_\lambda=\{\sigma\in\mathfrak{S}_{kn}\colon \lambda_{\sigma(i)}=\lambda_i,\ 1\leqslant i\leqslant kn\}$. The proposition follows from the following simple lemma.

Lemma 6.9. *Let λ be a partition whose depth is at most kn. Then, the monomial symmetric function $m_\lambda(x)$ has the following expression*

$$m_\lambda(x)=\frac{1}{\operatorname{wrdet}_k V_{n,k}(x)}\sum_{\sigma\in\mathfrak{S}_{kn}/\mathfrak{S}_\lambda}D_{n,k}\left(x;\boldsymbol{\delta}_{n,k}+\sum_{i=1}^{kn}\lambda_{\sigma(i)}\boldsymbol{e}_i\right).$$

Proof. For any $\sigma\in\mathfrak{S}_{kn}$, we have

$$D_{n,k}\left(x;\boldsymbol{\delta}_{n,k}+\sum_{i=1}^{kn}\lambda_{\sigma(i)}\boldsymbol{e}_i\right)=\sum_{\tau\in\mathfrak{S}_{kn}}\left(-\frac{1}{k}\right)^{kn-\nu_{kn}(\tau)}\prod_{i=1}^{kn}x_{\tau(i)}^{n-1-\lfloor\frac{i-1}{k}\rfloor}\cdot\prod_{i=1}^{kn}x_{\tau(i)}^{\lambda_{\sigma(i)}}.$$

Hence it follows that

$$\begin{aligned}&\sum_{\sigma\in\mathfrak{S}_{kn}}D_{n,k}\left(x;\boldsymbol{\delta}_{n,k}+\sum_{i=1}^{kn}\lambda_{\sigma(i)}\boldsymbol{e}_i\right)\\&\quad=\sum_{\tau\in\mathfrak{S}_{kn}}\left(-\frac{1}{k}\right)^{kn-\nu_{kn}(\tau)}\prod_{i=1}^{kn}x_{\tau(i)}^{n-1-\lfloor\frac{i-1}{k}\rfloor}\cdot\left(\sum_{\sigma\in\mathfrak{S}_{kn}}\prod_{i=1}^{kn}x_{\tau(i)}^{\lambda_{\sigma(i)}}\right)\\&\quad=\operatorname{wrdet}_k V_{n,k}(x)|\mathfrak{S}_\lambda|m_\lambda(x).\end{aligned}$$

Therefore we obtain

$$m_\lambda(x) = \frac{1}{\mathrm{wrdet}_k V_{n,k}(x)} \frac{1}{|\mathfrak{S}_\lambda|} \sum_{\sigma \in \mathfrak{S}_{kn}} D_{n,k}\left(x; \boldsymbol{\delta}_{n,k} + \sum_{i=1}^{kn} \lambda_{\sigma(i)} \boldsymbol{e}_i\right)$$
$$= \frac{1}{\mathrm{wrdet}_k V_{n,k}(x)} \sum_{\sigma \in \mathfrak{S}_{kn}/\mathfrak{S}_\lambda} D_{n,k}\left(x; \boldsymbol{\delta}_{n,k} + \sum_{i=1}^{kn} \lambda_{\sigma(i)} \boldsymbol{e}_i\right).$$

This completes the proof. □

Since the Schur functions are written as a linear combination

$$s_\lambda(x) = \sum_{\substack{\mu \leqslant \lambda \\ |\mu| = |\lambda|}} K_{\lambda\mu} m_\mu(x)$$

of monomial symmetric functions, Proposition 6.8 follows immediately.

Corollary 6.10. *The power-sum symmetric functions $p_d(x)$, the complete symmetric functions $h_d(x)$ and the elementary symmetric functions $e_d(x)$ are expressed as*

$$p_d(x) = \frac{1}{\mathrm{wrdet}_k V_{n,k}(x)} \sum_{i=1}^{kn} D_{n,k}(x; \boldsymbol{\delta}_{n,k} + d\boldsymbol{e}_i),$$
$$h_d(x) = \frac{1}{\mathrm{wrdet}_k V_{n,k}(x)} \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_d \leqslant kn} D_{n,k}\left(x; \boldsymbol{\delta}_{n,k} + \sum_{j=1}^{d} \boldsymbol{e}_{i_j}\right),$$
$$e_d(x) = \frac{1}{\mathrm{wrdet}_k V_{n,k}(x)} \sum_{1 \leqslant i_1 < \cdots < i_d \leqslant kn} D_{n,k}\left(x; \boldsymbol{\delta}_{n,k} + \sum_{j=1}^{d} \boldsymbol{e}_{i_j}\right).$$

7. Generalities on (n, k)-sign and spherical functions

For $k, n \in \mathbb{N}$, we put

$$\mathfrak{R}_{n,k} := \left\{ f : [kn] \to [n] \;\middle|\; \left|f^{-1}(j)\right| = k,\ \forall j \in [n] \right\}.$$

We notice that $\mathfrak{R}_{n,1} = \mathfrak{S}_n$. We also notice that $\mathfrak{S}_{kn}$ acts on $\mathfrak{R}_{n,k}$ transitively from the right, and $\mathfrak{S}_n$ acts on $\mathfrak{R}_{n,k}$ from the left.

For $f \in \mathfrak{R}_{n,k}$, we define the (n,k)-*sign* of f by

$$\mathrm{sgn}_{n,k}(f) := \mathrm{wrdet}_k \left(\delta_{f(i),j}\right)_{\substack{1 \leqslant i \leqslant kn \\ 1 \leqslant j \leqslant n}}.$$

We see that

$$\mathrm{sgn}_{n,k}(\tau \cdot f) = \mathrm{sgn}(\tau)^k \, \mathrm{sgn}_{n,k}(f)$$

for $\tau \in \mathfrak{S}_n$. Using this sign for $f \in \mathfrak{R}_{n,k}$ and the very definition (4.6) of the wreath determinant we have the

Lemma 7.1. *Let $k,n \in \mathbb{N}$. Then the equality*

$$\operatorname{wrdet}_k A = \sum_{f\in\mathfrak{R}_{n,k}} \operatorname{sgn}_{n,k}(f) \prod_{i\in[kn]} a_{if(i)}$$

holds for any $A=(a_{ij})\in \operatorname{Mat}_{kn,n}$.

We define the element $\iota_{n,k}\in\mathfrak{R}_{n,k}$ by

$$\iota_{n,k}\big((i-1)k+j\big)=i \quad (1\leqslant i\leqslant n,\ 1\leqslant j\leqslant k).$$

The stabilizer of $\iota_{n,k}$ in $\mathfrak{S}_{kn}$ is S_k^n. Hence, it follows that

$$\begin{aligned}\operatorname{sgn}_{n,k}(f) &= \sum_{w\in\mathfrak{S}_{kn}} \left(-\frac{1}{k}\right)^{kn-\nu_{kn}(w)} \prod_{i=1}^{n}\prod_{j=1}^{k} \delta_{fw((i-1)k+j),i}\\ &= \sum_{w\in\mathfrak{S}_{kn}} \left(-\frac{1}{k}\right)^{kn-\nu_{kn}(w)} \delta_{fw,\iota_{n,k}}\\ &= \sum_{w\in S_k^n} \left(-\frac{1}{k}\right)^{kn-\nu_{kn}(g(f)w)},\end{aligned}$$

where $g(f)\in\mathfrak{S}_{kn}$ is defined by $f=\iota_{n,k}\cdot g(f)$. Therefore, if we regard a standard tableau $T=(t_{ij})\in\operatorname{STab}((k^n))$ as an element of $\mathfrak{R}_{n,k}$ by the assignment $T:[kn]\ni t_{ij}\mapsto i\in[n]$, then $\operatorname{sgn}_{n,k}(T)=\operatorname{wrdet}_k I(T)$. Hence, the result of Theorem 5.3 can be expressed also as

$$\operatorname{wrdet}_k A = \sum_{T\in\operatorname{STab}((k^n))} \operatorname{sgn}_{n,k}(T)\det_T(A).$$

We consider the injection

$$\omega:\mathfrak{S}_n^k\ni(w_1,\dots,w_k)\mapsto\big((i-1)k+j\mapsto w_j(i)\big)\in\mathfrak{R}_{n,k},$$

and denote its image by $\mathfrak{R}_{n,k}^{\times}$. By Lemmas 4.6 and 7.1, we have

$$\left(\frac{k!}{k^k}\right)^n \sum_{w\in\mathfrak{S}_n^k} \operatorname{sgn}(w) \prod_{i=1}^{n}\prod_{j=1}^{k} a_{i,\omega(w)((i-1)k+j)} = \sum_{f\in\mathfrak{R}_{n,k}} \operatorname{sgn}_{n,k}(f) \prod_{i=1}^{n}\prod_{j=1}^{n} a_{i,f((i-1)k+j)} \tag{7.1}$$

for $(a_{ij})_{1\leqslant i,j\leqslant n}\in\operatorname{Mat}_n$. Comparing the coefficients in both sides, we obtain the

Corollary 7.2. *For any $f\in\mathfrak{R}_{n,k}$, the equality*

$$\operatorname{sgn}_{n,k}(f)=\operatorname{sgn}(w)\left(\frac{k!}{k^k}\right)^n \frac{|(f\cdot S_k^n)\cap\mathfrak{R}_{n,k}^{\times}|}{|f\cdot S_k^n|}$$

holds for $w\in\mathfrak{S}_n^k$ such that $\omega(w)\in(f\cdot S_k^n)\cap\mathfrak{R}_{n,k}^{\times}$. The sign $\operatorname{sgn}(w)$ does not depend on the choice of w.

Proof. Fix an element $f \in \mathfrak{R}_{n,k}$. We notice that the monomial $\prod_{i=1}^n \prod_{j=1}^n a_{i,f((i-1)k+j)}$ in the right-hand side of (7.1) depends only on the orbit $f \cdot S_k^n$. We also notice that the function $\mathrm{sgn}_{n,k}$ is constant on each S_k^n-orbit. Hence the coefficient of the monomial $\prod_{i=1}^n \prod_{j=1}^n a_{i,f((i-1)k+j)}$ in the right-hand side is $\mathrm{sgn}_{n,k}(f)|f \cdot S_k^n|$. For any $w = (w_1, \dots, w_k) \in \mathfrak{S}_n^k$ such that $\omega(w) \in f \cdot S_k^n$, the sign $\mathrm{sgn}(w) = \mathrm{sgn}(w_1 \dots w_k)$ gives the same value, which can be verified by counting the inversion numbers. It follows that the coefficient of the monomial $\prod_{i=1}^n \prod_{j=1}^n a_{i,f((i-1)k+j)}$ in the left-hand side is $\mathrm{sgn}(w)|(f \cdot S_k^n) \cap \mathfrak{R}_{n,k}^\times|$ for any $w \in (f \cdot S_k^n) \cap \mathfrak{R}_{n,k}^\times$. Thus we have the desired conclusion. □

As a corollary of the discussion above, we obtain the

Proposition 7.3.

(1) *Put*

$$m_{ij}(f) = \bigl|\bigl\{l \in [k] \bigm| f\bigl((i-1)k + l\bigr) = j\bigr\}\bigr|.$$

Then

$$\bigl|f \cdot S_k^n\bigr| = \frac{k!^n}{\prod_{i,j} m_{ij}(f)!}.$$

(2) *The equality*

$$\mathrm{sgn}_{n,k}(f)\det(A)^k = \sum_{h \in \mathfrak{R}_{n,k}} \mathrm{sgn}_{n,k}(h) \prod_{i=1}^{kn} a_{f(i)h(i)}$$

holds for any $f \in \mathfrak{R}_{n,k}$ and $A = (a_{ij})_{1 \leqslant i,j \leqslant n} \in \mathrm{Mat}_n$. (When $k = 1$, this is just the definition of the determinant.)

(3) *For $f \in \mathfrak{R}_{n,k}$, put*

$$P_f(x_{11}, \dots, x_{nk}) := \frac{1}{|\mathfrak{S}_k^n|} \sum_{(\sigma_1, \dots, \sigma_n) \in \mathfrak{S}_k^n} \prod_{i=1}^n \prod_{j=1}^k x_{f((i-1)k+j), \sigma_i(j)}.$$

Then

$$\frac{|(f \cdot S_k^n) \cap \mathfrak{R}_{n,k}^\times|}{|f \cdot S_k^n|} = \text{the coefficient of} \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant k}} x_{ij} \text{ in } P_f(x_{11}, \dots, x_{nk}).$$

It is convenient to express an element $f \in \mathfrak{R}_{n,k}$ as an $n \times k$ matrix whose (i, j)-entry is given by $f((i-1)k + j)$, that is,

$$f = \begin{pmatrix} f(1) & \dots & f(k) \\ \vdots & \ddots & \vdots \\ f((n-1)k+1) & \dots & f(nk) \end{pmatrix}.$$

If $f_1, f_2 \in \mathfrak{R}_{n,k}$ and $f_2 = f_1 \cdot \sigma$ for some $\sigma \in S_k^n$, then each row vector of f_2 is a permutation of the corresponding row vector of f_1.

Example 7.4. Let us calculate $\mathrm{sgn}_{n,k}(U_4) = \mathrm{wrdet}_2\, I(U_4)$ for U_4 (regarding as an element in $\mathfrak{R}_{3,2}$) given in Example 5.4. In the matrix notation,

$$U_4 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array} = \begin{Bmatrix} 1 \mapsto 1 & 2 \mapsto 2 \\ 3 \mapsto 1 & 4 \mapsto 3 \\ 5 \mapsto 2 & 6 \mapsto 3 \end{Bmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{pmatrix}.$$

It follows that

$$U_4 \cdot S_2^3 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 3 & 2 \end{pmatrix} \right\}$$

and

$$\left(U_4 \cdot S_2^3\right) \cap \mathfrak{R}_{3,2}^{\times} = \left\{ \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 3 & 2 \end{pmatrix} \right\}.$$

Since

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{pmatrix} = \omega\big((2,3), (1,2)\big)$$

and $\mathrm{sgn}((2,3),(1,2)) = 1$ (where (i,j) denotes the transposition of i and j), we get

$$\mathrm{wrdet}_2\, I(U_4) = \left(\frac{2!}{2^2}\right)^3 \times \frac{2}{8} = \frac{1}{32}.$$

We remark that

$$\begin{pmatrix} m_{11}(U_4) & m_{12}(U_4) & m_{13}(U_4) \\ m_{21}(U_4) & m_{22}(U_4) & m_{23}(U_4) \\ m_{31}(U_4) & m_{32}(U_4) & m_{33}(U_4) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and we see that

$$\frac{2!^3}{\prod_{1 \leqslant i,j \leqslant 3} m_{ij}(U_4)!} = \frac{2!^3}{1!1!0!1!0!1!0!1!1!} = 8 = \left|U_4 \cdot S_2^3\right|$$

as we counted above. We also note that

$$P_{U_4}(x_{11}, \dots, x_{32}) = \frac{1}{8}(x_{11}x_{22} + x_{12}x_{21})(x_{11}x_{32} + x_{12}x_{31})(x_{21}x_{32} + x_{22}x_{31}),$$

and the coefficient of $x_{11}x_{21}x_{31}x_{12}x_{22}x_{32}$ of P_{U_4} is $\frac{2}{8} = \frac{1}{4}$.

Let us put

$$\varphi_{n,k}(g) = \frac{\det_k(g \cdot \mathbf{1}_k^{\oplus n})}{\det_k(\mathbf{1}_k^{\oplus n})} = k^{kn} \frac{1}{|S_k^n|} \sum_{\sigma \in S_k^n} \left(-\frac{1}{k}\right)^{kn - \nu_{kn}(g^{-1}\sigma)} \tag{7.2}$$

for $g \in \mathfrak{S}_{kn}$. We note that $\varphi_{n,k}(g^{-1}) = \varphi_{n,k}(g)$ since $\nu_{kn}(g^{-1}\sigma) = \nu_{kn}(g\sigma^{-1})$. By Lemma 4.7 and its S_k^n-invariance of $\mathbf{1}_k^{\oplus n}$, it follows that

$$\varphi_{n,k}(h_1 g h_2) = \chi_{n,k}(h_1 h_2)^k \varphi_{n,k}(g)$$

for $g \in \mathfrak{S}_{kn}$ and $h_1, h_2 \in \mathfrak{S}_k \wr \mathfrak{S}_n$. In particular, $\varphi_{n,k}$ is a S_k^n-biinvariant (or S_k^n-zonal spherical) function on $\mathfrak{S}_{kn}$. We note that the rightmost side of (7.2) can be considered as an analogue of the integral expression of the zonal spherical function of a Riemannian symmetric space due to Harish-Chandra (see, e.g. [2]).

Lemma 7.5. *The $\chi_{n,k}^k$-spherical function $\varphi_{n,k}$ relative to the wreath product $\mathfrak{S}_k \wr \mathfrak{S}_n$ on $\mathfrak{S}_{kn}$ is expressed as a matrix element of the (unitary) representation $M_{n,k}^{T_{kn},\det} (\cong \mathcal{J}_{kn}^{\lambda})$ of $\mathfrak{S}_{kn}$:*

$$\varphi_{n,k}(g) = \frac{\langle g \cdot \mathrm{wrdet}_k(X), \mathrm{wrdet}_k(X)\rangle}{\langle \mathrm{wrdet}_k(X), \mathrm{wrdet}_k(X)\rangle},$$

where $\langle\, ,\, \rangle$ denotes the invariant inner product on $M_{n,k}^{T_{kn},\det}$. In particular, $\varphi_{n,k}$ is a positive definite function.

Proof. Consider the projection

$$P_{n,k} = \frac{1}{|S_k^n|} \sum_{\sigma \in S_k^n} \sigma \in \mathbb{C}[\mathfrak{S}_{kn}].$$

By (5.3), for each $g \in \mathfrak{S}_{kn}$, there exists a constant $C(g)$ such that

$$P_{n,k}\, g \cdot \mathrm{wrdet}_k(X) = C(g)\, \mathrm{wrdet}_k(X). \tag{7.3}$$

Since $P_{n,k}$ is self-adjoint with respect to $\langle\, ,\, \rangle$ and $\mathrm{wrdet}_k(X)$ is S_k^n-invariant, it follows that

$$\begin{aligned}\big\langle g \cdot \mathrm{wrdet}_k(X), \mathrm{wrdet}_k(X)\big\rangle &= \big\langle g \cdot \mathrm{wrdet}_k(X), P_{n,k} \cdot \mathrm{wrdet}_k(X)\big\rangle \\ &= \big\langle P_{n,k} g \cdot \mathrm{wrdet}_k(X), \mathrm{wrdet}_k(X)\big\rangle \\ &= C(g)\big\langle \mathrm{wrdet}_k(X), \mathrm{wrdet}_k(X)\big\rangle.\end{aligned}$$

To determine $C(g)$, let us calculate the coefficient of $\prod_{p=1}^n \prod_{l=1}^k x_{(p-1)k+l,p}$ in the both sides of (7.3). It is immediate to see that the coefficient in the right-hand side is $C(g)(\frac{k!}{k^k})^n = C(g)\det_k(\mathbf{1}_k^{\oplus n})$. We look at the left-hand side:

$$\begin{aligned}P_{n,k} g \cdot \mathrm{wrdet}_k(X) &= \frac{1}{|S_k^n|} \sum_{\sigma \in S_k^n} \sigma g \cdot \mathrm{wrdet}_k(X) \\ &= \frac{1}{|S_k^n|} \sum_{\sigma \in S_k^n} \sum_{h \in \mathfrak{S}_{kn}} \left(-\frac{1}{k}\right)^{kn-\nu_{kn}(h)} \prod_{p=1}^n \prod_{l=1}^k x_{(\sigma g h)((p-1)k+l),p} \\ &= \sum_{h \in \mathfrak{S}_{kn}} \left(\frac{1}{|S_k^n|} \sum_{\sigma \in S_k^n} \left(-\frac{1}{k}\right)^{kn-\nu_{kn}(g^{-1}\sigma^{-1}h)}\right) \prod_{p=1}^n \prod_{l=1}^k x_{h((p-1)k+l),p}.\end{aligned}$$

Hence the coefficient of $\prod_{p=1}^n \prod_{l=1}^k x_{(p-1)k+l,p}$ in $P_{n,k} g \cdot \mathrm{wrdet}_k(X)$ is equal to

$$\sum_{h\in S_k^n}\frac{1}{|S_k^n|}\sum_{\sigma\in S_k^n}\left(-\frac{1}{k}\right)^{kn-\nu_{kn}(g^{-1}\sigma^{-1}h)}=\sum_{\sigma\in S_k^n}\left(-\frac{1}{k}\right)^{kn-\nu_{kn}(g^{-1}\sigma)}=\det_k\bigl(g\cdot\mathbf{1}_k^{\oplus n}\bigr).$$

Thus we have

$$C(g)=\frac{\det_k(g\cdot\mathbf{1}_k^{\oplus n})}{\det_k(\mathbf{1}_k^{\oplus n})}=\varphi_{n,k}(g).$$

This completes the proof. □

Remark 7.6. By specializing the Frobenius character formula for $\mathfrak{S}_N$, we have

$$\alpha^{N-\nu_N(g)}=\sum_{\lambda\vdash N}\frac{f^\lambda}{N!}f_\lambda(\alpha)\chi^\lambda(g)\quad(g\in\mathfrak{S}_N),$$

where $f_\lambda(\alpha)$ denotes the content polynomial defined by

$$f_\lambda(\alpha)=\prod_{(i,j)\in\lambda}\bigl(1+(j-i)\alpha\bigr).$$

Since

$$f_\lambda\left(-\frac{1}{k}\right)=\prod_{(i,j)\in\lambda}\left(1-\frac{1}{k}(j-i)\right)=\frac{1}{k^{kn}}\prod_{(j,i)\in\lambda'}\bigl(k+(i-j)\bigr)=\frac{(kn)!}{f^\lambda}\frac{|\mathrm{SSTab}_k(\lambda')|}{k^{kn}},$$

it follows that

$$\left(-\frac{1}{k}\right)^{kn-\nu_{kn}(g)}=\sum_{\lambda\vdash kn}\frac{|\mathrm{SSTab}_k(\lambda')|}{k^{kn}}\chi^\lambda(g).$$

Hence the function $\varphi_{n,k}$ is a linear combination

$$\varphi_{n,k}(g)=\sum_{\lambda\vdash kn}\bigl|\mathrm{SSTab}_k(\lambda')\bigr|\phi_{n,k}^\lambda(g)$$

of S_k^n-zonal spherical functions

$$\phi_{n,k}^\lambda(g)=\frac{1}{|S_k^n|}\sum_{\sigma\in S_k^n}\chi^\lambda\bigl(g^{-1}\sigma\bigr)$$

with nonnegative (integral) coefficients. Therefore, it is immediate to see again that $\varphi_{n,k}$ is a positive definite function.

Remark 7.7. Since $\langle\mathrm{ind}_{S_k^n}^{\mathfrak{S}_{kn}}1_{S_k^n},\mathcal{J}_{kn}^\lambda\rangle=K_{\lambda,(k^n)}$ for $\lambda\vdash kn$, the pair $(\mathfrak{S}_{kn},S_k^n)$ is not a Gelfand pair in general. Further, although one can verify that the pair $(\mathfrak{S}_{kn},\mathfrak{S}_k\wr\mathfrak{S}_n)$ is a Gelfand pair when $k=2$ (see p. 401 in [7], in fact, the wreath product $\mathfrak{S}_2\wr\mathfrak{S}_n$ is isomorphic to the *hyperoctahedral* group of degree n), it is not the case for a general k. Actually, when $n=3$, by looking at the Schur function expansion of the plethysm $h_3\circ h_k$ (see p. 141 in [7]), it follows that the induced representation $\mathrm{ind}_{\mathfrak{S}_k\wr\mathfrak{S}_3}^{\mathfrak{S}_{3k}}1_{\mathfrak{S}_k\wr\mathfrak{S}_3}$ is not multiplicity free when $k\geqslant 18$.

For a standard tableau $T \in \mathrm{STab}((k^n))$, we define

$$D_T(X) = \mathrm{wrdet}_k\big(g(T)^{-1} \cdot X\big),$$

where $g(T)$ is a permutation given in (5.2). We see that

$$\begin{aligned} D_T(X) &= \sum_{S \in \mathrm{STab}((k^n))} \mathrm{wrdet}_k\big(g(T)^{-1} I(S)\big) \det_S(X) \\ &= \left(\frac{k!}{k^k}\right)^n \sum_{S \in \mathrm{STab}((k^n))} \varphi_{n,k}\big(g(T)^{-1} g(S)\big) \det_S(X). \end{aligned}$$

We now define the $f^{(k^n)} \times f^{(k^n)}$ matrix $\Xi_{n,k}$ by

$$\Xi_{n,k} = \big(\varphi_{n,k}\big(g(T)^{-1} g(S)\big)\big)_{S,T \in \mathrm{STab}((k^n))}. \tag{7.4}$$

Since $\varphi_{n,k}(g) = \varphi_{n,k}(g^{-1})$, one finds that the matrix $\Xi_{n,k}$ is symmetric. Moreover, we notice that $\det \Xi_{n,k} \geqslant 0$ by Lemma 7.5, because $\varphi_{n,k}$ is a positive definite function. Then the following conjecture looks quite reasonable.

Conjecture 7.8. *The matrix $\Xi_{n,k}$ is positive definite*; *in particular, one has* $\det \Xi_{n,k} > 0$. *In other words, $\{D_T(X)\}_{T \in \mathrm{STab}((k^n))}$ gives another basis of the space $M_{n,k}^{T_{kn},\det} = \mathbb{C}[\mathfrak{S}_{kn}] \cdot \mathrm{wrdet}_k$.*

We try to examine the first few examples which may support the above conjecture.

Example 7.9. We have

$$\det \Xi_{2,2} = \frac{1}{3}\left(\frac{3}{2}\right)^2, \qquad \det \Xi_{3,2} = \frac{2}{3}\left(\frac{3}{4}\right)^5, \qquad \det \Xi_{2,3} = \frac{3}{2}\left(\frac{2}{3}\right)^5,$$
$$\det \Xi_{4,2} = \frac{2^6 5}{3}\left(\frac{3}{8}\right)^{14}, \qquad \det \Xi_{2,4} = \frac{3}{2^6 5}\left(\frac{5}{6}\right)^{14}.$$

We notice here that

$$f^{(2^2)} = 2, \qquad f^{(2^3)} = f^{(3^2)} = 5, \qquad f^{(2^4)} = f^{(4^2)} = 14.$$

Appendix A. Laplace expansion of α-determinants

Proposition A.1 *(Laplace expansion). For a given n by n matrix $X = (x_{ij})_{1 \leqslant i,j \leqslant n}$, we have*

$$\det^{(\alpha)} X = \sum_{p=1}^{n} \alpha^{1-\delta_{pq}} x_{pq} \det^{(\alpha)} X_{pq},$$

where X_{pq} is an $n-1$ by $n-1$ matrix obtained by the following procedure: (1) *remove qth column vector and qth row vector in X,* (2) *if $p \neq q$, then replace the row vector $(x_{p1}, \ldots, x_{pn})$ in X by $(x_{q1}, \ldots, x_{qn})$.*

Proof. We have

$$\begin{aligned}\det^{(\alpha)} X &= \sum_{p=1}^{n} \sum_{\substack{g\in\mathfrak{S}_n\\ g(q)=p}} \alpha^{n-\nu_n(g)} \prod_{i=1}^{n} x_{g(i)i}\\ &= \sum_{p=1}^{n} x_{pq} \sum_{\substack{g\in\mathfrak{S}_n\\ g(q)=q}} \alpha^{n-\nu_n((p,q)\cdot g)} \prod_{1\leqslant i(\neq q)\leqslant n} x_{(p,q)\cdot g(i)i}\\ &= \sum_{p=1}^{n} \alpha^{1-\delta_{pq}} x_{pq} \sum_{\substack{g\in\mathfrak{S}_n\\ g(q)=q}} \alpha^{(n-1)-\nu_{n-1}(g)} \prod_{1\leqslant i(\neq q)\leqslant n} x_{(p,q)\cdot g(i)i}\\ &= \sum_{p=1}^{n} \alpha^{1-\delta_{pq}} x_{pq} \det^{(\alpha)} X_{pq}.\end{aligned}$$

Here we use the fact that $\nu_n((p,q)\cdot g) = \nu_{n-1}(g) + \delta_{pq}$ if $g(q) = q$ (see the proof of Lemma 2.1). □

Example A.2 *($n = 4$).* For

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14}\\ x_{21} & x_{22} & x_{23} & x_{24}\\ x_{31} & x_{32} & x_{33} & x_{34}\\ x_{41} & x_{42} & x_{43} & x_{44}\end{pmatrix},$$

we have

$$X_{12} = \begin{pmatrix} x_{21} & x_{23} & x_{24}\\ x_{31} & x_{33} & x_{34}\\ x_{41} & x_{43} & x_{44}\end{pmatrix}, \qquad X_{22} = \begin{pmatrix} x_{11} & x_{13} & x_{14}\\ x_{31} & x_{33} & x_{34}\\ x_{41} & x_{43} & x_{44}\end{pmatrix},$$

$$X_{32} = \begin{pmatrix} x_{11} & x_{13} & x_{14}\\ x_{21} & x_{23} & x_{24}\\ x_{41} & x_{43} & x_{44}\end{pmatrix}, \qquad X_{42} = \begin{pmatrix} x_{11} & x_{13} & x_{14}\\ x_{31} & x_{33} & x_{34}\\ x_{21} & x_{23} & x_{24}\end{pmatrix}.$$

Hence we have

$$\begin{aligned}&\det^{(\alpha)} \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14}\\ x_{21} & x_{22} & x_{23} & x_{24}\\ x_{31} & x_{32} & x_{33} & x_{34}\\ x_{41} & x_{42} & x_{43} & x_{44}\end{pmatrix}\\ &= \alpha x_{12} \det^{(\alpha)} \begin{pmatrix} x_{21} & x_{23} & x_{24}\\ x_{31} & x_{33} & x_{34}\\ x_{41} & x_{43} & x_{44}\end{pmatrix} + x_{22} \det^{(\alpha)} \begin{pmatrix} x_{11} & x_{13} & x_{14}\\ x_{31} & x_{33} & x_{34}\\ x_{41} & x_{43} & x_{44}\end{pmatrix}\\ &\quad + \alpha x_{32} \det^{(\alpha)} \begin{pmatrix} x_{11} & x_{13} & x_{14}\\ x_{21} & x_{23} & x_{24}\\ x_{41} & x_{43} & x_{44}\end{pmatrix} + \alpha x_{42} \det^{(\alpha)} \begin{pmatrix} x_{11} & x_{13} & x_{14}\\ x_{31} & x_{33} & x_{34}\\ x_{21} & x_{23} & x_{24}\end{pmatrix}.\end{aligned}$$

References

[1] W. Fulton, Young Tableaux, London Math. Soc. Stud. Texts, vol. 35, Cambridge University Press, Cambridge, 1997.

[2] S. Helgason, Groups and Geometric Analysis, Academic Press, New York, 1984.
[3] R. Howe, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, in: The Schur Lectures, Tel Aviv, 1992, in: Israel Math. Conf. Proc., vol. 8, Bar-Ilan Univ., Ramat Gan, 1995, pp. 1–182.
[4] R. Howe, $(\mathrm{GL}_n, \mathrm{GL}_m)$-duality and symmetric plethysm, Proc. Indian Acad. Sci. Math. Sci. 97 (1–3) (1987) 85–109.
[5] K. Kimoto, S. Matsumoto, M. Wakayama, Alpha-determinant cyclic modules and Jacobi polynomials, in preparation.
[6] K. Kimoto, M. Wakayama, Quantum α-determinant cyclic modules of $\mathcal{U}_q(\mathfrak{gl}_n)$, J. Algebra 313 (2007) 922–956.
[7] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford University Press, Oxford, 1995.
[8] S. Matsumoto, M. Wakayama, Alpha-determinant cyclic modules of $\mathfrak{gl}_n(\mathbb{C})$, J. Lie Theory 16 (2006) 393–405.
[9] T. Shirai, Y. Takahashi, Random point fields associated with certain Fredholm determinants I: Fermion, Poisson and boson point processes, J. Funct. Anal. 205 (2003) 414–463.
[10] R.P. Stanley, Enumerative Combinatorics, vol. 1, second ed., Cambridge Stud. Adv. Math., vol. 49, Cambridge University Press, Cambridge, 1997.
[11] D. Vere-Jones, A generalization of permanents and determinants, Linear Algebra Appl. 111 (1988) 119–124.
[12] H. Weyl, The Classical Groups. Their Invariants and Representations, second ed., Princeton University Press, Princeton, 1946.