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Automorphisms of abelian group extensions

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A B S T R A C T

Let $1 \to N \to G \to H \to 1$ be an abelian extension. The purpose of this paper is to study the problem of extending automorphisms of N and lifting automorphisms of H to certain automorphisms of G.

1. Introduction

Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be a short exact sequence of groups, i.e., an extension of a group N by the group $H \simeq G/N$. If N is abelian, then such an extension is called an *abelian extension*. Our aim in this paper is to construct certain exact sequences, similar to the one due to C. Wells [9], and apply them to study extensions and liftings of automorphisms in abelian extensions. More precisely, we study, for abelian extensions, the following well-known problem (see [3,7,8]):

Problem. Let N be a normal subgroup of G. Under what conditions (i) can an automorphism of N be extended to an automorphism of G; (ii) an automorphism of G/N is induced by an automorphism of G?

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Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be an abelian extension. We fix a left transversal $t : H \to G$ for H in G such that $t(1) = 1$, so that every element of G can be written uniquely as $t(x)n$ for some $x \in H$ and $n \in N$. Given an element $x \in H$, we define an action of x on N by setting $n^x = t(x)^{-1}nt(x)$; we thus have a homomorphism $\alpha : H \to \mathrm{Aut}(N)$ enabling us to view N as a right H-module.

A pair $(\theta, \phi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H)$ is called *compatible* if $\theta^{-1}x^\alpha\theta = (x^\phi)^\alpha$ for all $x \in H$. Let C denote the group of all compatible pairs. Let $\mathrm{C}_1 = \{\theta \in \mathrm{Aut}(N) \mid (\theta, 1) \in \mathrm{C}\}$ and $\mathrm{C}_2 = \{\phi \in \mathrm{Aut}(H) \mid (1, \phi) \in \mathrm{C}\}$.

We denote by $\mathrm{Aut}^{N,H}(G)$ the group of all automorphisms of G which centralize N (i.e., fix N element-wise) and induce identity on H. By $\mathrm{Aut}^N(G)$ and $\mathrm{Aut}_N(G)$ we denote respectively the group of all automorphisms of G which centralize N and the group of all automorphisms α of G which normalize N (i.e., $\alpha(N) = N$). By $\mathrm{Aut}_N^H(G)$ we denote the group of all automorphisms of G which normalize N and induce identity on H.

Observe that an automorphism $\gamma \in \mathrm{Aut}_N(G)$ induces automorphisms $\theta \in \mathrm{Aut}(N)$ and $\phi \in \mathrm{Aut}(H)$ given by $\theta(n) = \gamma(n)$ for all $n \in N$ and $\phi(x) = \gamma(t(x))N$ for all $x \in H$. We can thus define a homomorphism $\tau : \mathrm{Aut}_N(G) \to \mathrm{Aut}(N) \times \mathrm{Aut}(H)$ by setting $\tau(\gamma) = (\theta, \phi)$. We denote the restrictions of τ to $\mathrm{Aut}_N^H(G)$ and $\mathrm{Aut}^N(G)$ by τ_1 and τ_2 respectively.

With the above notation, there exists the following exact sequence, first constructed by C. Wells [9], which relates automorphisms of group extensions with group cohomology:

$$1 \to \mathrm{Z}^1_\alpha\big(H, \mathrm{Z}(N)\big) \to \mathrm{Aut}_N(G) \xrightarrow{\tau} C \to \mathrm{H}^2_\alpha\big(H, \mathrm{Z}(N)\big).$$

Recently P. Jin [7] gave an explicit description of this sequence for automorphisms of G inducing identity on H and obtained some interesting results regarding extensions of automorphisms of N to automorphisms of G inducing identity on H. We continue in the present work this line of investigation.

In Section 2, we establish our exact sequences.

Theorem 1. *If $1 \to N \to G \xrightarrow{\pi} H \to 1$ is an abelian extension, then there exist the following two exact sequences*:

$$1 \to \mathrm{Aut}^{N,H}(G) \to \mathrm{Aut}_N^H(G) \xrightarrow{\tau_1} \mathrm{C}_1 \xrightarrow{\lambda_1} \mathrm{H}^2(H, N) \tag{1.1}$$

and

$$1 \to \mathrm{Aut}^{N,H}(G) \to \mathrm{Aut}^N(G) \xrightarrow{\tau_2} \mathrm{C}_2 \xrightarrow{\lambda_2} \mathrm{H}^2(H, N). \tag{1.2}$$

For the definitions of maps λ_1 and λ_2, see (2.9) and (2.10). It may be noted that these maps are not necessarily homomorphisms (see Remark 2.12).

We say that an extension $1 \to N \to G \xrightarrow{\pi} H \to 1$ is *central* if $N \leqslant Z(G)$, the center of G; for such extensions we construct a more general exact sequence.

Theorem 2. *If $1 \to N \to G \xrightarrow{\pi} H \to 1$ is a central extension, then there exists an exact sequence*

$$1 \to \mathrm{Aut}^{N,H}(G) \to \mathrm{Aut}_N(G) \xrightarrow{\tau} \mathrm{Aut}(N) \times \mathrm{Aut}(H) \xrightarrow{\lambda} \mathrm{H}^2(H, N). \tag{1.3}$$

The map λ will be defined in the proof of Theorem 2 in Section 2.

As a consequence of Theorem 1, we readily get the following result:

Corollary 3. *Let N be an abelian normal subgroup of G with $\mathrm{H}^2(G/N, N)$ trivial. Then*

(1) *every element of C_1 can be extended to an automorphism of G centralizing H*;
(2) *every element of C_2 can be lifted to an automorphism of G centralizing N.*

Clearly, if G is finite and the map $x \mapsto x^{|G/N|}$ is an isomorphism of N, then $\mathrm{H}^2(G/N, N) = 1$. In particular, for the class of finite groups G such that $|G/N|$ is coprime to $|N|$, we have $\mathrm{H}^2(G/N, N) = 1$ and hence Corollary 3 holds true for this class of groups.

In Section 3, we apply Theorem 1 to reduce the problem of lifting of automorphisms of H to G to the problem of lifting of automorphisms of Sylow subgroups of H to automorphisms of their pre-images in G, and prove the following result:

Theorem 4. *Let N be an abelian normal subgroup of a finite group G. Then the following hold*:

(1) *An automorphism ϕ of G/N lifts to an automorphism of G centralizing N provided the restriction of ϕ to some Sylow p-subgroup P/N of G/N, for each prime number p dividing $|G/N|$, lifts to an automorphism of P centralizing N.*
(2) *If an automorphism ϕ of G/N lifts to an automorphism of G centralizing N, then the restriction of ϕ to a characteristic subgroup P/N of G/N lifts to an automorphism of P centralizing N.*

We mention below two corollaries to illustrate Theorem 4. These corollaries show that, in many cases, the hypothesis of Theorem 4 is naturally satisfied.

Corollary 5. *Let N be an abelian normal subgroup of a finite group G such that G/N is nilpotent. Then an automorphism ϕ of G/N lifts to an automorphism of G centralizing N if, and only if the restriction of ϕ to each Sylow subgroup P/N of G/N lifts to an automorphism of P centralizing N.*

An automorphism α of a group G is said to be *commuting automorphism* if $x\alpha(x) = \alpha(x)x$ for all $x \in G$. It follows from [4, Remark 4.2] that each Sylow subgroup of a finite group G is kept invariant by every commuting automorphism of G. Thus we have the following result:

Corollary 6. *Let N be an abelian normal subgroup of a finite group G. Then a commuting automorphism ϕ of G/N lifts to an automorphism of G centralizing N if, and only if the restriction of ϕ to each Sylow subgroup P/N of G/N lifts to an automorphism of P centralizing N.*

It may be remarked that if N is an abelian subgroup of G, then the map ω constructed by Jin in [7, Theorem A] is trivial and hence, in our notation, gives the following exact sequence:

$$1 \to \mathrm{Z}^1_\alpha(H, N) \to \mathrm{Aut}^H_N(G) \to \mathrm{C}_1 \to 1$$

which, in turn, implies that any element of C_1 can be extended to an automorphism of G centralizing H. The latter statement, however, is not true, in general, as illustrated by the following class of examples:

Let G_1 be a finite p-group, where p is an odd prime. Then there exists a group G in the isoclinism class (in the sense of P. Hall [6]) of G_1 such that G has no non-trivial abelian direct factor. Let $N = Z(G)$, the center of G. Then it follows from [1, Corollary 2] that $|\mathrm{Aut}^H_N(G)|$ is p^r for some $r \geqslant 1$. Let $\theta \in N$ be the automorphism inverting elements of N. Then the order of θ is 2 and therefore θ cannot be extended to an automorphism of G centralizing H. Thus Theorem D of Jin does not shed any light in case N is abelian. However, using our sequence (1.1), one can see that the following result holds:

Theorem 7. *Let N be an abelian normal subgroup of a finite group G. Then an automorphism θ of N extends to an automorphism of G centralizing G/N if, and only if, for some Sylow p-subgroup P/N of G/N, for each prime number p dividing $|G/N|$, θ extends to an automorphism of P centralizing P/N.*

Finally, in Section 4, we refine our sequences (1.1)–(1.3), and show that these sequences split in case the given exact sequence $1 \to N \to G \to H \to 1$ splits (Theorem 8). We also give examples to show that the converse is not true, in general.

2. Construction of sequences

Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be an abelian extension. For any two elements $x, y \in H$, we have $\pi(t(xy)) = xy = \pi(t(x))\pi(t(y)) = \pi(t(x)t(y))$. Thus there exists a unique element (say) $\mu(x, y) \in N$ such that $t(xy)\mu(x, y) = t(x)t(y)$. Observe that μ is a map from $H \times H$ to N such that $\mu(1, x) = \mu(x, 1) = 1$ and

$$\mu(xy, z)\mu(x, y)^{t(z)} = \mu(x, yz)\mu(y, z), \tag{2.1}$$

for all $x, y, z \in H$; in other words, $\mu : H \times H \to N$ is a normalized 2-cocycle.

We begin by recalling a result of Wells [9]; since we are dealing with abelian extensions, the proof of this result in the present case is quite easy. However, for the reader's convenience, we include a proof here.

Lemma 2.2. *(See [9].) Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be an abelian extension. If $\gamma \in \mathrm{Aut}_N(G)$, then there is a triplet $(\theta, \phi, \chi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H) \times N^H$ such that for all $x, y \in H$ and $n \in N$ the following conditions are satisfied*:

(1) $\gamma(t(x)n) = t(\phi(x))\chi(x)\theta(n)$,
(2) $\mu(\phi(x), \phi(y))\theta(\mu(x, y)^{-1}) = (\chi(x)^{-1})^{t(\phi(y))}\chi(y)^{-1}\chi(xy)$,
(3) $\theta(n^{t(x)}) = \theta(n)^{t(\phi(x))}$.

[*Here N^H denotes the group of all maps ψ from H to N such that $\psi(1) = 1$.*]

Conversely, if $(\theta, \phi, \chi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H) \times N^H$ is a triplet satisfying equations of (2) *and* (3), *then γ defined by* (1) *is an automorphism of G normalizing N.*

Proof. Every automorphism $\gamma \in \mathrm{Aut}_N(G)$ determines a pair $(\theta, \phi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H)$ such that γ restricts to θ on N and induces ϕ on H. For any $x \in H$, we have $\pi(\gamma(t(x))) = \phi(x)$. Thus

$$\gamma\big(t(x)\big) = t\big(\phi(x)\big)\chi(x), \tag{2.3}$$

for some element $\chi(x) \in N$. Since $\chi(x)$ is unique for a given $x \in H$, it follows that χ is a map from H to N. Notice that $\chi(1) = 1$. Let $g \in G$. Then $g = t(x)n$ for some $x \in H$ and $n \in N$. Applying γ, we have

$$\gamma(g) = \gamma\big(t(x)\big)\theta(n) = t\big(\phi(x)\big)\chi(x)\theta(n). \tag{2.4}$$

Let $x, y \in H$. Then $t(xy)\mu(x, y) = t(x)t(y)$. On applying γ we get $\gamma(t(xy))\theta(\mu(x, y)) = \gamma(t(x))\gamma(t(y))$, since γ restricts to θ on N. By (2.3), we have $\gamma(t(x)) = t(\phi(x))\chi(x)$, $\gamma(t(y)) = t(\phi(y))\chi(y)$ and $\gamma(t(xy)) = t(\phi(xy))\chi(xy)$, and consequently

$$t\big(\phi(xy)\big)\chi(xy)\theta\big(\mu(x, y)\big) = t\big(\phi(x)\big)\chi(x)t\big(\phi(y)\big)\chi(y).$$

This, in turn, gives

$$\mu\big(\phi(x), \phi(y)\big)\theta\big(\mu(x, y)^{-1}\big) = \big(\chi(x)^{-1}\big)^{t(\phi(y))}\chi(y)^{-1}\chi(xy). \tag{2.5}$$

For $x \in H$ and $n \in N$, we have

$$\begin{aligned}\theta\big(n^{t(x)}\big) &= \gamma\big(n^{t(x)}\big) = \theta(n)^{\gamma(t(x))} \\ &= \theta(n)^{t(\phi(x))\chi(x)} = \theta(n)^{t(\phi(x))},\end{aligned} \tag{2.6}$$

since θ is the restriction of γ and $\chi(x)$ commutes with $\theta(n)$. Thus, given an element $\gamma \in \mathrm{Aut}_N(G)$, there is a triplet $(\theta, \phi, \chi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H) \times N^H$ satisfying equations of (1), (2) and (3).

Conversely, let $(\theta, \phi, \chi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H) \times N^H$ be a triplet satisfying equations of (2) and (3). We proceed to verify that γ defined by (1) is an automorphism of G normalizing N. Let $g_1 = t(x_1)n_1$ and $g_2 = t(x_2)n_2$ be elements of G, where $x_1, x_2 \in H$ and $n_1, n_2 \in N$. Then

$$\begin{aligned}\gamma(g_1g_2) &= \gamma\big(t(x_1)n_1t(x_2)n_2\big)\\ &= \gamma\big(t(x_1x_2)\mu(x_1,x_2)n_1^{t(x_2)}n_2\big)\\ &= t\big(\phi(x_1x_2)\big)\chi(x_1x_2)\theta\big(\mu(x_1,x_2)n_1^{t(x_2)}n_2\big)\\ &= t\big(\phi(x_1x_2)\big)\mu\big(\phi(x_1),\phi(x_2)\big)\chi(x_1)^{t(\phi(x_2))}\chi(x_2)\theta(n_1)^{t(\phi(x_2))}\theta(n_2)\\ &= t\big(\phi(x_1)\big)\chi(x_1)\theta(n_1)t\big(\phi(x_2)\big)\chi(x_2)\theta(n_2)\\ &= \gamma(g_1)\gamma(g_2).\end{aligned}$$

Hence γ is a homomorphism. Let $g = t(x)n$ be an element of G. Since ϕ and θ are onto, there exist elements $x' \in H$ and $n' \in N$ such that $\phi(x') = x$ and $\theta(n') = n$. We then have $\gamma(t(x')\theta^{-1}(\chi(x')^{-1})n') = g$. Hence γ is onto.

Finally, let $\gamma(t(x)n) = 1$. Then $t(\phi(x)) \in N$, and it easily follows that $t(x)n = 1$; consequently γ is one–one. Also $\gamma(n) = \theta(n)$ for $n \in N$. Therefore $\gamma \in \mathrm{Aut}_N(G)$. □

Remark 2.7. If $1 \to N \to G \xrightarrow{\pi} H \to 1$ is a central extension, then the action of H on N becomes trivial, and therefore Lemma 2.2 takes the following simpler form which we will use in the proof of Theorem 2.

Lemma 2.2′. *Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be a central extension. If $\gamma \in \mathrm{Aut}_N(G)$, then there exists a triplet $(\theta, \phi, \chi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H) \times N^H$ such that for all $x, y \in H$ and $n \in N$ the following conditions are satisfied*:

(1′) $\gamma(t(x)n) = t(\phi(x))\chi(x)\theta(n)$,
(2′) $\mu(\phi(x), \phi(y))\theta(\mu(x, y)^{-1}) = \chi(x)^{-1}\chi(y)^{-1}\chi(xy)$.

Conversely, if $(\theta, \phi, \chi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H) \times N^H$ is a triplet satisfying equation of (2′)*, then γ defined by* (1′) *is an automorphism of G normalizing N.*

For $\theta \in \mathrm{C}_1$ and $\phi \in \mathrm{C}_2$, we define maps $k_\theta, k_\phi : H \times H \to N$ by setting, for $x, y \in H$,

$$k_\theta(x, y) = \mu(x, y)\theta\big(\mu(x, y)^{-1}\big)$$

and

$$k_\phi(x, y) = \mu\big(\phi(x), \phi(y)\big)\mu(x, y)^{-1}.$$

Notice that, for $\theta \in \mathrm{C}_1$, we have $\theta(n^{t(x)}) = \theta(n)^{t(x)}$ for all $x \in H$ and $n \in N$. Similarly, for $\phi \in \mathrm{C}_2$, we have $n^{t(x)} = n^{t(\phi(x))}$ for all $x \in H$ and $n \in N$.

Lemma 2.8. *The maps k_θ and k_ϕ are normalized 2-cocycles.*

Proof. For $x, y, z \in H$, we have

$$\begin{aligned} k_\theta(xy,z)k_\theta(x,y)^{t(z)} &= \mu(xy,z)\theta\big(\mu(xy,z)^{-1}\big)\big(\mu(x,y)\theta\big(\mu(x,y)^{-1}\big)\big)^{t(z)} \\ &= \mu(xy,z)\mu(x,y)^{t(z)}\theta\big(\mu(xy,z)^{-1}\big(\mu(x,y)^{-1}\big)^{t(z)}\big) \\ &= \mu(x,yz)\mu(y,z)\theta\big(\mu(x,yz)^{-1}\mu(y,z)^{-1}\big) \quad \text{by (2.1)} \\ &= \mu(x,yz)\theta\big(\mu(x,yz)^{-1}\big)\mu(y,z)\theta\big(\mu(y,z)^{-1}\big) \\ &= k_\theta(x,yz)k_\theta(y,z), \end{aligned}$$

and $k_\theta(x,1) = 1 = k_\theta(1,x)$. Hence k_θ is a normalized 2-cocycle.

We next show that k_ϕ is a normalized 2-cocycle. For $x, y, z \in H$, we have

$$\begin{aligned} k_\phi(xy,z)k_\phi(x,y)^{t(z)} &= \mu\big(\phi(xy),\phi(z)\big)\mu(xy,z)^{-1}\big(\mu\big(\phi(x),\phi(y)\big)\mu(x,y)^{-1}\big)^{t(z)} \\ &= \mu\big(\phi(xy),\phi(z)\big)\mu\big(\phi(x),\phi(y)\big)^{t(z)}\mu(xy,z)^{-1}\big(\mu(x,y)^{-1}\big)^{t(z)} \\ &= \mu\big(\phi(x),\phi(yz)\big)\mu\big(\phi(y),\phi(z)\big)\mu(x,yz)^{-1}\mu(y,z)^{-1} \quad \text{by (2.1)} \\ &= \mu\big(\phi(x),\phi(yz)\big)\mu(x,yz)^{-1}\mu\big(\phi(y),\phi(z)\big)\mu(y,z)^{-1} \\ &= k_\phi(x,yz)k_\phi(y,z), \end{aligned}$$

and $k_\phi(x,1) = 1 = k_\phi(1,x)$. Thus the map k_ϕ is a normalized 2-cocycle. This completes the proof of the lemma. □

Define $\lambda_1 : \mathrm{C}_1 \to \mathrm{H}^2(H,N)$ by setting, for $\theta \in \mathrm{C}_1$,

$$\lambda_1(\theta) = [k_\theta], \quad \text{the cohomology class of } k_\theta; \tag{2.9}$$

similarly, define $\lambda_2 : \mathrm{C}_2 \to \mathrm{H}^2(H,N)$ by setting, for $\phi \in \mathrm{C}_2$,

$$\lambda_2(\phi) = [k_\phi], \quad \text{the cohomology class of } k_\phi. \tag{2.10}$$

To justify this definition, we need the following:

Lemma 2.11. *The maps λ_1 and λ_2 are well defined.*

Proof. To show that the maps λ_1 and λ_2 are well defined, we need to show that these maps are independent of the choice of transversals. Let $t, s : H \to N$ be two transversals with $t(1) = 1 = s(1)$. Then there exist maps $\mu, \nu : H \times H \to N$ such that for $x, y \in H$ we have $t(xy)\mu(x,y) = t(x)t(y)$ and $s(xy)\nu(x,y) = s(x)s(y)$. For $x \in H$, since $t(x)$ and $s(x)$ satisfy $\pi(t(x)) = x = \pi(s(x))$, there exists a unique element (say) $\lambda(x) \in N$ such that $t(x) = s(x)\lambda(x)$. We thus have a map $\lambda : H \to N$ with $\lambda(1) = 1$. For $x, y \in H$, $t(xy) = s(xy)\lambda(xy)$. This gives $t(x)t(y)\mu(x,y)^{-1} = s(x)s(y)\nu(xy)^{-1}\lambda(xy)$. Putting $t(u) = s(u)\lambda(u)$, where $u = x, y$, we have $\lambda(x)^{s(y)}\lambda(y)\lambda(xy)^{-1} = \mu(x,y)\nu(x,y)^{-1}$. Since $\lambda(1) = 1$, $\mu(x,y)\nu(x,y)^{-1} \in \mathrm{B}^2(H,N)$, the group of 2-coboundaries.

Similarly,

$$\begin{aligned} \theta\big(\mu(x,y)\big)\theta\big(\nu(x,y)^{-1}\big) &= \theta\big(\mu(x,y)\nu(x,y)^{-1}\big) \\ &= \theta\big(\lambda(x)^{s(y)}\lambda(y)\lambda(xy)^{-1}\big) \end{aligned}$$

$$= \theta\big(\lambda(x)^{s(y)}\big)\theta\big(\lambda(y)\big)\theta\big(\lambda(xy)^{-1}\big)$$
$$= \theta\big(\lambda(x)\big)^{s(y)}\theta\big(\lambda(y)\big)\theta\big(\lambda(xy)^{-1}\big)$$
$$= \lambda'(x)^{s(y)}\lambda'(y)\lambda'(xy)^{-1} \in \mathrm{B}^2(H, N),$$

where $\lambda' = \theta\lambda$. This proves that λ_1 is independent of the choice of a transversal.

Next we prove that λ_2 is well defined. It is sufficient to show that

$$\mu\big(\phi(x), \phi(y)\big)\nu\big(\phi(x), \phi(y)\big)^{-1} \in \mathrm{B}^2(H, N).$$

Just as above, we have

$$\lambda\big(\phi(x)\big)^{s(\phi(y))}\lambda\big(\phi(y)\big)\lambda\big(\phi(xy)\big)^{-1} = \mu\big(\phi(x), \phi(y)\big)\nu\big(\phi(x), \phi(y)\big)^{-1}.$$

Putting $\lambda\phi = \lambda''$, we get

$$\lambda''(x)^{s(\phi(y))}\lambda''(y)\lambda''(xy)^{-1} = \mu\big(\phi(x), \phi(y)\big)\nu\big(\phi(x), \phi(y)\big)^{-1}.$$

Since $n^{s(x)} = n^{s(\phi(x))}$ and $\lambda''(1) = 1$, $\mu(\phi(x), \phi(y))\nu(\phi(x), \phi(y))^{-1} \in \mathrm{B}^2(H, N)$. This proves that λ_2 is also independent of the choice of a transversal, and the proof of the lemma is complete. □

Proof of Theorem 1. Let $1 \to N \to G \to H \to 1$ be an abelian extension. Clearly both the sequences (1.1) and (1.2) are exact at the first two terms. To complete the proof it only remains to show the exactness at the third term of the respective sequences.

First consider (1.1). Let $\gamma \in \mathrm{Aut}_N^H(G)$. Then $\theta \in \mathrm{C}_1$, where θ is the restriction of γ to N. For $x, y \in H$, we have, by Lemma 2.2(2), $k_\theta(x, y) = (\chi(x)^{-1})^{t(y)}\chi(y)^{-1}\chi(xy)$. Thus $k_\theta \in \mathrm{B}^2(H, N)$ and hence $\lambda_1(\theta) = 1$. Conversely, if $\theta \in \mathrm{C}_1$ is such that $\lambda_1(\theta) = 1$, then for $x, y \in H$, we have $k_\theta(x, y) = (\chi(x)^{-1})^{t(y)}\chi(y)^{-1}\chi(xy)$, where $\chi : H \to N$ with $\chi(1) = 1$. Therefore γ defined by Lemma 2.2(1) is an element of $\mathrm{Aut}_N^H(G)$. Hence the sequence (1.1) is exact.

Next let us consider the sequence (1.2). Let $\gamma \in \mathrm{Aut}^N(G)$. Then $\phi \in \mathrm{C}_2$, where ϕ is induced by γ on H. For $x, y \in H$, we have $k_\phi(x, y) = (\chi(x)^{-1})^{t(\phi(y))}\chi(y)^{-1}\chi(xy)$ by Lemma 2.2(2). Since $n^{t(\phi(y))} = n^{t(y)}$ for all $n \in N$ and $y \in H$, we have $k_\phi \in \mathrm{B}^2(H, N)$ and hence $\lambda_2(\phi) = 1$. Conversely, if $\phi \in \mathrm{C}_2$ is such that $\lambda_2(\phi) = 1$, then, for $x, y \in H$, we have $k_\phi(x, y) = (\chi(x)^{-1})^{t(y)}\ \chi(y)^{-1}\chi(xy)$, where $\chi : H \to N$ is a map with $\chi(1) = 1$. Therefore γ defined by Lemma 2.2(1) is an element of $\mathrm{Aut}_H^N(G)$. Hence the sequence (1.2) is exact, and the proof of Theorem 1 is complete. □

Proof of Theorem 2. The sequence (1.3) is clearly exact at $\mathrm{Aut}^{N,H}$ and $\mathrm{Aut}_N(G)$. We construct the map λ, and show the exactness at $\mathrm{Aut}(N) \times \mathrm{Aut}(H)$. For $(\theta, \phi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H)$, define $k_{\theta,\phi} : H \times H \to N$ by setting, for $x, y \in H$,

$$k_{\theta,\phi}(x, y) = \mu\big(\phi(x), \phi(y)\big)\theta\big(\mu(x, y)\big)^{-1}.$$

Observe that for $x, y, z \in H$, we have $k_{\theta,\phi}(x, 1) = 1 = k_{\theta,\phi}(1, x)$ and

$$k_{\theta,\phi}(xy, z)k_{\theta,\phi}(x, y) = k_{\theta,\phi}(x, yz)k_{\theta,\phi}(y, z).$$

Thus $k_{\theta,\phi} \in \mathrm{Z}^2(H, N)$, the group of normalized 2-cocycles. Define $\lambda(\theta, \phi) = [k_{\theta,\phi}]$, the cohomology class of $k_{\theta,\phi}$ in $H^2(H, N)$. Proceeding as in the proof of Lemma 2.11, one can prove that λ is well defined. If $(\theta, \phi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H)$ is induced by some $\gamma \in \mathrm{Aut}_N(G)$, then by Lemma 2.2′, we have

$$k_{\theta,\phi}(x,y) = \chi(x)^{-1}\chi(y)^{-1}\chi(xy),$$

where $\chi : H \to N$ is a map with $\chi(1) = 1$. Thus $k_{\theta,\phi}(x,y) \in \mathrm{B}^2(H,N)$. Hence $\lambda(\theta,\phi) = 1$.

Conversely, if $(\theta,\phi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H)$ is such that $[k_{\theta,\phi}] = 1$, then $k_{\theta,\phi}(x,y) = \chi(x)^{-1}\chi(y)^{-1}\chi(xy)$ for some $\chi : H \to N$ with $\chi(1) = 1$. By Lemma 2.2′ there exists $\gamma \in \mathrm{Aut}_N(G)$ inducing θ and ϕ. Thus the sequence (1.3) is exact. □

Remark 2.12. The maps λ_1 and λ_2 are not homomorphisms, but they turn out to be derivations with respect to natural actions of C_1 and C_2 respectively on $\mathrm{H}^2(H,N)$. There is an action of C_1 on $\mathrm{Z}^2(H,N)$ given by $(\theta,k) \mapsto k^\theta$ for $\theta \in \mathrm{C}_1$ and $k \in \mathrm{Z}^2(H,N)$, where $k^\theta(x,y) = \theta(k(x,y))$ for $x, y \in H$. Note that if $k \in \mathrm{B}^2(H,N)$, then $k(x,y) = \chi(x)^{t(y)}\chi(y)\chi(xy)^{-1}$. Since $\theta(\chi(x)^{t(y)}) = \theta(\chi(x))^{t(y)}$, we have

$$k^\theta(x,y) = \theta\big(\chi(x)\big)^{t(y)}\theta\big(\chi(y)\big)\theta\big(\chi(xy)\big)^{-1}.$$

Putting $\chi' = \theta\chi$, we have $k^\theta(x,y) = \chi'(x)^{t(y)}\chi'(y)\chi'(xy)^{-1}$. Thus the action keeps $\mathrm{B}^2(H,N)$ invariant and hence induces an action on $\mathrm{H}^2(H,N)$ given by $(\theta,[k]) \mapsto [k^\theta]$.

One can see that for $\theta_1, \theta_2 \in \mathrm{C}_1$ and $x, y \in H$, we have

$$\begin{aligned} k_{\theta_1\theta_2}(x,y) &= \mu(x,y)\theta_1\theta_2\big(\mu(x,y)^{-1}\big) \\ &= \mu(x,y)\theta_1\big(\mu(x,y)^{-1}\big)\theta_1\big(\mu(x,y)\theta_2\big(\mu(x,y)^{-1}\big)\big) \\ &= k_{\theta_1}(x,y)k_{\theta_2}^{\theta_1}(x,y). \end{aligned}$$

Thus $k_{\theta_1\theta_2} = k_{\theta_1}k_{\theta_2}^{\theta_1}$ and hence $\lambda_1(\theta_1\theta_2) = \lambda_1(\theta_1)\lambda_1(\theta_2)^{\theta_1}$. Consequently λ_1 is a derivation with respect to this action.

Similarly, there is an action of C_2 on $\mathrm{Z}^2(H,N)$ given by $(\phi,k) \mapsto k^\phi$ for $\phi \in \mathrm{C}_2$ and $k \in \mathrm{Z}^2(H,N)$, where $k^\phi(x,y) = k(\phi(x),\phi(y))$ for $x, y \in H$. If $k \in \mathrm{B}^2(H,N)$, then $k(x,y) = \chi(x)^{t(y)}\chi(y)\ \chi(xy)^{-1}$. Since $\chi(\phi(x))^{t(\phi(y))} = \chi(\phi(x))^{t(y)}$, we have

$$k^\phi(x,y) = \chi\big(\phi(x)\big)^{t(y)}\chi\big(\phi(y)\big)\chi\big(\phi(xy)\big)^{-1}.$$

Putting $\chi' = \chi\phi$, we have $k^\phi(x,y) = \chi'(x)^{t(y)}\chi'(y)\chi'(xy)^{-1}$. Thus the action keeps $\mathrm{B}^2(H,N)$ invariant and hence induces an action on $\mathrm{H}^2(H,N)$ given by $(\phi,[k]) \mapsto [k^\phi]$. Just as above, one can see that λ_2 is a derivation with respect to this action.

3. Applications

In this section we give some applications of our exact sequences to lifting and extension of automorphisms in abelian extensions.

Proof of Theorem 4. (1) Let N be an abelian normal subgroup of a finite group G. Suppose that the restriction $\phi|_P$ of ϕ to any Sylow subgroup P/N of G/N lifts to an automorphism of P centralizing N. Then the pair $(1,\phi|_P)$ is compatible, and hence, as it is easy to see, $(1,\phi)$ is also compatible. Applying sequence (1.2) of Theorem 1 to the abelian extension $1 \to N \to P \to P/N \to 1$, we have that the cohomology class $[k_{\phi|_P}] = 1$ in $\mathrm{H}^2(P/N,N)$. It follows from the construction of the cochain complex defining the group cohomology, that the map $\mathrm{H}^2(G/N,N) \to \mathrm{H}^2(P/N,N)$ induced by the inclusion $P/N \hookrightarrow G/N$ maps the class $[k_\phi]$ to $[k_{\phi|_P}]$. However, by [2, Chapter III, Proposition 9.5(ii)], we have $[G/N : P/N][k_\phi] = 1$. Since this holds for at least one Sylow p-subgroup P/N of G/N, for each prime number p dividing $|G/N|$, it follows that $[k_\phi] = 1$. By exactness of sequence (1.2) of Theorem 1, ϕ lifts to an automorphism γ of G centralizing N.

(2) Let γ be a lift of ϕ to an automorphism of G centralizing N. To complete the proof it only needs to be observed that if P/N is a characteristic subgroup of G/N, then P is invariant under γ. So the restriction of γ to P is the required lift. $\square$

The proof of Theorem 7 is similar to the above proof and we omit the details.

For the case of central extensions, the sequence (1.3) yields the following result:

Corollary 3.1. *Let N be a central subgroup of a finite group G. Then a pair $(\theta, \phi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(G/N)$ lifts to an automorphism of G provided for some Sylow p-subgroup P/N of G/N, for each prime number p dividing $|G/N|$, $(\theta, \phi|_P) \in \mathrm{Aut}(N) \times \mathrm{Aut}(P/N)$ lifts to an automorphism of P.*

4. Splitting of sequences

Let $1 \to N \to G \to H \to 1$ be an abelian extension. Let $\mathrm{C}_1^* = \{\theta \in \mathrm{C}_1 \mid \lambda_1(\theta) = 1\}$ and $\mathrm{C}_2^* = \{\phi \in \mathrm{C}_2 \mid \lambda_2(\phi) = 1\}$. Then it follows from Theorem 1 that the sequences

$$1 \to \mathrm{Aut}^{N,H}(G) \to \mathrm{Aut}_N^H(G) \to \mathrm{C}_1^* \to 1 \tag{4.1}$$

and

$$1 \to \mathrm{Aut}^{N,H}(G) \to \mathrm{Aut}^N(G) \to \mathrm{C}_2^* \to 1 \tag{4.2}$$

are exact.

Similarly, let $1 \to N \to G \to H \to 1$ be a central extension and $\mathrm{C}^* = \{(\theta, \phi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H) \mid \lambda(\theta, \phi) = 1\}$. Then it follows from Theorem 2 that the sequence

$$1 \to \mathrm{Aut}^{N,H}(G) \to \mathrm{Aut}_N(G) \to \mathrm{C}^* \to 1 \tag{4.3}$$

is exact.

Theorem 8. *Let G be a finite group and N an abelian normal subgroup of G such that the sequence $1 \to N \to G \to H \to 1$ splits. Then the sequences* (4.1) *and* (4.2) *split. Further, if N is a central subgroup of G, then the sequence* (4.3) *also splits.*

Proof. Since the sequence $1 \to N \to G \to H \to 1$ splits, we have $\mu \in \mathrm{B}^2(H, N)$. We can write $G = N \rtimes H$ as a semidirect product of N by H. Every element $g \in G$ can be written uniquely as $g = hn$ with $h \in H$ and $n \in N$.

We first show that the sequence (4.1) splits. Note that $\mathrm{C}_1^* = \{\theta \in \mathrm{Aut}(N) \mid \theta(n^h) = \theta(n)^h$ for all $n \in N$ and $h \in H\}$. Define a map $\psi_1 : \mathrm{C}_1^* \to \mathrm{Aut}_N^H(G)$ by $\psi_1(\theta) = \gamma_1$, where $\gamma_1 : G \to G$ is given by $\gamma_1(g) = \gamma_1(hn) = h\theta(n)$ for $g = hn$ in G. Then for $g_1 = h_1 n_1$, $g_2 = h_2 n_2$ in G, we have

$$\begin{aligned}\gamma_1(g_1 g_2) &= \gamma_1\big((h_1 n_1)(h_2 n_2)\big) = \gamma_1\big(h_1 h_2 n_1^{h_2} n_2\big) \\ &= h_1 h_2 \theta\big(n_1^{h_2} n_2\big) = h_1 h_2 \theta(n_1)^{h_2} \theta(n_2) \\ &= \gamma_1(g_1)\gamma_1(g_2),\end{aligned}$$

showing that γ_1 is a homomorphism. It is easy to see that γ_1 is an automorphism of G which normalizes N and induces identity on H. Notice that ψ_1 is a section in the sequence (4.1) and hence the sequence splits.

Next we show that the sequence (4.2) splits. Notice that $C_2^* = \{\phi \in \operatorname{Aut}(H) \mid n^{\phi(h)} = n^h$ for all $n \in N$ and $h \in H\}$. Define a map $\psi_2 : C_2^* \to \operatorname{Aut}^N(G)$ by setting $\psi_2(\phi) = \gamma_2$, where $\gamma_2 : G \to G$ is given by $\gamma_2(g) = \gamma_2(hn) = \phi(h)n$ for $g = hn$ in G. Then for $g_1 = h_1n_1$, $g_2 = h_2n_2$ in G, we have

$$\begin{aligned}\gamma_2(g_1g_2) &= \gamma_2\big((h_1n_1)(h_2n_2)\big) = \gamma_2\big(h_1h_2n_1^{h_2}n_2\big)\\ &= \phi(h_1h_2)n_1^{h_2}n_2 = \phi(h_1)\phi(h_2)n_1^{h_2}n_2\\ &= \gamma_2(g_1)\gamma_2(g_2).\end{aligned}$$

This shows that γ_2 is a homomorphism. It is not difficult to show that γ_2 is an automorphism of G which centralizes N. Notice that ψ_2 is a section in the sequence (4.2) and hence the sequence splits.

Finally, we consider the sequence (4.3). Since N is central, G is a direct product of H and N. Notice that $C^* = \operatorname{Aut}(N) \times \operatorname{Aut}(H)$. For a given pair $(\theta, \phi) \in \operatorname{Aut}(N) \times \operatorname{Aut}(H)$, we can define $f \in \operatorname{Aut}(G)$ by $f(hn) = \phi(h)\theta(n)$ for $g = hn$ in G. This gives rise to a section in the sequence (4.3) and hence the sequence splits. □

Remark 4.4. The converse of Theorem 8 is not true, in general, as is shown by the following examples.

(1) Let $1 \to N \to G \to H \to 1$ be an exact sequence, where G is a non-abelian finite group of nilpotency class 2 such that $N = Z(G) = [G, G]$ and $H \simeq G/N$. Notice that this sequence does not split under the natural action of H on N. For, if the sequence splits, then G is a direct product of N and H. This implies that G is abelian, which is a contradiction. In this case, $\operatorname{Aut}^{N,H}(G) = \operatorname{Autcent}(G) = \operatorname{Aut}_N^H(G)$, where $\operatorname{Autcent}(G)$ is the group of central automorphisms of G. Thus from the exactness of sequence (4.1), $C_1^* = 1$ and the sequence splits.

(2) Consider an exact sequence $1 \to N \to G \to H \to 1$, where G is an extra-special 2-group of order 2^{2n+1} with $n = 1$ or 2, and $N = Z(G) = [G, G]$. Notice that the sequence does not split. For this sequence we have $\operatorname{Aut}^{N,H}(G) = \operatorname{Inn}(G) = \operatorname{Autcent}(G)$ and $\operatorname{Aut}^N(G) = \operatorname{Aut}(G)$. Define a map $\rho : H \times H \to N$ by $\rho(t(x)N, t(y)N) = [t(x), t(y)]$. Notice that ρ is a bilinear map. Let $\gamma \in \operatorname{Aut}(G)$ and $\phi = \tau_2(\gamma)$. Now

$$\begin{aligned}\rho\big(\phi\big(t(x)N\big), \phi\big(t(y)N\big)\big) &= \rho\big(\gamma\big(t(x)\big)N, \gamma\big(t(y)\big)N\big) = \big[\gamma\big(t(x)\big), \gamma\big(t(y)\big)\big]\\ &= \gamma\big(\big[t(x), t(y)\big]\big) = \big[t(x), t(y)\big]\\ &= \rho\big(t(x)N, t(y)N\big).\end{aligned}$$

This shows that ϕ, viewed as a linear transformation of the $\mathbb{F}_2$-vector space H, is orthogonal. Thus $\phi \in O(2n, 2)$. This shows that $C_2^* \subset O(2n, 2)$. It is well known that $\operatorname{Aut}(G)/\operatorname{Inn}(G)$ is isomorphic to the full orthogonal group $O(2n, 2)$. Thus from the exactness of the sequence (4.2), we have $C_2^* = O(2n, 2)$. It follows from [5, Theorem 1] that the sequence (4.2) splits.

(3) Let $1 \to N \to G \to H \to 1$ be the exact sequence of example (2) above. Since $\operatorname{Aut}(N) = 1$, Eqs. (4.2) and (4.3) are the same. Hence the sequence (4.3) splits while $1 \to N \to G \to H \to 1$ does not split.

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